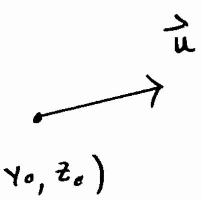
13.6 Directional Derivatives and Gradients

Let $f(x,y,z)$ be a function. Suppose you are standing at the point (x_0, y_0, z_0), and you are interested in the rate of change of f if you were to walk in the direction of some vector $\vec{u}$.



What you are interested in is the <u>directional derivative of f at (x_0, y_0, z_0) in the direction of the unit vector $\vec{u}$</u>.
This is denoted and given by

$$\boxed{D_{\vec{u}} f(x_0, y_0, z_0) = \vec{\nabla} f(x_0, y_0, z_0) \cdot \vec{u}}$$

$$\vec{\nabla} f(x,y,z) = \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z) \rangle$$

is called the gradient of f. It is a vector.

<u>Note</u>: In the boxed formula, <u>$\vec{u}$ must be a unit vector</u>.

OK. We know how to find the rate of change of f in all directions. The question now is: In which direction does f change the fastest and what is the maximum rate of change?

Translation: In which direction is $D_{\vec{u}} f(x_0, y_0, z_0)$ maximum and what is the maximum value of $D_{\vec{u}} f(x_0, y_0, z_0)$?

Answer: The maximum value of $D_{\vec{u}} f(x_0, y_0, z_0)$ occurs in the direction of $\vec{\nabla} f(x_0, y_0, z_0)$ and the maximum value of $D_{\vec{u}} f(x_0, y_0, z_0)$ is $\|\vec{\nabla} f(x_0, y_0, z_0)\|$.

(Also, the minimum value of $D_{\vec{u}} f(x_0, y_0, z_0)$ occurs in the direction of $-\vec{\nabla} f(x_0, y_0, z_0)$ and the minimum value is $-\|\vec{\nabla} f(x_0, y_0, z_0)\|$.)

Remark: All of the above applies to a function of two variables $f(x,y)$. Here $\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

It also applies to functions of 4, 5, 300, etc. variables.

Problem 8 from Review

(a) we are looking for

$$D_{\vec{u}} f(\tfrac{\pi}{2}, -2, 0) = \vec{\nabla} f(\tfrac{\pi}{2}, -2, 0) \cdot \vec{u}$$

A vector in the desired direction is

$$\vec{v} = \vec{PQ} = \langle 0, 1, 6 \rangle$$

To apply the formula, we need a unit vector in this direction

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 0, 1, 6 \rangle}{\sqrt{37}} = \langle 0, \tfrac{1}{\sqrt{37}}, \tfrac{6}{\sqrt{37}} \rangle$$

we still need to compute $\vec{\nabla} f(\tfrac{\pi}{2}, -2, 0)$

$$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$$

$$f_x(x,y,z) = e^{\sin(xy)} (\cos(xy))\, y$$

$$f_y(x,y,z) = e^{\sin(xy)} (\cos(xy))\, x$$

$$f_z(x,y,z) = 2z$$

$$f_x(\tfrac{\pi}{2}, -2, 0) = e^{\sin(-\pi)} \cos(-\pi)(-2) = 2$$

$$f_y(\tfrac{\pi}{2}, -2, 0) = e^{\sin(-\pi)} \cos(-\pi)\, \tfrac{\pi}{2} = -\tfrac{\pi}{2}$$

$$f_z(\tfrac{\pi}{2}, -2, 0) = 0$$

So, $\vec{\nabla} f(\frac{\pi}{2}, -2, 0) = \langle 2, \frac{-\pi}{2}, 0 \rangle$ and

$$D_{\vec{u}} f(\tfrac{\pi}{2}, -2, 0) = \langle 2, \tfrac{-\pi}{2}, 0 \rangle \cdot \langle 0, \tfrac{1}{\sqrt{37}}, \tfrac{6}{\sqrt{37}} \rangle$$

$$= \frac{-\pi}{2\sqrt{37}}$$

(b) and (c)

The maximum value of $D_{\vec{u}} f(\frac{\pi}{2}, -2, 0)$ occurs in the direction of

$$\vec{\nabla} f(\tfrac{\pi}{2}, -2, 0) = \langle 2, \tfrac{-\pi}{2}, 0 \rangle$$

Answer to (c)

and the maximum value is

$$\| \vec{\nabla} f(\tfrac{\pi}{2}, -2, 0) \| = \sqrt{4 + \frac{\pi^2}{4}} = \frac{\sqrt{16 + \pi^2}}{2}$$

Answer to (b)

13.7 Tangent Planes and Normal Lines

Fact: The gradient of a function at a point P is normal to the level surface of that function through P.

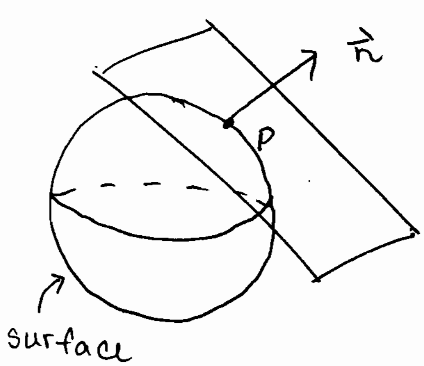
OR

Just remember: "gradient is normal to level surface."

Problem 9 from Review

(b) Our surface is actually the graph of the function. Our surface is

$$z = x \tan y$$



Picture of general situation

we need a vector normal to the surface.

We know "gradient is normal to level surface."

Q: Is our surface the level surface of some function?

A: $z = x \tan y$ is equivalent to

$x \tan y - z = 0$

If we let $F(x,y,z) = x \tan y - z$ then our surface is precisely the level surface $F(x,y,z) = 0$.

So, $\vec{\nabla} F(2, \frac{\pi}{4}, 2)$ will be normal to the surface at $(2, \frac{\pi}{4}, 2)$.

$\vec{\nabla} F(x,y,z) = \langle F_x, F_y, F_z \rangle = \langle \tan y, x \sec^2 y, -1 \rangle$

$\vec{\nabla} F(2, \frac{\pi}{4}, 2) = \langle 1, 4, -1 \rangle$

Eq of tangent plane to surface at $(2, \frac{\pi}{4}, 2)$

$\vec{n} = \vec{\nabla} F(2, \frac{\pi}{4}, 2) = \langle 1, 4, -1 \rangle$

Point $(2, \frac{\pi}{4}, 2)$

$$1(x-2) + 4\left(y - \frac{\pi}{4}\right) + -1(z-2) = 0$$

Eq of normal line to surface at $(2, \frac{\pi}{4}, 2)$

direction vector $\vec{v} = \langle 1, 4, -1 \rangle$

Point $(2, \frac{\pi}{4}, 2)$

Parametric Eq

$x = 2 + t$

$y = \frac{\pi}{4} + 4t$

$z = 2 - t$

Symmetric Eq

$$\frac{x-2}{1} = \frac{y - \frac{\pi}{4}}{4} = \frac{z-2}{-1}$$

(a) our surface $xy^2 + zy^2 + 4y - xz^2 = 18$

$\underline{\underline{1^\circ}}$ Express surface as level surface of some function F.

The above equation is equivalent to

$$xy^2 + zy^2 + 4y - xz^2 - 18 = 0$$

Let $F(x,y,z) = xy^2 + zy^2 + 4y - xz^2 - 18$

our surface is precisely the level surface

$F(x,y,z) = 0$.

$\underline{\underline{2^\circ}}$ "gradient is normal to level surface"

$$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle = \langle y^2 - z^2, \; 2xy + 2zy + 4, \; y^2 - 2xz \rangle$$

$$\vec{\nabla} F(-2,0,3) = \langle -9, 4, 12 \rangle$$

Eq of tangent plane to surface at $(-2,0,3)$

$\vec{n} = \langle -9, 4, 12)$

point $(-2, 0, 3)$

$$-9(x+2) + 4(y) + 12(z-3) = 0$$

Eq of normal line to surface at $(-2,0,3)$

direction vector $\vec{v} = \langle -9, 4, 12 \rangle$

point $(-2,0,3)$

parametric eq

$x = -2 - 9t$

$y = 4t$

$z = 3 + 12t$

Symmetric Eq

$$\frac{x+2}{-9} = \frac{y}{4} = \frac{z-3}{12}$$