13.8 Extrema of Functions of Two Variables

Let $f(x,y)$ be a function.

Main Problem: Find $\underbrace{\text{relative extrema}}_{\text{rel max \& rel min}}$ and saddle points.

Step 1: Find critical points by solving

$$\vec{\nabla} f = \langle f_x, f_y \rangle = 0$$

which is equivalent to the system

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

Step 2: Apply 2^{nd} Partials Test.

$$d = f_{xx} f_{yy} - (f_{xy})^2$$

Let (a,b) be one of the critical pts.

If $d(a,b) < 0 \Rightarrow$ saddle pt at (a,b)

If $d(a,b) > 0$ & $f_{xx}(a,b) > 0 \Rightarrow$ rel min at (a,b)

If $d(a,b) > 0$ & $f_{xx}(a,b) < 0 \Rightarrow$ rel max at (a,b)

If $d(a,b) = 0 \Rightarrow$ no information.

Problem 10 from Review

(b) $f(x,y) = x^3 + 12xy + y^3$

$f_x(x,y) = 3x^2 + 12y$ $\qquad$ $f_{xx}(x,y) = 6x$

$f_y(x,y) = 12x + 3y^2$ $\qquad$ $f_{yy}(x,y) = 6y$

$f_{xy}(x,y) = 12$

$\underline{\underline{1^\circ}}$ Find critical points

we solve

$$3x^2 + 12y = 0 \qquad (1)$$

$$12x + 3y^2 = 0 \qquad (2)$$

solving for y in (1) gives $y = -\frac{1}{4}x^2$

Plugging this into (2) yields

$$12x + 3\left(-\tfrac{1}{4}x^2\right)^2 = 0$$

$$12x + \tfrac{3}{16}x^4 = 0$$

$$3x\left(4 + \tfrac{1}{16}x^3\right) = 0$$

$x = 0$

corresponding $y = 0$

or

$4 + \frac{1}{16}x^3 = 0$

$x^3 = -64$

$x = -4$

corresponding $y = -4$

$\therefore$ Critical points are $(0,0)$ and $(-4,-4)$

$\underline{\underline{2^\circ}}$ Apply 2nd partials test

$$d = f_{xx} f_{yy} - (f_{xy})^2$$

$$d(x,y) = (6x)(6y) - (12)^2$$

$$= 36xy - 144$$

For critical point $(0,0)$

$$d(0,0) = -144 < 0$$

$\therefore$ saddle pt at $(0,0)$

For critical point $(-4,-4)$

$$d(-4,-4) > 0$$

$$f_{xx}(-4,-4) = -24 < 0$$

$\therefore$ rel max at $(-4,-4)$

(c) $f(x,y) = 3xy - x^2y - xy^2$

$f_x(x,y) = 3y - 2xy - y^2$

$f_y(x,y) = 3x - x^2 - 2xy$

$f_{xx}(x,y) = -2y$

$f_{yy}(x,y) = -2x$

$f_{xy}(x,y) = 3 - 2x - 2y$

$\underline{\underline{1^o}}$. Find critical points

we solve

$$3y - 2xy - y^2 = 0 \quad (1)$$
$$3x - x^2 - 2xy = 0 \quad (2)$$

From (1) we have that

$$y(3-2x-y) = 0$$

So,

$y = 0$ or $3 - 2x - y = 0$

Plugging $y=0$ into (2) gives

$$3x - x^2 = 0$$
$$x(3-x) = 0$$

$x = 0$ or $x = 3$

$(0,0)$ & $(3,0)$ are critical pts

$$y = 3 - 2x$$

Plugging this into (2) yields

$$3x - x^2 - 2x(3-2x) = 0$$
$$3x - x^2 - 6x + 4x^2 = 0$$
$$3x^2 - 3x = 0$$
$$3x(x-1) = 0$$

$x = 0$, $y = 3$ or $x = 1$, $y = 1$

$(0,3)$ & $(1,1)$ are critical pts.

$\overline{|12}$

we have four critical points

$(0,0) \quad (3,0) \quad (0,3) \quad (1,1)$

$\underline{\underline{2^\circ}}$. Apply 2^{nd} Partials Test

$$d = f_{xx} f_{yy} - (f_{xy})^2$$

$$d(x,y) = (-2y)(-2x) - (3-2x-2y)^2$$

$$= 4xy - (3-2x-2y)^2$$

crit point $(0,0)$

$d(0,0) = -9 < 0$

saddle pt at $(0,0)$

Crit point $(3,0)$

$d(3,0) < 0$

saddle pt at $(3,0)$

Crit point $(0,3)$

$d(0,3) < 0$

saddle pt at $(0,3)$

Crit pt $(1,1)$

$d(1,1) = 4 - (-1)^2 = 3 > 0$

$f_{xx}(1,1) = -2 < 0$ $\Rightarrow$ rel max at $(1,1)$.

(a) $f(x,y) = x^3 + y^5 + 3x^2 - 9x - 5y - 8$

$f_x(x,y) = 3x^2 + 6x - 9$

$f_y(x,y) = 5y^4 - 5$

$f_{xx}(x,y) = 6x + 6$

$f_{yy}(x,y) = 20y^3$

$f_{xy}(x,y) = 0$

$\underline{\underline{1^\circ}}$ Find critical points

we solve

$$3x^2 + 6x - 9 = 0 \qquad (1)$$

$$5y^4 - 5 = 0 \qquad (2)$$

From (1) we have

$$3(x^2 + 2x - 3) = 0$$

$$3(x+3)(x-1) = 0$$

$x = -3$ or $x = 1$

plugging this into (2) gives us

$$5y^4 - 5 = 0$$

$$y^4 = 1$$

$y = 1$ or $y = -1$

$(-3, 1)$ & $(-3, -1)$ are crit pts.

$(1, 1)$ and $(1, -1)$ are critical pts.

we have four critical pts

$(-3, 1)$ $(-3, -1)$ $(1, 1)$ $(1, -1)$

$\underline{\underline{2^\circ}}$ Apply 2^{nd} partials Test

$d = f_{xx} f_{yy} - (f_{xy})^2$

$d(x,y) = (6x + 6)(20y^3)$

Crit pt $(-3, 1)$

$d(-3, 1) < 0$

saddle pt at $(-3, 1)$

Crit pt $(-3, -1)$

$d(-3, -1) > 0$

$f_{xx}(-3, -1) < 0$

rel max at $(-3, -1)$

Crit pt $(1, 1)$

$d(1, 1) > 0$

$f_{xx}(1, 1) > 0$

rel min at $(1, 1)$

Crit pt $(1, -1)$

$d(1, -1) < 0$

saddle pt at $(1, -1)$.

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