# Complex Numbers

A **complex number** is a number of the form

$$a + bi$$

where $a$ and $b$ are real numbers and
$i$ is a symbol with the property $i^2 = -1$
$(i = \sqrt{-1})$

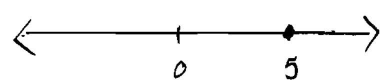
Ex. $\frac{-\sqrt{3}}{4} + \frac{1}{4}i$ is a complex number.

In Calc I, we learned how to add, subtract, multiply, and divide complex numbers.

## 1°. Polar Form

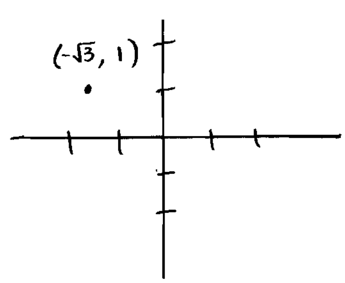
Every real number $x$ corresponds to a point on the real line.

Ex. $x = 5$

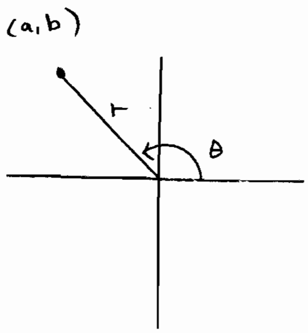



Similarly, the complex number $z = a + bi$ corresponds to the point $(a, b)$ in the plane.

Ex. $z = -\sqrt{3} + i$



**Recall.** we can describe points in the plane using either rectangular coordinates or polar coordinates. Thus, the point $(a,b)$ also has a polar coordinate description $(r,\theta)$.




$(*)\quad a = r\cos\theta$

$(**)\quad b = r\sin\theta$

$(\#)\quad r^2 = a^2 + b^2$

$(\#\#)\quad \tan\theta = \dfrac{b}{a}$

**Upshot.** Given a complex number $z = a + bi$, we can plot $(a,b)$ in the plane, find $r$ and $\theta$ and, using $(*)$ and $(**)$, write $z$ as:

$$z = a + bi$$

$$= (r\cos\theta) + (r\sin\theta)\,i$$

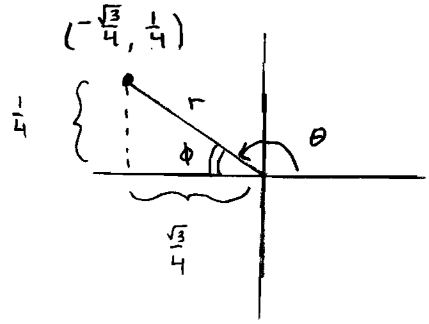
$$\boxed{z = r(\cos\theta + i\sin\theta)}$$ **Polar Form.**

$\theta$ is called the argument of $z$

$r$ is called the modulus of $z$.

Ex. Express $\frac{-\sqrt{3}}{4} + \frac{1}{4}i$ in polar form.

Soln.




$$r^2 = \left(-\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{3}{16} + \frac{1}{16} = \frac{1}{4}$$

$$r = \frac{1}{2}$$

$$\tan \phi = \frac{\frac{1}{4}}{\frac{\sqrt{3}}{4}} = \frac{1}{\sqrt{3}}$$

$$\phi = \frac{\pi}{6}$$

So, $\theta = \frac{5\pi}{6}$

$$z = \frac{1}{2}\left(\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}\right)$$

Why do we care about the polar form of a complex number?

## $\underline{2}^{\circ}$. Multiplication and Division

Let $z_1$ and $z_2$ be complex numbers. If we write $z_1$ and $z_2$ in polar form, ie

$$z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

then

$$z_1 z_2 = r_1 r_2 \left( \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left( \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right)$$

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To multiply two complex numbers, we multiply the moduli and add the arguments. ($\Delta$)

To divide two complex numbers, we divide the moduli and subtract the arguments. ($\Delta\Delta$)

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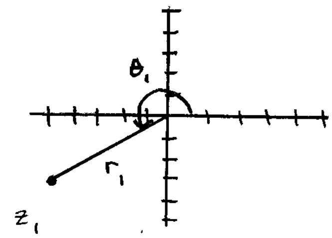
This gives us a feel for what complex multiplication / divison is doing geometrically. Consider the following Example.

Example




Let $z_1 = -5 - 3i$ and $z_2 = 1 + i$

using just our Calc I knowledge, it's not clear what $z_1 z_2 = (-5-3i)(1+i)$ is without actually doing the calculation. we don't have an intuitive feel for complex multiplication.

But, now that we have the polar form ...
check that $z_2$ in polar form is $z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$
what is the effect of multiplying $z_1$ by $z_2$?
what happens going from $z_1$ to $z_1 z_2$?

Well by (A), the modulus of $z_1$ will be multiplied by $\sqrt{2}$, and $\frac{\pi}{4}$ will be added to the argument of $z_1$.

What does this mean geometrically?
Note that the modulus is the distance to the origin (it is the length of the line segment in the figure above)
So, going from $z_1$ to $z_1 z_2$, multiplying the modulus of $z_1$ by $\sqrt{2}$ means stretching the line segment by a factor of $\sqrt{2}$. Adding $\frac{\pi}{4}$ to the argument of $z_1$ means rotating the line segment counter-clockwise by $\frac{\pi}{4}$.

Upshot. Complex multiplication / division is effectively a stretch or shrink and a rotation.

## $\underline{\underline{3^\circ}}$ De Moivre's Thm (obtained by applying $(\Delta)$ repeatedly.)

If $z = r(\cos\theta + i\sin\theta)$ and $n$ is a positive integer then

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

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To take the $n^{th}$ power of a complex number, we take the $n^{th}$ power of the modulus and multiply the argument by $n$.

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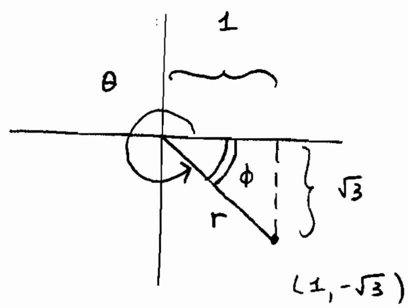
Ex. Find $(1-\sqrt{3}\,i)^5$ using De Moivre's Thm

Soln. we need $z = 1-\sqrt{3}\,i$ in polar form.




$$r^2 = 1^2 + (-\sqrt{3})^2 = 4$$

$$r = 2$$

$$\tan\phi = \frac{\sqrt{3}}{1}$$

$$\phi = \frac{\pi}{3}$$

So, $\theta = \dfrac{5\pi}{3}$

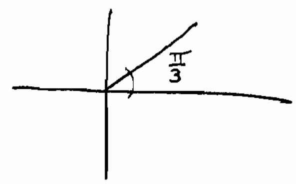
$$z = 2\left(\cos\left(\tfrac{5\pi}{3}\right) + i\sin\left(\tfrac{5\pi}{3}\right)\right)$$

By DeMoivre's Thm,

$$z^5 = 2^5 \left( \cos\left(\frac{25\pi}{3}\right) + i \sin\left(\frac{25\pi}{3}\right) \right)$$

$$= 32 \left( \cos\left(8\pi + \frac{\pi}{3}\right) + i \sin\left(8\pi + \frac{\pi}{3}\right) \right)$$

$8\pi + \frac{\pi}{3}$ is 4 revolutions $+ \frac{\pi}{3}$




$$= 32 \left( \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right)$$

$$= 32 \left( \frac{1}{2} + \frac{\sqrt{3}}{2} i \right)$$

$$= 16 + 16\sqrt{3}\, i$$

Observe that multiplying $(1 - \sqrt{3}\, i)^5$ out without DeMoivre's Thm and without a calculator would be a pain!

## $4^\circ$ $n^{th}$ roots of a complex number

Let $z = r(\cos\theta + i\sin\theta)$ be a complex number, and let $n$ be a positive integer. Then $z$ has $n$ distinct $n^{th}$ roots given by

$$u_k = r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

for $k = 0, 1, 2, \dots, n-1$.

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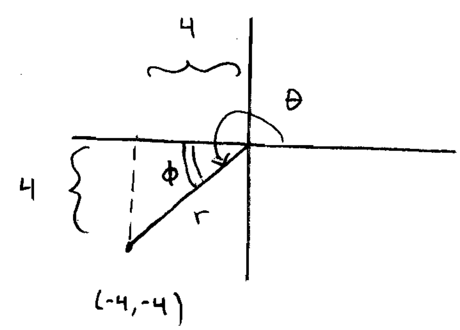
__Ex__ Find the sixth roots of $-4-4i$

__Soln__. We need $z = -4-4i$ in polar form.




$$r^2 = 16 + 16 = 32$$

$$r = \sqrt{32} = 4\sqrt{2}$$

$$\tan\phi = \frac{4}{4} = 1$$

$$\phi = \frac{\pi}{4}$$

So, $\theta = \frac{5\pi}{4}$

Thus, $z = 4\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$

We apply the Thm. Here $n = 6$

$k = 0, 1, 2, 3, 4, 5$

$$u_0 = (4\sqrt{2})^{1/6}\left(\cos\left(\frac{\frac{5\pi}{4}+0}{6}\right) + i\sin\left(\frac{\frac{5\pi}{4}+0}{6}\right)\right)$$

$$u_1 = (4\sqrt{2})^{1/6}\left(\cos\left(\frac{\frac{5\pi}{4}+2\pi}{6}\right) + i\sin\left(\frac{\frac{5\pi}{4}+2\pi}{6}\right)\right)$$

$$u_2 = (4\sqrt{2})^{1/6}\left(\cos\left(\frac{\frac{5\pi}{4}+4\pi}{6}\right) + i\sin\left(\frac{\frac{5\pi}{4}+4\pi}{6}\right)\right)$$

$$u_3 = (4\sqrt{2})^{1/6}\left(\cos\left(\frac{\frac{5\pi}{4}+6\pi}{6}\right) + i\sin\left(\frac{\frac{5\pi}{4}+6\pi}{6}\right)\right)$$

$$u_4 = (4\sqrt{2})^{1/6}\left(\cos\left(\frac{\frac{5\pi}{4}+8\pi}{6}\right) + i\sin\left(\frac{\frac{5\pi}{4}+8\pi}{6}\right)\right)$$

$$u_5 = (4\sqrt{2})^{1/6}\left(\cos\left(\frac{\frac{5\pi}{4}+10\pi}{6}\right) + i\sin\left(\frac{\frac{5\pi}{4}+10\pi}{6}\right)\right)$$

The only reasonable simplification from here is to simplify the arguments.

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The End