Exam 1 Review Solutions

Problem 1

(a) This is done in "Parametric Curves" Notes pg 10 - 11. The answer is

Horizontal tangents occur when

$$t = 0, 2$$

Vertical tangents occur when

$$t = 3.$$

(b)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3 - 16t}{6t^2 - 6t - 36} \quad \text{at } t = 1 \quad \frac{dy}{dx} = \frac{-12}{-36} = \frac{1}{3}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{4t^3 - 16t}{6t^2 - 6t - 36}\right]}{6t^2 - 6t - 36}$$

$$= \frac{\frac{(6t^2 - 6t - 36)(12t^2 - 16) - (4t^3 - 16t)(12t - 6)}{(6t^2 - 6t - 36)^2}}{6t^2 - 6t - 36}$$

$$= \frac{(6t^2-6t-36)(12t^2-16)-(4t^3-16t)(12t-6)}{(6t^2-6t-36)^3}$$

when $t = 1$ $\quad \dfrac{d^2y}{dx^2} = \dfrac{(-36)(-4)-(-12)(6)}{(-36)^3}$

$$= \frac{216}{-46656}$$

The curve is concave down.

$\boxed{E}$

(Problem 2 appears after Problem 5)

Problem 4

(a)(i) $\dfrac{dy}{dt} = \frac{3}{2}(4t+9)^{1/2}(4) = 6(4t+9)^{1/2}$

$\dfrac{dx}{dt} = 4t$

$$S = \int_0^1 \sqrt{(4t)^2 + (6(4t+9)^{1/2})^2}\, dt$$

$$= \int_0^1 \sqrt{16t^2 + 36(4t+9)}\, dt$$

$$= \int_0^1 \sqrt{16t^2 + 144t + 324}\, dt$$

$$= \int_0^1 \sqrt{4(4t^2 + 36t + 81)}\, dt$$

$$= \int_0^1 2\sqrt{(2t+9)^2}\, dt$$

$$= \int_0^1 2\,|2t+9|\, dt$$

$$= \int_0^1 2(2t+9)\, dt \qquad \left(\begin{array}{l} \text{On } 0 \leq t \leq 1, \quad 2t+9 \geq 0. \\ \text{So, } |2t+9| = 2t+9. \end{array}\right)$$

$$= \int_0^1 (4t + 18)\, dt$$

$$= \left[4\frac{t^2}{2} + 18t\right]_0^1 = \underline{\underline{20}}$$

(ii) very similar to problem in "Parametric curves"

(b) $$\frac{dr}{d\theta} = 3\left(\cos^2\frac{\theta}{3}\right)\left(-\sin\frac{\theta}{3}\right)\frac{1}{3}$$

$$= -\cos^2\frac{\theta}{3}\sin\frac{\theta}{3}$$

$$S = \int_0^{3\pi} \sqrt{\left(\cos^3\frac{\theta}{3}\right)^2 + \left(-\cos^2\frac{\theta}{3}\sin\frac{\theta}{3}\right)^2}\, d\theta$$

$$= \int_0^{3\pi} \sqrt{\cos^6\frac{\theta}{3} + \cos^4\frac{\theta}{3}\sin^2\frac{\theta}{3}}\, d\theta$$

$$= \int_0^{3\pi} \sqrt{\cos^4 \frac{\theta}{3} \left(\underbrace{\cos^2 \frac{\theta}{3} + \sin^2 \frac{\theta}{3}}_{1} \right)} \, d\theta$$

$$= \int_0^{3\pi} \left| \cos^2 \frac{\theta}{3} \right| d\theta$$

$$= \int_0^{3\pi} \cos^2 \frac{\theta}{3} \, d\theta \qquad \left(\begin{array}{l} \cos^2 \frac{\theta}{3} \geq 0 \\ \text{so, } \left|\cos^2 \frac{\theta}{3}\right| = \cos^2 \frac{\theta}{3} \end{array} \right)$$

$$= \int_0^{3\pi} \frac{1 + \cos\left(\frac{2\theta}{3}\right)}{2} \, d\theta$$

$$= \frac{1}{2} \int_0^{3\pi} \left(1 + \cos\left(\frac{2\theta}{3}\right) \right) d\theta$$

$$= \frac{1}{2} \left[\theta + \frac{3}{2} \sin\left(\frac{2\theta}{3}\right) \right]_0^{3\pi}$$

$$= \frac{3\pi}{2}$$

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Problem 5

(a) We transform the polar eq $r = 3 - 3\cos\theta$ into parametric form as follows

$$x = r\cos\theta = 3\cos\theta - 3\cos^2\theta$$

$$y = r\sin\theta = 3\sin\theta - 3\sin\theta\cos\theta$$

Now we use ideas from 10.3 to solve the problem.

In particular, slope $= \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$$\frac{dy}{d\theta} = 3\cos\theta - 3\left[\sin\theta(-\sin\theta) + \cos\theta(\cos\theta)\right]$$

$$= 3\cos\theta - 3\cos^2\theta + 3\sin^2\theta$$

$$\frac{dx}{d\theta} = 3(-\sin\theta) - 3\cdot 2\cos\theta(-\sin\theta)$$

$$= -3\sin\theta + 6\sin\theta\cos\theta$$

So, $$\frac{dy}{dx} = \frac{3\cos\theta - 3\cos^2\theta + 3\sin^2\theta}{-3\sin\theta + 6\sin\theta\cos\theta}$$

At $\theta = \frac{\pi}{4}$

$$\frac{dy}{dx} = \frac{3\left(\frac{1}{\sqrt{2}}\right) - 3\left(\frac{1}{\sqrt{2}}\right)^2 + 3\left(\frac{1}{\sqrt{2}}\right)^2}{-3\left(\frac{1}{\sqrt{2}}\right) + 6\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{\frac{3}{\sqrt{2}}}{-\frac{3}{\sqrt{2}} + 3}$$

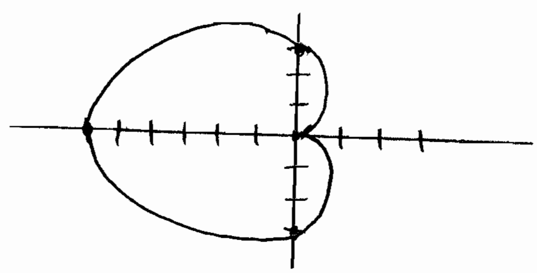
$$= \frac{3}{-3 + 3\sqrt{2}}$$

(c). For a polar curve, arc length is given by

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

The limits are not given to us. So, we find them by sketching a picture.

θ	r
0	0
$\frac{\pi}{2}$	3
π	6
$\frac{3\pi}{2}$	3



$$\text{Arc Length} = 2\Bigg[\underbrace{\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}_{\text{length of half of cardiod.}} \Bigg]$$

$$= 2\int_0^{\pi} \sqrt{(3-3\cos\theta)^2 + (3\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{9 - 18\cos\theta + \underbrace{9\cos^2\theta + 9\sin^2\theta}_{9(\cos^2\theta + \sin^2\theta) = 9}}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{18 - 18\cos\theta}\, d\theta$$

$$= 2\sqrt{18}\int_0^{\pi} \sqrt{1-\cos\theta}\, d\theta$$

$$= 6\sqrt{2}\int_0^{\pi} \sqrt{1-\cos\theta} \cdot \frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}\, d\theta$$

$$= 6\sqrt{2}\int_0^{\pi} \frac{\sqrt{1-\cos^2\theta}}{\sqrt{1+\cos\theta}}\, d\theta$$

$$= 6\sqrt{2}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\, d\theta$$

$\left(\sqrt{1-\cos^2\theta} = \sqrt{\sin^2\theta} = |\sin\theta| \right.$
On $0 \le \theta \le \pi$, $\sin\theta \ge 0$.
so, $\left. |\sin\theta| = \sin\theta \right)$

$u = 1 + \cos\theta$

$du = -\sin\theta \; d\theta$

$$= 6\sqrt{2}\int_2^{0} -u^{-1/2}\, d\theta$$

$$= 6\sqrt{2}\left[-\frac{u^{1/2}}{1/2} \right]_2^0 = \underline{\underline{24}}$$

Problem 2

$$\frac{dy}{d\theta} = 3\cos\theta - 3\left[\sin\theta(-\sin\theta) + \cos\theta(\cos\theta)\right]$$

$$= 3\cos\theta - 3\cos^2\theta + 3\sin^2\theta$$

$$= 3\cos\theta - 3\cos^2\theta + 3(1-\cos^2\theta)$$ Trig identity

$$= 3 + 3\cos\theta - 6\cos^2\theta$$

$$= 3(1 + \cos\theta - 2\cos^2\theta)$$

$$= 3(1 + 2\cos\theta)(1 - \cos\theta)$$

$\frac{dy}{d\theta} = 0$ when $1 + 2\cos\theta = 0$ or $1 - \cos\theta = 0$

Solving these yield :

$$1 + 2\cos\theta = 0 \qquad\qquad 1 - \cos\theta = 0$$

$$\cos\theta = -\frac{1}{2} \qquad\qquad \cos\theta = 1$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \qquad\qquad \theta = 0$$

$$\frac{dx}{d\theta} = 3(-\sin\theta) - 3\cdot 2\cos\theta(-\sin\theta)$$

$$= -3\sin\theta + 6\sin\theta\cos\theta$$

$$= -3\sin\theta(1 - 2\cos\theta)$$

$\frac{dx}{d\theta} = 0$ when $\sin\theta = 0$ or $1 - 2\cos\theta = 0$

$\theta = 0, \pi$ $\quad$ $\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Thus, horizontal tangents occur when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

vertical tangents occur when $\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$

$\blacksquare$