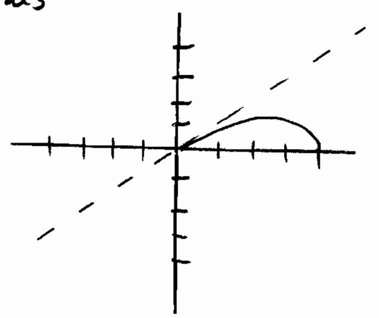
Problem 6

rose w/ 23 petals

$r = 4\cos 23\theta$



$\theta = 0 \qquad r = 4$

$\theta = \frac{\pi}{46} \qquad r = 0$

Area inside one petal

$$= 2\left[\int_0^{\frac{\pi}{46}} \tfrac{1}{2}(4\cos 23\theta)^2\, d\theta\right]$$

$$= \int_0^{\frac{\pi}{46}} 16\cos^2 23\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{46}} 16 \cdot \frac{1-\cos 46\theta}{2}\, d\theta$$

$$= \int_0^{\frac{\pi}{46}} (8 - 8\cos 46\theta)\, d\theta$$

$$= \left[8\theta - 8\,\frac{\sin 46\theta}{46}\right]_0^{\frac{\pi}{46}}$$

$$= \left(\frac{8\pi}{46} - 0\right) - (0 - 0)$$

$$= \frac{4\pi}{23}$$

$\boxed{E}$

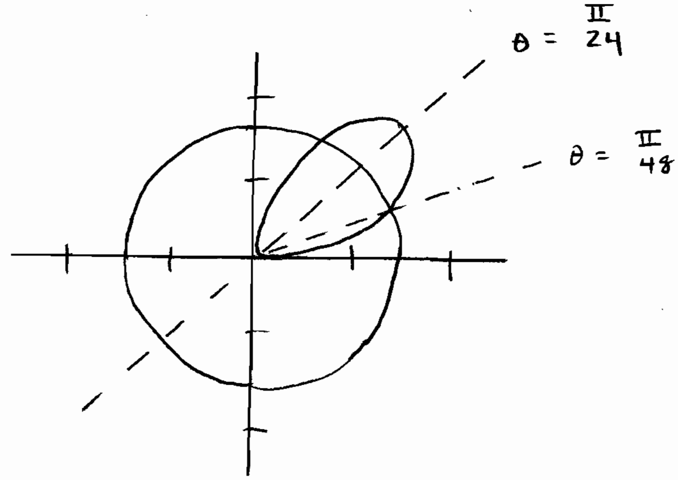
Problem 7

(a) $r = 2\sin 12\theta$

rose w/ 24 petals

$\theta = 0 \qquad r = 0$

$\theta = \frac{\pi}{24} \qquad r = 2$

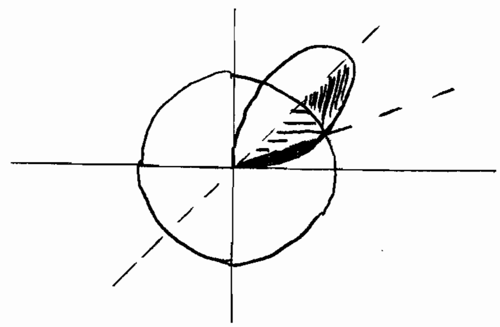


$2\sin 12\theta = \sqrt{2}$

$\sin 12\theta = \frac{\sqrt{2}}{2}$

$12\theta = \frac{\pi}{4}, \quad \frac{3\pi}{4}$

$\theta = \frac{\pi}{48}$



Area inside one petal and outside $r=\sqrt{2}$

$$= 2\left[\int_{\frac{\pi}{48}}^{\frac{\pi}{24}} \frac{1}{2}(2\sin 12\theta)^2\,d\theta - \int_{\frac{\pi}{48}}^{\frac{\pi}{24}} \frac{1}{2}(\sqrt{2})^2\,d\theta\right]$$

$$= \int_{\frac{\pi}{48}}^{\frac{\pi}{24}} (4\sin^2 12\theta - 2)\,d\theta$$

$$= \int_{\frac{\pi}{48}}^{\frac{\pi}{24}} \left(4\,\frac{1-\cos 24\theta}{2} - 2\right)d\theta$$

$$= \int_{\frac{\pi}{48}}^{\frac{\pi}{24}} (2 - 2\cos 24\theta - 2)\,d\theta$$

$$= \int_{\frac{\pi}{48}}^{\frac{\pi}{24}} \left(-2\cos 24\theta \right)\, d\theta$$

$$= \left[-2 \, \frac{\sin 24\theta}{24} \right]_{\frac{\pi}{48}}^{\frac{\pi}{24}}$$

$$= -\frac{\sin \pi}{12} - \left(-\frac{\sin \frac{\pi}{2}}{12} \right)$$

$$= \frac{1}{12}$$

Desired Area $= 24 \cdot \frac{1}{12} = 2$

b)

common interior of one petal and circle $= 2 \left[\overbrace{\int_0^{\frac{\pi}{48}} \frac{1}{2} (2\sin 12\theta)^2 \, d\theta} + \overbrace{\int_{\frac{\pi}{48}}^{\frac{\pi}{24}} \frac{1}{2} (\sqrt{2})^2 \, d\theta} \right]$

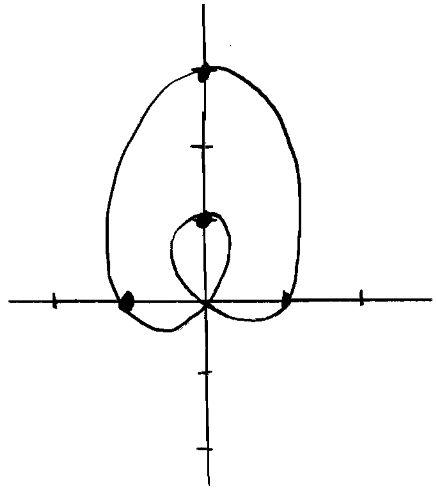
$= (**)$

Desired Area $= 24 \, (**)$

Problem 8

(a)

θ	r
0	1
$\frac{\pi}{2}$	3
π	1
$\frac{3\pi}{2}$	-1



$$1 + 2\sin\theta = 0$$

$$\sin\theta = \frac{-1}{2}$$

$$\theta = \frac{7\pi}{6}, \quad \frac{11\pi}{6}$$

Area of inner loop $= 2\left[\int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \frac{1}{2}(1 + 2\sin\theta)^2 \, d\theta\right]$

$$= \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} (1 + 4\sin\theta + 4\sin^2\theta) \, d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \left(1 + 4\sin\theta + 4\left(\frac{1 - \cos 2\theta}{2}\right)\right) d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} (1 + 4\sin\theta + 2 - 2\cos 2\theta) \, d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \left(3 + 4\sin\theta - 2\cos 2\theta\right) d\theta$$

$$= \left[3\theta - 4\cos\theta - 2\,\frac{\sin 2\theta}{2} \right]_{\frac{7\pi}{6}}^{\frac{3\pi}{2}}$$

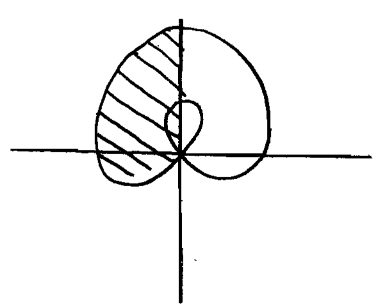
$$= \left(\frac{9\pi}{2} - 0 - 0\right) - \left(\frac{7\pi}{2} - 4\left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \pi - 2\sqrt{3} - \frac{\sqrt{3}}{2}$$

$$= \pi - \frac{3\sqrt{3}}{2}$$

(b)

Area between inner and outer loops $= 2\left[\int_{\frac{\pi}{2}}^{\frac{7\pi}{6}} \frac{1}{2}(1 + 2\sin\theta)^2 \, d\theta \right] - \underbrace{\left(\pi - \frac{3\sqrt{3}}{2}\right)}_{\text{area of inner loop (part a)}}$



Problem 9

(a)

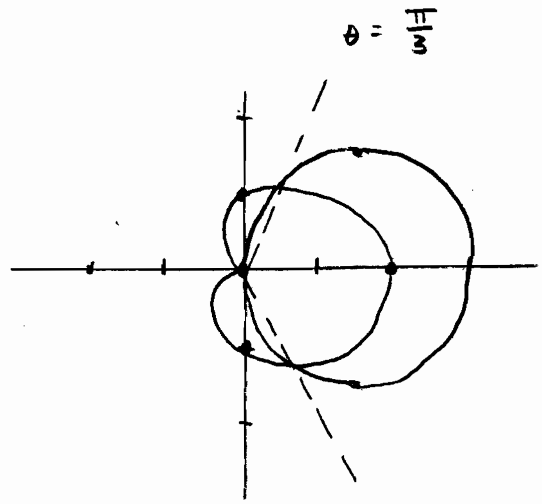
$r = 1 + \cos\theta$

θ	r
0	2
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	1

$r = 3\cos\theta$

$\theta = 0 \quad r = 3$

$\theta = \frac{\pi}{2} \quad r = 0$

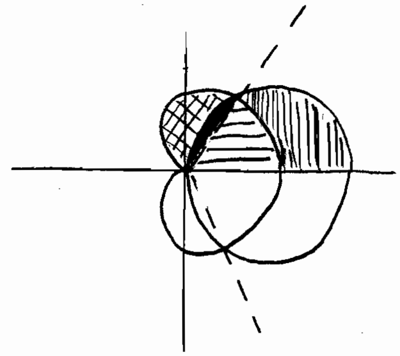


$3\cos\theta = 1 + \cos\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$



Desired Area =

$$2\left[\underbrace{\int_0^{\frac{\pi}{3}} \frac{1}{2}(3\cos\theta)^2 d\theta} - \underbrace{\int_0^{\frac{\pi}{3}} \frac{1}{2}(1+\cos\theta)^2\, d\theta}\right]$$

$$= \int_0^{\frac{\pi}{3}} \left(9\cos^2\theta - (1 + 2\cos\theta + \cos^2\theta)\right)\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \left(8\cos^2\theta - 2\cos\theta - 1\right)\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \left(8\left(\frac{1+\cos 2\theta}{2}\right) - 2\cos\theta - 1\right)\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \left(4 + 4\cos 2\theta - 2\cos\theta - 1 \right) d\theta$$

$$= \int_0^{\frac{\pi}{3}} \left(3 + 4\cos 2\theta - 2\cos\theta \right) d\theta$$

$$= \left[3\theta + 4\,\frac{\sin 2\theta}{2} - 2\sin\theta \right]_0^{\frac{\pi}{3}}$$

$$= \left(\pi + 2\cdot\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2} \right) - (0 + 0 - 0)$$

$$= \pi$$

(b)

Desired Area

$$= 2\left[\underbrace{\int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta}_{} - \underbrace{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(3\cos\theta)^2\,d\theta}_{} \right]$$

$+$

(c) Common Area $= 2\left[\underbrace{\int_0^{\frac{\pi}{3}} \frac{1}{2}(1+\cos\theta)^2\,d\theta}_{} + \underbrace{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(3\cos\theta)^2\,d\theta}_{} \right]$

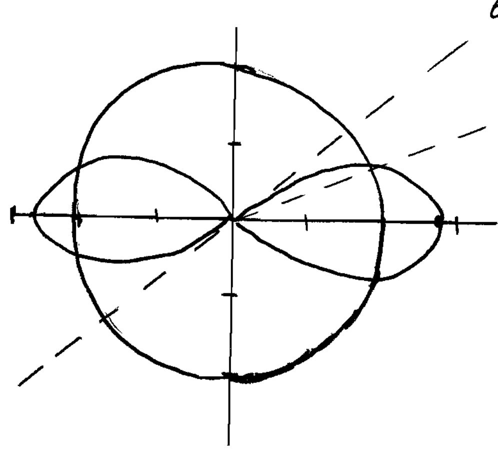
Problem 10

(a)

$r^2 = 8\cos 2\theta$ lemniscate

$\theta = 0$ $r = \sqrt{8}$

$\theta = \frac{\pi}{4}$ $r = 0$



$8\cos 2\theta = (2)^2$ ← note square

$\cos 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{3}$

$\theta = \frac{\pi}{6}$

note no square

Area inside lemniscate outside circle $= 4\left[\int_0^{\frac{\pi}{6}} \frac{1}{2}(8\cos 2\theta)\,d\theta - \int_0^{\frac{\pi}{6}} \frac{1}{2}(2)^2\,d\theta\right]$

$$= \int_0^{\frac{\pi}{6}} \left(2(8\cos 2\theta) - 2\cdot 4\right) d\theta$$

$$= \int_0^{\frac{\pi}{6}} (16\cos 2\theta - 8)\, d\theta$$

$$= \left[16\,\frac{\sin 2\theta}{2} - 8\theta\right]_0^{\frac{\pi}{6}}$$

$$= \left(8\cdot\frac{\sqrt{3}}{2} - 8\,\frac{\pi}{6}\right) - (0-0) = 4\sqrt{3} - \frac{4\pi}{3}.$$

(b)

$$\text{Common Area} = 4\left[\int_0^{\frac{\pi}{6}} \frac{1}{2}(2)^2\,d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2}(8\cos 2\theta)\,d\theta\right]$$