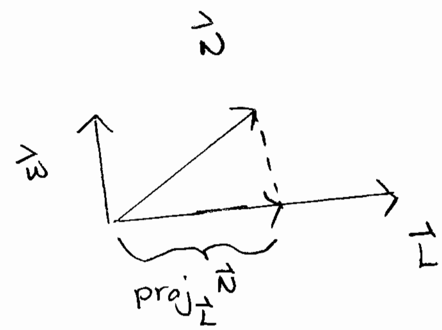
Problem 11

(a)



$$\text{proj}_{\vec{L}} \vec{N} = \left(\frac{\vec{N} \cdot \vec{L}}{\|\vec{L}\|^2} \right) \vec{L} = \frac{\langle 4, -4, 0 \rangle \cdot \langle 6, 3, 2 \rangle}{\| \langle 6, 3, 2 \rangle \|^2} \langle 6, 3, 2 \rangle$$

$$= \frac{24 - 12 + 0}{\left(\sqrt{36 + 9 + 4}\right)^2} \langle 6, 3, 2 \rangle$$

$$= \frac{12}{49} \langle 6, 3, 2 \rangle$$

vector component of $\vec{N}$ orthogonal to $\vec{L}$ $= \vec{w} = \vec{N} - \text{proj}_{\vec{L}} \vec{N}$

$$= \langle 4, -4, 0 \rangle - \frac{12}{49} \langle 6, 3, 2 \rangle$$

$$= \left\langle \frac{124}{49}, \frac{-232}{49}, \frac{-24}{49} \right\rangle$$

Two vectors are orthogonal if and only if their dot product is 0.

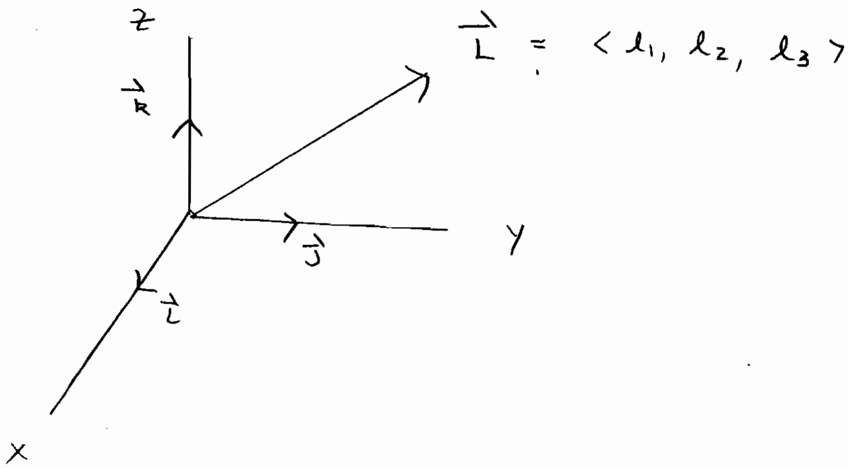
$$\vec{w} \cdot \vec{L} = \left\langle \frac{124}{49}, -\frac{232}{49}, -\frac{24}{49} \right\rangle \cdot \langle 6, 3, 2 \rangle = 0$$

Thus, $\vec{w}$ and $\vec{L}$ are indeed orthogonal.

(b)

α = angle between $\vec{L}$ and $\vec{i}$ (positive x-axis)
β = angle between $\vec{L}$ and $\vec{j}$ (positive y-axis)
γ = angle between $\vec{L}$ and $\vec{k}$ (positive z-axis)

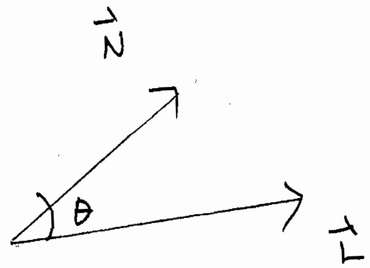
} direction angles



$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines.

$$\left.\begin{aligned}
\cos\alpha &= \frac{\vec{L}\cdot\vec{i}}{\|\vec{L}\|\,\|\vec{i}\|} = \frac{l_1}{\|\vec{L}\|} = \frac{6}{7} \\
\cos\beta &= \frac{l_2}{\|\vec{L}\|} = \frac{3}{7} \\
\cos\gamma &= \frac{l_3}{\|\vec{L}\|} = \frac{2}{7}
\end{aligned}\right\} \text{direction cosines}$$

(c)



Recall $\quad \vec{N} \cdot \vec{L} = \|\vec{N}\| \, \|\vec{L}\| \cos\theta$

So,

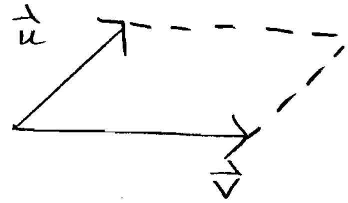
$$\cos\theta = \frac{\vec{N} \cdot \vec{L}}{\|\vec{N}\| \, \|\vec{L}\|} = \frac{\langle 4, -4, 0 \rangle \cdot \langle 6, 3, 2 \rangle}{\| \langle 4, -4, 0 \rangle \| \, \| \langle 6, 3, 2 \rangle \|}$$

$$= \frac{24 - 12}{\sqrt{32} \, \sqrt{49}}$$

$$= \frac{12}{7\sqrt{32}}$$

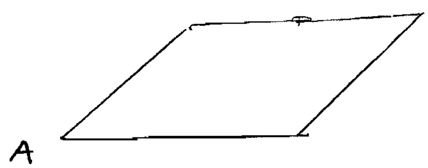
$\boxed{E}$

Problem 12



Area of parallelogram with adjacent sides $\vec{u}$ & $\vec{v}$ $= \| \vec{u} \times \vec{v} \|$

(a) $A(2,-1,1)$ $B(5,1,4)$ $C(0,1,1)$ $D(3,3,4)$



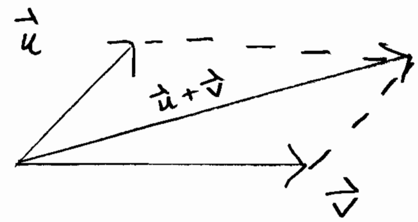
$\vec{AB} = \langle 3, 2, 3 \rangle$

$\vec{AC} = \langle -2, 2, 0 \rangle$

$\vec{AD} = \langle 1, 4, 3 \rangle$

We need the cross product of two vectors. The question is which two? Two of them are going to be the adjacent sides of the parallelogram. These we want. The other is going to be the diagonal. We don't want this.

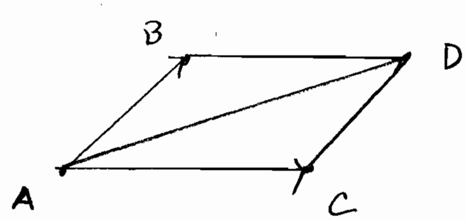
Remember the geometric interpretation of the sum of two vectors



$\vec{u}+\vec{v}$ is the diagonal of the parallelogram having $\vec{u}$ and $\vec{v}$ as adjacent sides.

Back to the problem at hand.
The diagonal is the vector that is the sum of the other two

Observe that $\vec{AB} + \vec{AC} = \vec{AD}$. So, $\vec{AD}$ is the diagonal. The correct picture is



verify that we have a parallelogram:

$\vec{AB} = \langle 3, 2, 3 \rangle$ $\qquad$ $\vec{CD} = \langle 3, 2, 3 \rangle$

$\vec{AC} = \langle -2, 2, 0 \rangle$ $\qquad$ $\vec{BD} = \langle -2, 2, 0 \rangle$

$\vec{AB} = \vec{CD}$ & $\vec{AC} = \vec{BD}$

we indeed have a parallelogram.

Area of parallelogram

$$\|\vec{AB} \times \vec{AC}\| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 3 \\ -2 & 2 & 0 \end{vmatrix} \right\|$$

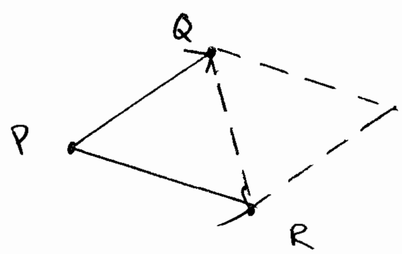
$$= \|(-6)\vec{i} - (6)\vec{j} + (10)\vec{k}\|$$

$$= \|\langle -6, -6, 10 \rangle\|$$

$$= \sqrt{36 + 36 + 100}$$

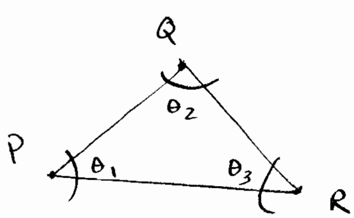
$$= \sqrt{172}$$

(b)



Area $\Delta PQR = \frac{1}{2}$ (area of parallelogram)

$$= \frac{1}{2} \|\vec{PQ} \times \vec{PR}\|$$



$$\cos\theta_1 = \frac{\vec{PQ} \cdot \vec{PR}}{\|\vec{PQ}\|\,\|\vec{PR}\|}$$

$$\cos\theta_2 = \frac{\vec{QP} \cdot \vec{QR}}{\|\vec{QP}\|\,\|\vec{QR}\|}$$

$$\cos\theta_3 = \frac{\vec{RQ} \cdot \vec{RP}}{\|\vec{RQ}\|\,\|\vec{RP}\|}$$

length of sides are $\|\vec{PQ}\|$, $\|\vec{PR}\|$, $\|\vec{QR}\|$.

Problem 13

Recall that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

Let $\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 6 & 0 \\ -2 & 1 & 1 \end{vmatrix} = 6\vec{i} - \vec{j} + 13\vec{k}$

$\vec{w}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

$$\frac{\vec{w}}{\|\vec{w}\|} = \frac{\langle 6, -1, 13 \rangle}{\sqrt{206}} = \left\langle \frac{6}{\sqrt{206}}, \frac{-1}{\sqrt{206}}, \frac{13}{\sqrt{206}} \right\rangle$$

is a unit vector orthogonal to both $\vec{u}$ & $\vec{v}$.

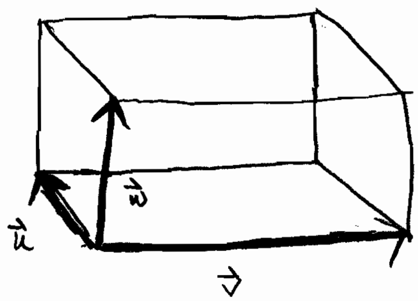
$\vec{v} \times \vec{u}$ is also orthogonal to both $\vec{u}$ and $\vec{v}$. You can repeat the calculation or recall that $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$. So, $-\frac{\vec{w}}{\|\vec{w}\|}$ is the other unit vector orthogonal to $\vec{u}$ and $\vec{v}$.

$\boxed{\equiv}$

Problem 14

The volume of a parallelepiped with vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ as adjacent edges is

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| \qquad \text{Triple scalar product}$$



$\vec{u} = \vec{PQ} = \langle 1, -1, 2 \rangle$

$\vec{v} = \vec{PR} = \langle 3, 0, 6 \rangle$

$\vec{w} = \vec{PS} = \langle 2, -2, -3 \rangle$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{vmatrix} = 12\vec{i} - (-21)\vec{j} + -6\vec{k}$$

$$= \langle 12, 21, -6 \rangle$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \langle 1, -1, 2 \rangle \cdot \langle 12, 21, -6 \rangle$$

$$= 12 - 21 - 12$$

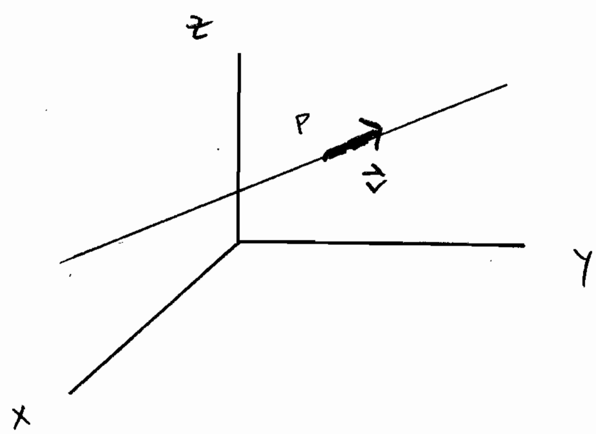
$$= -21$$

$$\text{Volume} = |\vec{u} \cdot (\vec{v} \times \vec{w})| = 21$$

E

Problem 15

A line in space is completely determined if you are given a point $P(x_0, y_0, z_0)$ on the line and a direction vector $\vec{v} = \langle a, b, c \rangle$ parallel to the line.



The line is given by the parametric equations

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

Symmetric Equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

(a)

$$x = 2 + 5t$$
$$y = 4 - t$$
$$z = -3$$

$$\frac{x-2}{5} = \frac{y-4}{-1}, \quad z = -3$$

(b) . The direction vector is $\vec{v} = \langle 1, -5, 4 \rangle$

$x = 2 + t$

$y = 4 - 5t$

$z = -3 + 4t$

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}$$

(c) The direction vector is $\vec{v} = \langle 3, -2, 1 \rangle$

$x = -3 + 3t$

$y = 5 - 2t$

$z = 4 + t$

$$\frac{x+3}{3} = \frac{y-5}{-2} = \frac{z-4}{1}$$

(d) The direction vector is $\vec{v} = \langle -2, 2, 0 \rangle$

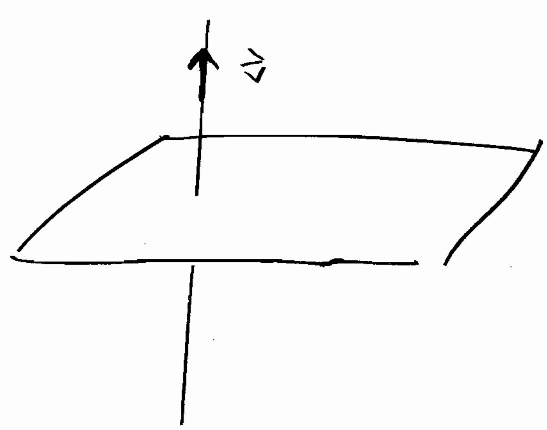
$x = -3 - 2t$

$y = 5 + 2t$

$z = 4$

$$\frac{x+3}{-2} = \frac{y-5}{2}, \quad z = 4$$

(e) we need a direction vector, our line is perpendicular to the plane.



So, we should choose a $\vec{v}$ that is perpendicular to the plane.

Note that the plane has normal vector

$\vec{n} = \langle 1, 3, 1 \rangle$

$\vec{n}$ is perpendicular to the plane which is precisely what we need.

So, let $\vec{v} = \langle 1, 3, 1 \rangle$ be our direction vector.

$x = 1 + t$

$y = 3t$

$z = 6 + t$

$$\frac{x-1}{1} = \frac{y}{3} = \frac{z-6}{1}$$

$\boxed{}$

Problems 16 & 17 are done in my "Lines & Planes" notes.