spring 2008

Exam 2 Review Solutions

Problem 1

(a) $\vec{r}\,''(t) = \langle 25e^{-5t},\ 12t,\ -16\cos 2t \rangle$

$\vec{r}\,'(0) = \langle 0, 3, -4 \rangle \quad (*)$

$\vec{r}(0) = \langle 3, 2, -5 \rangle \quad (\#)$

$$\vec{r}\,'(t) = \int \vec{r}\,''(t)\,dt$$

$$= \left\langle \int 25e^{-5t}\,dt,\ \int 12t\,dt,\ \int -16\cos 2t\,dt \right\rangle$$

$$= \langle -5e^{-5t} + c_1,\ 6t^2 + c_2,\ -8\sin 2t + c_3 \rangle$$

we use the initial conditions to recover the constants.

$$\vec{r}\,'(0) = \langle -5 + c_1,\ c_2,\ c_3 \rangle$$

$$= \langle 0, 3, -4 \rangle$$

$-5 + c_1 = 0 \Rightarrow c_1 = 5$

$c_2 = 3$

$c_3 = -4$

So, $\vec{r}\,'(t) = \langle -5e^{-5t} + 5,\ 6t^2 + 3,\ -8\sin 2t - 4 \rangle$

$$\vec{r}(t) = \int \vec{r}\,'(t)\, dt$$

$$= \left\langle \int (-5e^{-5t} + 5)\, dt, \quad \int (6t^2 + 3)\, dt, \quad \int (-8\sin 2t - 4)\, dt \right\rangle$$

$$= \left\langle e^{-5t} + 5t + D_1, \quad 2t^3 + 3t + D_2, \quad 4\cos 2t - 4t + D_3 \right\rangle$$

$$\vec{r}(0) = \langle 1 + D_1, \quad D_2, \quad 4 + D_3 \rangle$$

$$= \langle 3, 2, -5 \rangle$$

$$1 + D_1 = 3 \implies D_1 = 2$$

$$D_2 = 2$$

$$4 + D_3 = -5 \implies D_3 = -9$$

Hence,

$$\vec{r}(t) = \langle e^{-5t} + 5t + 2, \quad 2t^3 + 3t + 2, \quad 4\cos 2t - 4t - 9 \rangle$$

$\square$

(b) $\vec{r}'(t) = \left\langle \frac{t}{t^2-3},\ 0,\ -e^{-3t} \right\rangle$

$\vec{r}(2) = \langle -1, 1, 0 \rangle$

$$\vec{r}(t) = \int \vec{r}'(t)\,dt$$

$$= \left\langle \int \frac{t}{t^2-3}\,dt,\ \int 0\,dt,\ \int -e^{-3t}\,dt \right\rangle$$

$$\left(\begin{aligned} &u = t^2-3 \\ &du = 2t\,dt \\ &\tfrac{1}{2}du = t\,dt \\ &\int \tfrac{1}{2}\,\tfrac{1}{u}\,du \\ &= \tfrac{1}{2}\ln|u| + c_1 \\ &= \tfrac{1}{2}\ln|t^2-3| + c_1 \end{aligned}\right)$$

$$= \left\langle \tfrac{1}{2}\ln|t^2-3| + c_1,\ c_2,\ \tfrac{1}{3}e^{-3t} + c_3 \right\rangle$$

$$\vec{r}(2) = \left\langle c_1,\ c_2,\ \tfrac{1}{3}e^{-6} + c_3 \right\rangle$$

$$= \langle -1, 1, 0 \rangle$$

$$c_1 = -1$$

$$c_2 = 1$$

$$\frac{1}{3}e^{-6} + c_3 = 0 \quad \Rightarrow \quad c_3 = -\frac{1}{3}e^{-6}$$

Thus,

$$\vec{F}(t) = \left\langle \frac{1}{2}\ln|t^2 - 3| - 1,\ 1,\ \frac{1}{3}e^{-3t} - \frac{1}{3}e^{-6} \right\rangle$$

$\boxed{\equiv}$

Problem 2

$$\vec{r}(t) = \langle 4t , \cos 5t , \sin 5t \rangle$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$\vec{r}'(t) = \langle 4 , -5 \sin 5t , 5 \cos 5t \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{16 + \underbrace{25 \sin^2 5t + 25 \cos^2 5t}} = \sqrt{41}$$

$$= 25 (\sin^2 5t + \cos^2 5t)$$

$$= 25$$

$$\vec{T}(t) = \langle \frac{4}{\sqrt{41}} , \frac{-5}{\sqrt{41}} \sin 5t , \frac{5}{\sqrt{41}} \cos 5t \rangle$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

$$\vec{T}'(t) = \langle 0 , \frac{-25}{\sqrt{41}} \cos 5t , \frac{-25}{\sqrt{41}} \sin 5t \rangle$$

$$\|\vec{T}'(t)\| = \sqrt{0 + \frac{25^2}{41} \cos^2 5t + \frac{25^2}{41} \sin^2 5t}$$

$$= \sqrt{\frac{25^2}{41}}$$

$$= \frac{25}{\sqrt{41}}$$

$$\vec{N}(t) = \langle 0, -\cos 5t, -\sin 5t \rangle$$

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\frac{25}{\sqrt{41}}}{\sqrt{41}} = \frac{25}{41}$$

$\boxed{E}$

Problem 3

$$\vec{r}(t) = \langle t, \tfrac{\sqrt{3}}{2}t^2, t^3 \rangle$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$\vec{r}\,'(t) = \langle 1, \sqrt{3}\,t, 3t^2 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + 3t^2 + 9t^4}$$

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{1+3t^2+9t^4}}, \frac{\sqrt{3}\,t}{\sqrt{1+3t^2+9t^4}}, \frac{3t^2}{\sqrt{1+3t^2+9t^4}} \right\rangle$$

$$a_T(t) = \frac{\vec{v}(t)\cdot\vec{a}(t)}{\|\vec{v}(t)\|}$$

$$\vec{v}(t) = \langle 1, \sqrt{3}\,t, 3t^2 \rangle$$

$$\vec{a}(t) = \langle 0, \sqrt{3}, 6t \rangle$$

$$a_T(t) = \frac{3t + 18t^3}{\sqrt{1+3t^2+9t^4}}$$

$$a_N(t) = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|}$$

$$\vec{v}(t) \times \vec{a}(t) = (6\sqrt{3}t^2 - 3\sqrt{3}t^2)\vec{i} - (6t)\vec{j} + (\sqrt{3})\vec{k}$$

$$= \langle 3\sqrt{3}\,t^2, -6t, \sqrt{3} \rangle$$

$$\|\vec{v}(t) \times \vec{a}(t)\| = \sqrt{27t^4 + 36t^2 + 3}$$

$$a_N(t) = \frac{\sqrt{27t^4 + 36t^2 + 3}}{\sqrt{1 + 3t^2 + 9t^4}}$$

$$\kappa(t) = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3}$$

$$\kappa(t) = \frac{\sqrt{27t^4 + 36t^2 + 3}}{\left(\sqrt{1 + 3t^2 + 9t^4}\right)^3}$$

Problem 4

$$\vec{r}(t) = \langle t - \cos t, \ \ln(t+1), \ \frac{e^{-5t}}{5} \rangle$$

$$\vec{T}(0) = \frac{\vec{r}\,'(0)}{\|\vec{r}\,'(0)\|} \qquad K(0) = \frac{\|\vec{v}(0) \times \vec{a}(0)\|}{\|\vec{v}(0)\|^3}$$

$$a_T(0) = \frac{\vec{v}(0) \cdot \vec{a}(0)}{\|\vec{v}(0)\|}$$

$$a_N(0) = \frac{\|\vec{v}(0) \times \vec{a}(0)\|}{\|\vec{v}(0)\|}$$

$$\vec{v}(t) = \langle 1 + \sin t, \ \frac{1}{t+1}, \ -e^{-5t} \rangle$$

$$\vec{a}(t) = \langle \cos t, \ \frac{-1}{(t+1)^2}, \ 5e^{-5t} \rangle$$

1°

$$\vec{r}\,'(0) = \langle 1, 1, -1 \rangle$$

$$\|\vec{r}\,'(0)\| = \sqrt{1+1+1} = \sqrt{3}$$

$$\vec{T}(0) = \langle \frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ \frac{-1}{\sqrt{3}} \rangle$$

$\underline{\underline{2^\circ}}$

$$\vec{v}(0) = \langle 1, 1, -1 \rangle$$

$$\vec{a}(0) = \langle 1, -1, 5 \rangle$$

$$a_T(0) = \frac{1 - 1 - 5}{\sqrt{1+1+1}} = \frac{-5}{\sqrt{3}}$$

$\underline{\underline{3^\circ}}$

$$\vec{v}(0) \times \vec{a}(0) = (5-1)\vec{i} - (5+1)\vec{j} + (-1-1)\vec{k}$$

$$= \langle 4, -6, -2 \rangle$$

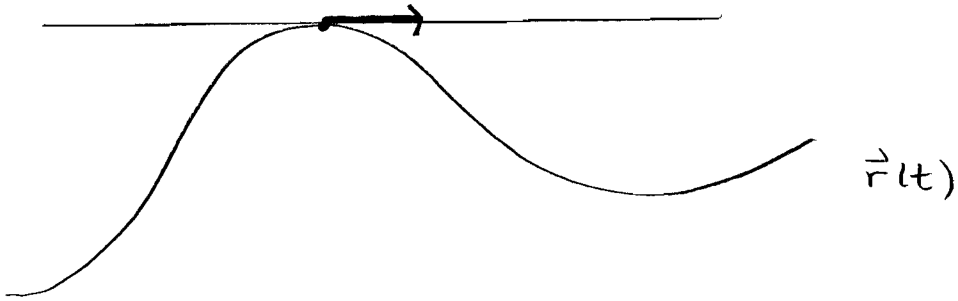
$$\|\vec{v}(0) \times \vec{a}(0)\| = \sqrt{16 + 36 + 4} = \sqrt{56}$$

$$a_N(0) = \frac{\sqrt{56}}{\sqrt{3}}$$

$\underline{\underline{4^\circ}}$

$$\kappa(0) = \frac{\sqrt{56}}{(\sqrt{3})^3} = \frac{\sqrt{56}}{3\sqrt{3}}$$

$\underline{\underline{5^o}}$



we need a point and a direction vector.

when $t = 0$ we are at the point $P(-1, 0, \frac{1}{5})$.

we know $\vec{r}\,'(t)$ is tangent to the curve.

So $\vec{r}\,'(0) = \langle 1, 1, -1 \rangle$ is our desired direction vector.

The equation of the required line is

$$x = -1 + t$$

$$y = t$$

$$z = \frac{1}{5} - t$$

Remark

In $\underline{\underline{3^\circ}}$ we could have also used

$$a_N(t) = \sqrt{\|\vec{a}(t)\|^2 - (a_T(t))^2}$$

$$a_N(0) = \sqrt{\|\vec{a}(0)\|^2 - (a_T(0))^2}$$

$$= \sqrt{\|\langle 1, -1, 5 \rangle\|^2 - \left(\frac{-5}{\sqrt{3}}\right)^2}$$

$$= \sqrt{\left(\sqrt{1+1+25}\right)^2 - \frac{25}{3}}$$

$$= \sqrt{27 - \frac{25}{3}}$$

$$= \sqrt{\frac{56}{3}}$$

The cross product formula is probably better since we'll need to compute the cross product to find curvature anyway.

But, this might be a good way to check your work if you have time.

$\boxed{\equiv}$

$$s = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

Problem 5

(a) $\vec{r}(t) = \langle \frac{2}{3} t^{3/2}, \sqrt{2}\, t, 2t^{1/2} \rangle$

$\vec{r}\,'(t) = \langle t^{1/2}, \sqrt{2}, t^{-1/2} \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{t + 2 + \frac{1}{t}}$

$$s = \int_4^9 \sqrt{t + 2 + \frac{1}{t}}\, dt$$

$$= \int_4^9 \sqrt{\frac{t^2 + 2t + 1}{t}}\, dt$$

$$= \int_4^9 \sqrt{\frac{(t+1)^2}{t}}\, dt \qquad \left(\sqrt{\frac{(t+1)^2}{t}} = \frac{|t+1|}{\sqrt{t}} \right.$$

on $[4,9]$, $t+1 \geq 0$

So, $\left. \frac{|t+1|}{\sqrt{t}} = \frac{t+1}{\sqrt{t}} \right)$

$$= \int_4^9 \frac{t+1}{t^{1/2}}\, dt$$

$$= \int_4^9 (t^{1/2} + t^{-1/2})\, dt$$

$$= \left[\frac{t^{3/2}}{3/2} + \frac{t^{1/2}}{1/2} \right]_4^9$$

$$= \left[\frac{2}{3} t^{3/2} + 2t^{1/2} \right]_4^9 = \left(\frac{2}{3}(27) + 2(3) \right) - \left(\frac{2}{3}(8) + 2(2) \right)$$

$$= \frac{44}{3}$$

(b) $\vec{r}(t) = \langle 6 , 3\sqrt{2}\, t^2 , 2t^3 \rangle$

$\vec{r}\,'(t) = \langle 0 , 6\sqrt{2}\, t , 6t^2 \rangle$

$\| \vec{r}\,'(t) \| = \sqrt{72t^2 + 36t^4}$

$$s = \int_1^{\sqrt{2}} \sqrt{72t^2 + 36t^4}\, dt$$

$$= \int_1^{\sqrt{2}} \sqrt{36t^2 (2 + t^2)}\, dt$$

$$= \int_1^{\sqrt{2}} |6t| \sqrt{2+t^2}\, dt$$

$\left(\text{on } [1, \sqrt{2}] , \ 6t \geq 0 \quad \text{so, } |6t| = 6t . \right)$

$$= \int_1^{\sqrt{2}} 6t \sqrt{2+t^2}\, dt$$

$u = 2 + t^2$

$du = 2t\, dt$

$3\, du = 6t\, dt$

$$= \int_3^4 3u^{1/2}\, du$$

$$= \left[3 \frac{u^{3/2}}{3/2} \right]_3^4$$

$$= \left[2u^{3/2} \right]_3^4 = 2(8) - 2 \cdot 3^{3/2}$$

$$= 16 - 6\sqrt{3}$$

(c) $\vec{r}(t) = \langle \sqrt{2}\,t \,,\; e^{t} \,,\; e^{-t} \rangle$

$\vec{r}\,'(t) = \langle \sqrt{2} \,,\; e^{t} \,,\; -e^{-t} \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{2 + e^{2t} + e^{-2t}}$

$$S = \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}\, dt$$

Either you see right away

or

some more algebra

$$= \int_0^1 \sqrt{(e^{t} + e^{-t})^2}\, dt$$

$$= \int_0^1 |e^{t} + e^{-t}|\, dt$$

$$= \int_0^1 (e^{t} + e^{-t})\, dt$$

$$= \left[e^{t} - e^{-t} \right]_0^1$$

$$= (e - e^{-1}) - (1 - 1)$$

$$= e - \frac{1}{e}$$

$$\int_0^1 \sqrt{2 + e^{2t} + \frac{1}{e^{2t}}}\, dt$$

$$= \int_0^1 \sqrt{\frac{2e^{2t} + e^{4t} + 1}{e^{2t}}}\, dt$$

$$= \int_0^1 \sqrt{\frac{e^{4t} + 2e^{2t} + 1}{e^{2t}}}\, dt$$

$$= \int_0^1 \sqrt{\frac{(e^{2t} + 1)^2}{e^{2t}}}\, dt$$

$$= \int_0^1 \frac{|e^{2t} + 1|}{|e^{t}|}\, dt$$

$$= \int_0^1 \frac{e^{2t} + 1}{e^{t}}\, dt$$

$$= \int_0^1 (e^{t} + e^{-t})\, dt$$

⋮