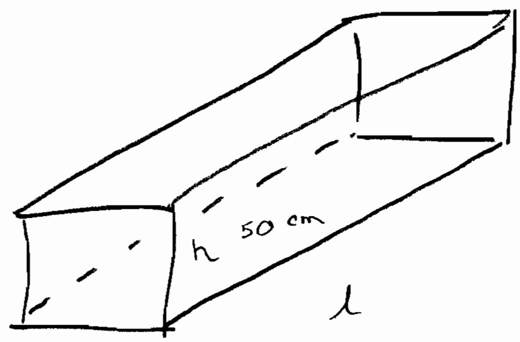
Problem 6

(a)



The surface area is given by

$$S = 2lw + 2lh + 2wh$$

We are given that

$$\Delta l = \pm(0.01)(80) = \pm 0.8$$

$$\Delta w = \pm(0.01)(60) = \pm 0.6$$

$$\Delta h = \pm(0.01)(50) = \pm 0.5$$

We want to estimate

$$\Delta S \approx dS = \frac{\partial S}{\partial l}\,dl + \frac{\partial S}{\partial w}\,dw + \frac{\partial S}{\partial h}\,dh$$

$$= (2w + 2h)\,dl + (2l + 2h)\,dw + (2l + 2w)\,dh$$

$$= (2(60) + 2(50))(\pm 0.8) + (2(80) + 2(50))(\pm 0.6)$$

$$+ (2(80) + 2(60))(\pm 0.5)$$

The maximum that this could be is $\pm\ 472\ cm^2$.

Maximum error is $\pm\ 472\ cm^2$

Maximum percent error

$$\frac{\Delta S}{S} = \frac{\pm\ 472\ cm^2}{23600\ cm^2} = \pm\ 0.02 = \pm\ 2\%.$$

(b). The volume is given by

$$V = lwh$$

We are given that

$$\frac{dl}{dt} = 2\ cm/s$$

$$\frac{dw}{dt} = 2\ cm/s$$

$$\frac{dh}{dt} = -5\ cm/s$$

We want to calculate $\frac{dV}{dt}$.

We need to relate $\frac{dV}{dt}$ to $\frac{dl}{dt}$, $\frac{dw}{dt}$, $\frac{dh}{dt}$

By the Chain Rule we have

$$\begin{tikzcd}[column sep=small] & V \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ l \arrow[d, dash] & w \arrow[d, dash] & h \arrow[d, dash] \\ t & t & t \end{tikzcd}$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial l}\frac{dl}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= (wh)\frac{dl}{dt} + (lh)\frac{dw}{dt} + (lw)\frac{dh}{dt}$$

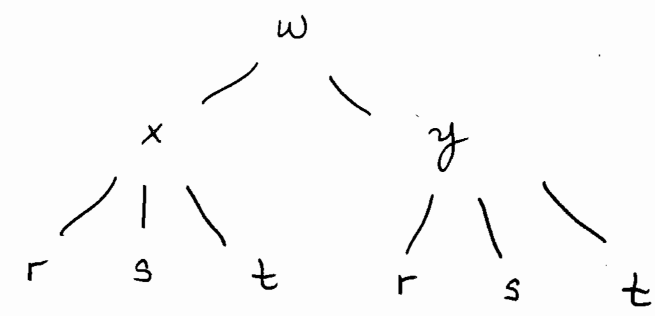
At the instant $l = 84$ $w = 64$ $h = 40$,

$$\frac{dV}{dt} = (64 \cdot 40)\,2 + (84 \cdot 40)\,2 + (84 \cdot 64)(-5)$$

$$= -15040 \ \mathrm{cm}^3/\mathrm{s}$$

Problem 7

(a)



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \qquad (*)$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \qquad (**)$$

$$\left.\begin{aligned}
\frac{\partial w}{\partial x} &= 5(x + x^2y^3 + y^6 + 1)^4 \, (1 + 2xy^3) \\
\frac{\partial w}{\partial y} &= 5(x + x^2y^3 + y^6 + 1)^4 \, (3x^2y^2 + 6y^5) \\
\frac{\partial x}{\partial s} &= (\sin r)(-\sin(\pi s - t))\,\pi \\
\frac{\partial y}{\partial s} &= \frac{-2t}{s^3} \\
\frac{\partial x}{\partial t} &= (\sin r)(-\sin(\pi s - t))(-1) \\
\frac{\partial y}{\partial t} &= \frac{1}{s^2}
\end{aligned}\right\} \qquad (***)$$

$$\frac{\partial w}{\partial s} = 5(x + x^2y^3 + y^6 + 1)^4 (1 + 2xy^3) \left(-\pi \sin r \sin(\pi s - t)\right)$$

$$+ 5(x + x^2y^3 + y^6 + 1)^4 (3x^2y^2 + 6y^5) \left(\frac{-2t}{s^3}\right)$$

$$\frac{\partial w}{\partial t} = 5(x + x^2y^3 + y^6 + 1)^4 (1 + 2xy^3) \left(\sin r \sin(\pi s - t)\right)$$

$$+ 5(x + x^2y^3 + y^6 + 1)^4 (3x^2y^2 + 6y^5) \left(\frac{1}{s^2}\right)$$

unless the directions tell you otherwise, after finding (***), you can just say something like " substitute everything back into (*) & (**) "

This last step is just rewriting everything.

(b) $\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r}$

$$= 5(x + x^2y^3 + y^6 + 1)^4 (1 + 2xy^3) \left(\cos(\pi s - t) \cos r\right)$$

$$+ 5(x + x^2y^3 + y^6 + 1)^4 (3x^2y^2 + 6y^5) (2r)$$

At $s = 1 \quad t = 0 \quad r = 0$

$x = 0$

$y = 0$

So,

$$\frac{\partial w}{\partial r} = 5(1)^4 (1)\ (-1)(1) + 0$$

$$= \underline{\underline{-5}}$$

(c) $z = \ln(x - yz) - y^3 z \cos z$

$$\Rightarrow\ 0 = \underbrace{\ln(x-yz) - y^3 z \cos z - z}_{F(x,y,z)}$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \qquad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

$$F_x(x,y,z) = \frac{1}{x - yz}$$

$$F_y(x,y,z) = \frac{1}{x-yz}(-z) - 3y^2 z \cos z$$

$$F_z(x,y,z) = \frac{1}{x-yz}(-y) - y^3\left[z(-\sin z) + \cos z\right] - 1$$

$$\frac{\partial z}{\partial x} = \frac{-\dfrac{1}{x-yz}}{\dfrac{-y}{x-yz} + y^3 z \sin z - y^3 \cos z - 1}$$

$$\frac{\partial z}{\partial y} = \frac{-\left(\dfrac{-z}{x-yz} - 3y^2 z \cos z\right)}{\dfrac{-y}{x-yz} + y^3 z \sin z - y^3 \cos z - 1}$$

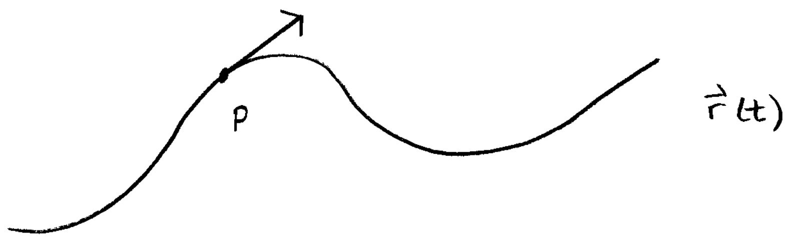
$\boxed{/\!/}$

Problems 8 & 9 are done in

" Notes on 13.6 & 13.7 "

Problem 10

(a)



Observe that the bee passes through the point $P(\frac{\pi}{2}, -2, 0)$ when $t = \frac{\pi}{2}$.

The direction we want is $\vec{r}\,'(\frac{\pi}{2})$

$$\vec{r}\,'(t) = \langle 1, \ 6\cos 3t, \ -6\sin 3t \rangle$$

$$\vec{r}\,'(\tfrac{\pi}{2}) = \langle 1, 0, 6 \rangle$$

(b). we are looking for the directional derivative of T at the point $P(\frac{\pi}{2}, -2, 0)$ in the direction of $\vec{v} = \langle 1, 0, 6 \rangle$

In symbols,

$$D_{\vec{u}} T(\tfrac{\pi}{2}, -2, 0) = \vec{\nabla} T(\tfrac{\pi}{2}, -2, 0) \cdot \vec{u}$$

To apply the formula, we need a unit vector in the direction of $\vec{v}$.

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, 0, 6 \rangle}{\sqrt{1+0+36}} = \left\langle \frac{1}{\sqrt{37}}, 0, \frac{6}{\sqrt{37}} \right\rangle$$

we need to compute $\vec{\nabla} T\left(\frac{\pi}{2}, -2, 0\right)$

This done in Problem 8

$$\vec{\nabla} T\left(\frac{\pi}{2}, -2, 0\right) = \left\langle 2, \frac{-\pi}{2}, 0 \right\rangle$$

So,

$$D_{\vec{u}} T\left(\frac{\pi}{2}, -2, 0\right) = \left\langle 2, \frac{-\pi}{2}, 0 \right\rangle \cdot \left\langle \frac{1}{\sqrt{37}}, 0, \frac{6}{\sqrt{37}} \right\rangle$$

$$= \frac{2}{\sqrt{37}} + 0 + 0$$

$$= \frac{2}{\sqrt{37}}$$

(c) Q: In what direction is $D_{\vec{u}} T\left(\frac{\pi}{2}, -2, 0\right)$ greatest (maximum)?

A: $\vec{\nabla} T\left(\frac{\pi}{2}, -2, 0\right) = \left\langle 2, -\frac{\pi}{2}, 0 \right\rangle$

(d) we want to estimate

$$\Delta T \approx dT = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz$$

$$= \left(e^{\sin(xy)}(\cos(xy))\ y\right)\ dx \ +$$

$$\left(e^{\sin(xy)}(\cos(xy))\ x\right)\ dy \ +$$

$$(2z)\ dz$$

$dx = -\frac{\pi}{4}$ $\qquad x = \frac{\pi}{2}$

$dy = -1$ $\qquad y = -2$

$dz = 1$ $\qquad z = 0$

Remember: we want to estimate the change using information at the initial / original point. That's why we plug in P, not Q.

$$\Delta T \approx \left(e^{\sin(-\pi)} \cos(-\pi)\ (-2)\right)\ \left(-\frac{\pi}{4}\right) +$$

$$\left(e^{\sin(-\pi)} \cos(-\pi)\ \frac{\pi}{2}\right)\ (-1) +$$

$$0$$

$$= (2)\left(-\frac{\pi}{4}\right) + \left(-\frac{\pi}{2}\right)(-1) + 0$$

$$= \underline{\underline{0}}$$

Note that $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$, $\frac{\partial T}{\partial z}$ at $P(\frac{\pi}{2}, -2, 0)$ should already have been computed in (b) & (c). So, using differentials is actually quick.