Problem 3

I do not know how to integrate $\cos(x^3)$.
So, maybe switching the order of integration would help.

1° Switch order of integration

right of $\qquad$ left of

$$\sqrt{y} \leq x \leq 2$$

above $\qquad$ below

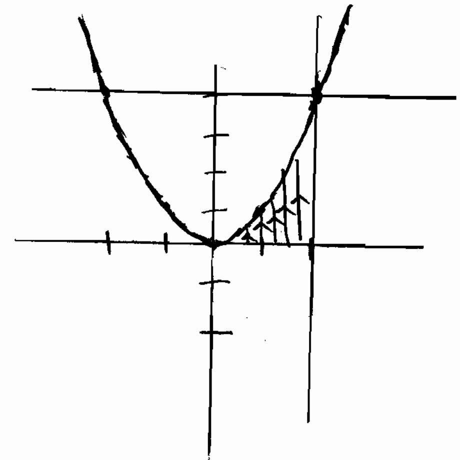
$$0 \leq y \leq 4$$

$x = \sqrt{y}$ $\qquad$ $y = x^2$

$x = 2$

$y = 0$

$y = 4$



original $\rightarrow\rightarrow\rightarrow$ $\qquad$ new $\uparrow\uparrow\uparrow$

$$\int_{x=0}^{x=2} \int_{y=0}^{y=x^2} \cos(x^3)\, dy\, dx$$

2° Evaluate new integral

$$\int_{x=0}^{x=2} \Big[\cos(x^3)\, y \Big]_{y=0}^{y=x^2} dx$$

$$= \int_{x=0}^{x=2} \cos(x^3)\, x^2\, dx$$

$$= \Big[\tfrac{1}{3} \sin(x^3) \Big]_{x=0}^{x=2}$$

$$= \tfrac{1}{3} \sin(8)$$

$$\left(\begin{aligned} &\int \cos(x^3)\, x^2\, dx \\ &u = x^3 \\ &du = 3x^2\, dx \\ &\tfrac{1}{3} du = x^2\, dx \\ &\int \tfrac{1}{3} \cos u\, du \\ &= \tfrac{1}{3} \sin u \\ &= \tfrac{1}{3} \sin(x^3) \end{aligned} \right)$$

Problem 4

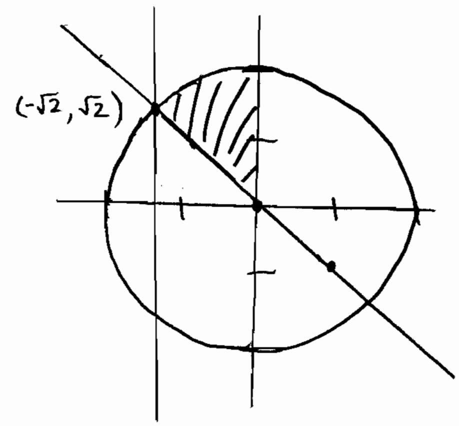
(a) $$\int_{-\sqrt{2}}^{0} \int_{-x}^{\sqrt{4-x^2}} 15xy^2 \, dy \, dx$$

above $\quad$ below

$-x \leq y \leq \sqrt{4-x^2}$

right of $\quad$ left of

$-\sqrt{2} \leq x \leq 0$



$y = -x$

$y = \sqrt{4-x^2}$

$x^2+y^2=4$

In polar coordinates the region is

$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$

$0 \leq r \leq 2$

Our integral becomes

$$\int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{4}} \int_{r=0}^{r=2} 15(r\cos\theta)(r\sin\theta)^2 \, r \, dr \, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \int_{0}^{2} 15 r^4 \cos\theta \sin^2\theta \, dr \, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \left[3r^5 \cos\theta \sin^2\theta \right]_{r=0}^{r=2} d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} 96 \cos\theta \sin^2\theta \, d\theta$$

$$= \left[96 \frac{\sin^3\theta}{3} \right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \qquad (u = \sin\theta)$$

$$= 32\left(\sin\frac{3\pi}{4}\right)^3 - 32\left(\sin\frac{\pi}{2}\right)^3$$

$$= 32\left(\frac{1}{\sqrt{2}}\right)^3 - 32$$

$\square$

(b) $$\int_0^{\sqrt{3}} \int_{\frac{1}{\sqrt{3}}y}^{\sqrt{4-y^2}} (x^2+y^2)^{2008} \, dx \, dy$$

right of $\qquad$ left of

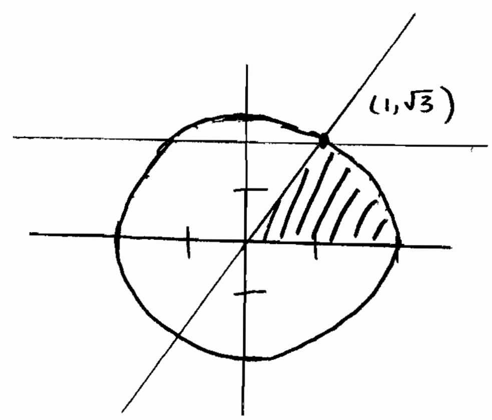
$$\frac{1}{\sqrt{3}}y \leq x \leq \sqrt{4-y^2}$$

$x = \frac{1}{\sqrt{3}}y \qquad y = \sqrt{3}x$

$x = \sqrt{4-y^2} \qquad x^2+y^2=4$

above $\qquad$ below

$$0 \leq y \leq \sqrt{3}$$



In polar coordinates the region is

$$0 \leq \theta \leq \frac{\pi}{3}$$

$$0 \leq r \leq 2$$

Our integral becomes

$$\int_{\theta=0}^{\theta=\frac{\pi}{3}} \int_{r=0}^{r=2} (r^2)^{2008} \, r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \int_0^{2} r^{4017} \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \left[\frac{r^{4018}}{4018} \right]_0^{2} d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{2^{4018}}{4018} \, d\theta$$

$$= \left[\frac{2^{4018}}{4018} \theta \right]_0^{\frac{\pi}{3}}$$

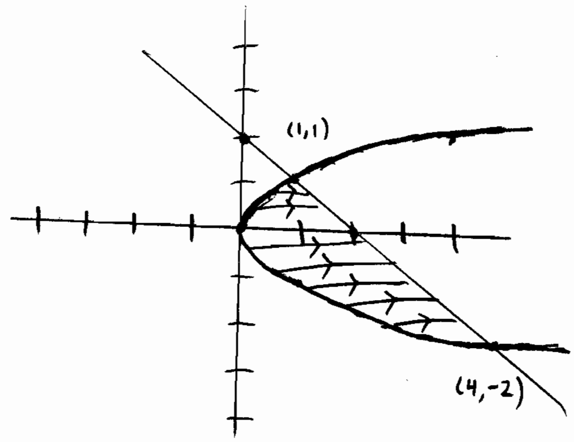
$$= \frac{2^{4018}}{4018} \frac{\pi}{3}$$

Problem 5

(a) we begin by sketching the region R

$$x - y^2 = 0 \Rightarrow x = y^2 \quad (*)$$

$$x + y = 2 \Rightarrow y = 2 - x \quad (**)$$



If the intersection points aren't clear from your drawing, you can find them algebraically by solving (*), (**) simultaneously.

$$y = 2 - y^2$$

$$y^2 + y - 2 = 0$$

$$(y+2)(y-1) = 0$$

$y = -2$, $x = 4$ $y = 1$, $x = 1$

Now, we'll set up the iterated integral

$$\iint_R 12xy \, dA = \int_{y=-2}^{y=1} \int_{x=y^2}^{x=2-y} 12xy \, dx \, dy$$

(we're using)

$$= \int_{y=-2}^{y=1} \left[6x^2 y \right]_{x=y^2}^{x=2-y} dy$$

$$= \int_{y=-2}^{y=1} \left(6(2-y)^2 y - 6(y^2)^2 y \right) dy$$

$$= \int_{y=-2}^{y=1} \left(6(4 - 4y + y^2)y - 6y^5 \right) dy$$

$$= \int_{y=-2}^{y=1} \left(24y - 24y^2 + 6y^3 - 6y^5 \right) dy$$

$$= \left[12y^2 - 8y^3 + \frac{6}{4}y^4 - y^6 \right]_{y=-2}^{y=1}$$

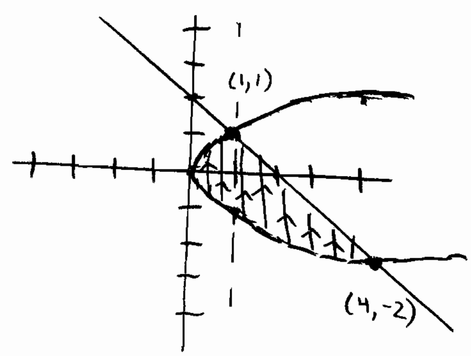
$$= \left(12 - 8 + \frac{6}{4} - 1\right) - \left(48 + 64 + 24 - 64\right)$$

$$= \frac{3}{2} - 69$$

$$= \frac{-135}{2}$$

(b) Now let us reverse the order of integration. Again, the purpose of this is to test your understanding.

same picture



We will be using $\uparrow\uparrow\uparrow$

Note that we have to split up the region.

$$\underbrace{\int_{x=0}^{x=1} \int_{y=-\sqrt{x}}^{y=\sqrt{x}} 12xy \, dy \, dx}_{I_1} + \underbrace{\int_{x=1}^{x=4} \int_{y=-\sqrt{x}}^{y=2-x} 12xy \, dy \, dx}_{I_2}$$

$$I_1 = \int_{x=0}^{x=1} \left[6xy^2 \right]_{y=-\sqrt{x}}^{y=\sqrt{x}} \, dx$$

$$= \int_{x=0}^{x=1} 0 \, dx$$

$$= 0$$

$$I_2 = \int_{x=1}^{x=4} \left[6xy^2 \right]_{y=-\sqrt{x}}^{y=2-x} \, dx$$

$$= \int_{x=1}^{x=4} \left(6x(2-x)^2 - 6x(-\sqrt{x})^2 \right) dx$$

$$= \int_{x=1}^{x=4} \left(6x(4-4x+x^2) - 6x^2 \right) dx$$

$$= \int_{x=1}^{x=4} \left(24x - 30x^2 + 6x^3 \right) dx$$

$$= \left[12x^2 - 10x^3 + \frac{6}{4}x^4 \right]_{x=1}^{x=4}$$

$$= (192 - 640 + 384) - \left(12 - 10 + \frac{6}{4}\right)$$

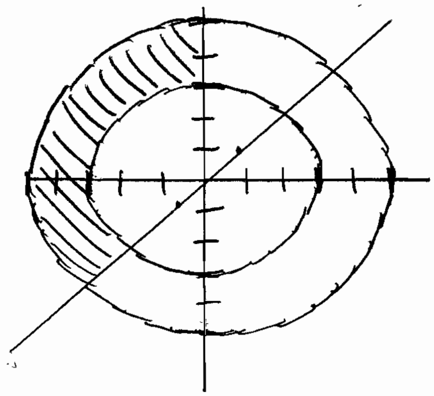
$$= -66 - \frac{6}{4}$$

$$= \frac{-135}{2}$$

So, $I_1 + I_2 = \frac{-135}{2}$ as expected.

$\square$

(c)



In polar coordinates this region is

$$\frac{\pi}{2} \le \theta \le \frac{5\pi}{4}$$

$$3 \le r \le 5$$

$$\iint_R e^{25-x^2-y^2}\, dA = \int_{\theta=\frac{\pi}{2}}^{\theta=\frac{5\pi}{4}} \int_{r=3}^{r=5} e^{25-r^2}\, r \, dr\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{5\pi}{4}} \left[-\frac{1}{2} e^{25-r^2} \right]_{r=3}^{r=5} d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{5\pi}{4}} \left(-\frac{1}{2} e^{0} - \left(-\frac{1}{2} e^{16}\right)\right) d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{5\pi}{4}} \left(-\frac{1}{2} + \frac{1}{2} e^{16}\right) d\theta$$

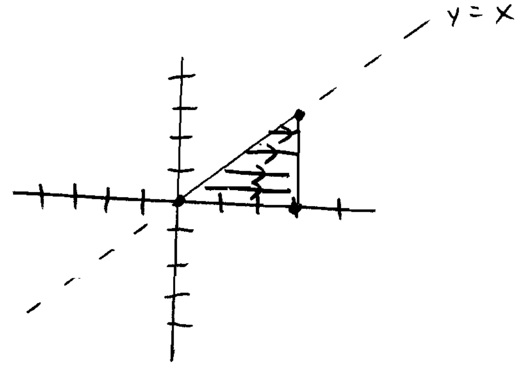
$$= \left[\left(-\frac{1}{2} + \frac{1}{2} e^{16}\right) \theta \right]_{\frac{\pi}{2}}^{\frac{5\pi}{4}}$$

$$= \left(-\frac{1}{2} + \frac{1}{2} e^{16}\right) \frac{3\pi}{4}$$

$$\left(\begin{aligned} &\int e^{25-r^2}\, r\, dr \\ &u = 25 - r^2 \\ &du = -2r\, dr \\ &-\tfrac{1}{2}\, du = r\, dr \\ &\int -\tfrac{1}{2} e^{u}\, du \\ &= -\tfrac{1}{2} e^{u} \\ &= -\tfrac{1}{2} e^{25-r^2} \end{aligned} \right)$$

$\boxed{F}$

Problem 6



(a) $f_x = 2$

$f_y = -3$

$$\text{Surface Area} = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$

$$= \iint_R \sqrt{1+4+9}\, dA$$

$$= \int_{y=0}^{y=3} \int_{x=y}^{x=3} \sqrt{14}\, dx\, dy \qquad \left(\text{using } \rightrightarrows\right)$$

$$\left(\text{using } \uparrow\uparrow\uparrow \text{ would be } \int_{x=0}^{x=3} \int_{y=0}^{y=x} \sqrt{14}\, dy\, dx\right)$$

$$= \int_{y=0}^{y=3} \Big[\sqrt{14}\, x \Big]_{x=y}^{x=3} dy$$

$$= \int_{y=0}^{y=3} \left(\sqrt{14}(3) - \sqrt{14}\, y \right) dy$$

$$= \left[3\sqrt{14}\, y - \frac{\sqrt{14}}{2} y^2 \right]_0^3$$

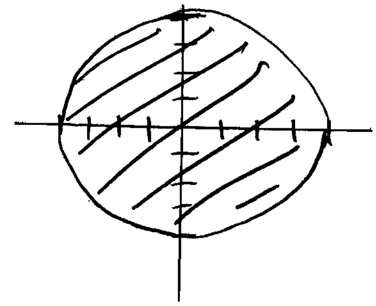
$$= \left(9\sqrt{14} - \frac{\sqrt{14}}{2} 9 \right) - 0$$

$$= \frac{9\sqrt{14}}{2}$$

$\square$

(b) $f_x = y$

$f_y = x$



In polar coordinates

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 4$

Surface Area $= \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$

$= \iint_R \sqrt{1 + y^2 + x^2}\, dA$

$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} \sqrt{1+r^2}\; r\, dr\, d\theta$

$= \int_0^{2\pi} \left[\frac{1}{3}(1+r^2)^{3/2} \right]_{r=0}^{r=4} d\theta$

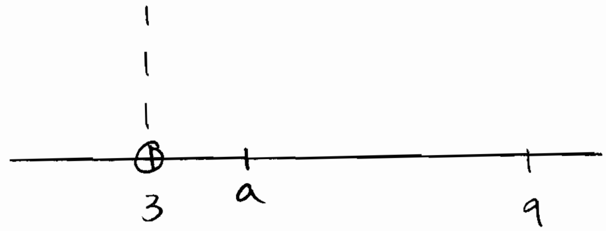
$= \int_0^{2\pi} \left(\frac{1}{3}(17)^{3/2} - \frac{1}{3} \right) d\theta$

$= \left[\left(\frac{1}{3}(17)^{3/2} - \frac{1}{3} \right) \theta \right]_0^{2\pi}$

$= \left(\frac{1}{3}(17)^{3/2} - \frac{1}{3} \right) 2\pi$

$$\left(\begin{array}{c} \int \sqrt{1+r^2}\; r\, dr \\ u = 1 + r^2 \\ du = 2r\, dr \\ \frac{1}{2} du = r\, dr \\ \int \frac{1}{2} u^{1/2}\, du \\ = \frac{1}{2} \frac{u^{3/2}}{3/2} \\ = \frac{1}{3}(1+r^2)^{3/2} \end{array} \right)$$

Problem 7



(a)

$$\int_3^9 \frac{x}{(x^2-9)^{4/3}}\,dx$$

vertical asymptote at $x = 3$

$$= \lim_{a \to 3^+} \int_a^9 \frac{x}{(x^2-9)^{4/3}}\,dx$$

$$= \lim_{a \to 3^+} \left[-\frac{3}{2}(x^2-9)^{-1/3} \right]_a^9$$

$$= \lim_{a \to 3^+} \left(-\frac{3}{2}(72)^{-1/3} - \left(-\frac{3}{2}(a^2-9)^{-1/3} \right) \right)$$

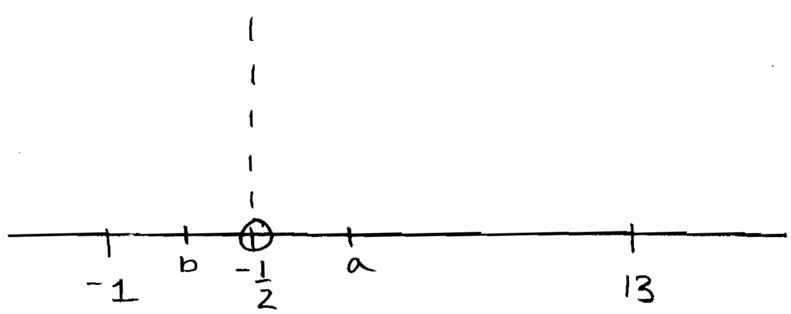
$$= \lim_{a \to 3^+} \left(\frac{-3}{2(72)^{1/3}} + \frac{3}{2(a^2-9)^{1/3}} \right)$$

$$= \infty$$

The integral diverges.

$$\left(\begin{aligned} &\int \frac{x}{(x^2-9)^{4/3}}\,dx \\ &u = x^2-9 \\ &du = 2x\,dx \\ &\tfrac{1}{2}\,du = x\,dx \\ &\int \tfrac{1}{2} u^{-4/3}\,du \\ &= \tfrac{1}{2}\,\frac{u^{-1/3}}{-1/3} \\ &= -\tfrac{3}{2}(x^2-9)^{-1/3} \end{aligned} \right)$$

(b)



vertical asymptote at $x = -\frac{1}{2}$

$$\int_{-1}^{13} \frac{dx}{\sqrt[3]{2x+1}} = \int_{-1}^{-\frac{1}{2}} \frac{dx}{\sqrt[3]{2x+1}} + \int_{-\frac{1}{2}}^{13} \frac{dx}{\sqrt[3]{2x+1}}$$

$$= \underbrace{\lim_{b \to -\frac{1}{2}^-} \int_{-1}^{b} (2x+1)^{-1/3} dx}_{(*)} + \underbrace{\lim_{a \to -\frac{1}{2}^+} \int_{a}^{13} (2x+1)^{-1/3} dx}_{(**)}$$

We compute each limit in turn.

(*)

$$\lim_{b \to -\frac{1}{2}^-} \int_{-1}^{b} (2x+1)^{-1/3} dx$$

$$= \lim_{b \to -\frac{1}{2}^-} \left[\frac{3}{4} (2x+1)^{2/3} \right]_{-1}^{b}$$

$$= \lim_{b \to -\frac{1}{2}^-} \left(\frac{3}{4} (2b+1)^{2/3} - \frac{3}{4} (-1)^{2/3} \right)$$

$$= \lim_{b \to -\frac{1}{2}^-} \left(\frac{3}{4} (2b+1)^{2/3} - \frac{3}{4} \right)$$

$$= -\frac{3}{4}$$

$$\left(\begin{aligned} &\int (2x+1)^{-1/3} dx \\ &u = 2x+1 \\ &du = 2dx \\ &\int \frac{1}{2} u^{-1/3} du \\ &= \frac{1}{2} \frac{u^{2/3}}{2/3} \\ &= \frac{3}{4} (2x+1)^{2/3} \end{aligned} \right)$$

<u>Note</u>. If we had got ∞, $-\infty$, or (limit does not exist) here instead of $-\frac{3}{4}$, then would be no need to compute (**). we could conclude that the integral diverges.

(**)

$$\lim_{a \to -\frac{1}{2}^+} \int_a^{13} (2x+1)^{-1/3}\, dx$$

$$= \lim_{a \to -\frac{1}{2}^+} \left[\frac{3}{4}(2x+1)^{2/3} \right]_a^{13}$$

$$= \lim_{a \to -\frac{1}{2}^+} \left(\frac{3}{4}(27)^{2/3} - \frac{3}{4}(2a+1)^{2/3} \right)$$

$$= \frac{27}{4}$$

The integral converges and

$$\int_{-1}^{13} \frac{dx}{\sqrt[3]{2x+1}} = -\frac{3}{4} + \frac{27}{4} = \underline{\underline{6}}$$

$\boxed{E}$