Problem 8

(a) Observe that $\lim_{k \to \infty} \frac{1}{\arctan k} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \neq 0$

By the Divergence Test, the series diverges.

$\boxed{E}$

(b) obvious candidate for Limit Comparison Test, Diverges. see Example 5.3 pg 12 of "Series" notes.

(c) Geometric series, common ratio $\frac{-8}{9}$

converges, sum $\frac{-4}{51}$. (see Example 2.5 pg 5).

(d) Integral Test, converges, remember you must verify the 3 conditions (see Example 4.1 pg 9).

(e) Telescoping Series

$$\sum_{k=1}^{\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+9} \right)$$

$$S_k = \underbrace{\left(\frac{1}{5} - \cancel{\frac{1}{13}} \right)}_{k=1} + \underbrace{\left(\frac{1}{9} - \cancel{\frac{1}{17}} \right)}_{k=2} + \underbrace{\left(\cancel{\frac{1}{13}} - \cancel{\frac{1}{21}} \right)}_{k=3} + \underbrace{\left(\cancel{\frac{1}{17}} - \frac{1}{25} \right)}_{k=4}$$

$$+ \underbrace{\left(\cancel{\frac{1}{21}} - \frac{1}{29} \right)}_{k=5} + \cdots + \underbrace{\left(\cancel{\frac{1}{4(k-1)+1}} - \frac{1}{4(k-1)+9} \right)}_{k-1} + \underbrace{\left(\cancel{\frac{1}{4k+1}} - \frac{1}{4k+9} \right)}_{k}$$

$$S_k = \frac{1}{5} + \frac{1}{9} - \frac{1}{4(k-1)+9} - \frac{1}{4k+9}$$

$$\lim_{k \to \infty} S_k = \frac{1}{5} + \frac{1}{9} = \frac{14}{45}$$

So, series converges and sum is $\frac{14}{45}$

$$\sum_{k=1}^{\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+9} \right) = \frac{14}{45}$$

<u>Alternatively</u> $\sum_{k=1}^{\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+9} \right) = \sum_{k=1}^{\infty} \frac{(4k+9) - (4k+1)}{(4k+1)(4k+9)}$

$$= \sum_{k=1}^{\infty} \frac{8}{16k^2 + 37k + 9}$$

Now use LCT and compare with $\sum \frac{1}{k^2}$.

Although you would be able to conclude that the series converges, note that using LCT will not give you the actual sum.

$\boxed{}$

(f) Obvious candidate for LCT

$$\sum_{k=1}^{\infty} \frac{\sqrt{k^3+1}}{k^3-2k^2+5} \quad (*)$$

Consider $\displaystyle\sum \frac{k^{3/2}}{k^3} = \sum \frac{1}{k^{3/2}}$

we apply LCT

$$\lim_{k\to\infty} \frac{\sqrt{k^3+1}}{k^3-2k^2+5} \frac{k^{3/2}}{1} = \lim_{k\to\infty} \frac{\sqrt{k^3+1}}{k^3-2k^2+5} \cdot \frac{\sqrt{k^3}}{1}$$

$$= \lim_{k\to\infty} \frac{\sqrt{k^6+k^3}}{k^3-2k^2+5} \cdot \frac{\frac{1}{k^3}}{\frac{1}{k^3}}$$

$$= \lim_{k\to\infty} \frac{\sqrt{k^6+k^3}}{1-\frac{2}{k}+\frac{5}{k^3}} \quad \frac{1}{\sqrt{k^6}}$$

$$= \lim_{k\to\infty} \frac{\sqrt{1+\frac{1}{k^3}}}{1-\frac{2}{k}+\frac{5}{k^3}}$$

$$= 1 \quad \text{(finite \& positive)}$$

By LCT both series behave the same.

$\displaystyle\sum \frac{1}{k^{3/2}}$ is a convergent p-series $\left(p = 3/2 > 1\right)$

So, $(*)$ also converges.

$\boxed{E}$

(g) Direct Comparison Test, converges,

See Example 5.1 pg 11

(h) Geometric series, common ratio $\frac{-25}{27}$, converges

sum $\dfrac{\frac{125}{729}}{1-\left(\frac{-25}{27}\right)}$ (See Example 2.6 pg 6)

(i) Integral Test, diverges,

(See the <u>first part</u> of Example 7.4 pg 17)

Here to show that f is decreasing, you can use the derivative argument as done in the notes

<u>or</u> you can argue that

x is increasing, $\ln x$ is increasing.

So, $x \ln x$ is increasing. Hence, $\frac{1}{x \ln x}$ must be decreasing.

(j) Divergence Test

Observe that $\lim_{n\to\infty} \dfrac{n}{(\ln n)^2} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \dfrac{1}{2(\ln n)\frac{1}{n}}$ L'hôpital

$= \lim_{n\to\infty} \dfrac{n}{2 \ln n}$ $\frac{\infty}{\infty}$

$$= \lim_{n\to\infty} \frac{1}{2 \cdot \frac{1}{n}} \quad \text{L'hôpital}$$

$$= \lim_{n\to\infty} \frac{n}{2}$$

$$= \infty \quad (\neq 0)$$

The series diverges by the Divergence Test.

$\boxed{E}$

(k) Divergence Test

Observe that

$$\lim_{k\to\infty} \frac{\sqrt{5k^6 + 3k + 21}}{7k^3 + 14k^2 + 93} \cdot \frac{\frac{1}{k^3}}{\frac{1}{k^3}}$$

$$= \lim_{k\to\infty} \frac{\sqrt{5k^6 + 3k + 21}}{7 + \frac{14}{k} + \frac{93}{k^3}} \quad \frac{1}{\sqrt{k^6}}$$

$$= \lim_{k\to\infty} \frac{\sqrt{5 + \frac{3}{k^5} + \frac{21}{k^6}}}{7 + \frac{14}{k} + \frac{93}{k^3}} = \frac{\sqrt{5}}{7} \neq 0$$

The series diverges by the Divergence Test.

$\boxed{E}$

(l) $$\sum_{n=1}^{\infty} \frac{2n+5}{\sqrt[3]{n^4+n^2}} \quad (**)$$

Obvious candidate for LCT

Consider $$\sum \frac{n}{n^{4/3}} = \sum \frac{1}{n^{1/3}}$$

we apply LCT

$$\lim_{n\to\infty} \frac{2n+5}{\sqrt[3]{n^4+n^2}} \cdot \frac{n^{1/3}}{1} = \lim_{n\to\infty} \frac{2n^{4/3}+5n^{1/3}}{\sqrt[3]{n^4+n^2}} \cdot \frac{\frac{1}{n^{4/3}}}{\frac{1}{n^{4/3}}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{5}{n}}{\sqrt[3]{n^4+n^2} \cdot \frac{1}{\sqrt[3]{n^4}}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{5}{n}}{\sqrt[3]{1+\frac{1}{n^2}}}$$

$$= 2 \quad \text{(finite \& positive)}$$

By LCT both series behave the same.

$\sum \frac{1}{n^{1/3}}$ is a divergent p-series $(p = 1/3 \leq 1)$

So, $(**)$ also diverges

$\boxed{E}$

(m) Geometric series, common ratio $-\frac{5}{4}$, diverges

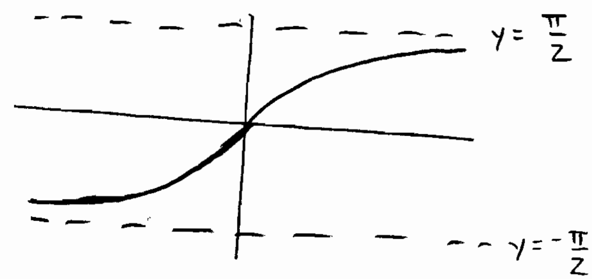
see Example 2.7 pg 6.

(n) Divergence Test, diverges, see Example 3.1 pg 8.

(o) Direct comparison Test, diverges

see Example 5.2 pg 11 and use the fact

that $\frac{-\pi}{2} \leq \arctan n \leq \frac{\pi}{2}$



(p) Direct Comparison Test.

$$\sum_{n=1}^{\infty} \frac{3^n}{5^n + n} \quad (\#)$$

The dominant terms in the numerator and denominator are 3^n and 5^n respectively

So, $(\#)$ looks like $\sum \frac{3^n}{5^n} = \sum \left(\frac{3}{5}\right)^n$

which we recognize is a convergent geometric series.

we think $(\#)$ also converges.

To use DCT we need to find a convergent series above our series. $\sum \frac{3^n}{5^n}$ is a natural candidate. We just need to verify

$$\frac{3^n}{5^n+n} \overset{?}{\leq} \frac{3^n}{5^n}$$

$$3^n 5^n \overset{?}{\leq} 3^n (5^n + n)$$

$$3^n 5^n \overset{?}{\leq} 3^n 5^n + 3^n n$$

The last inequality is obviously true. So indeed

$$\frac{3^n}{5^n+n} \leq \frac{3^n}{5^n}.$$

So, (#) converges by DCT.

$\boxed{E_1}$

(q) Divergence Test

Observe that

$$\lim_{k\to\infty} \sec\left(\frac{3k+1}{2+k^2} \cdot \frac{\frac{1}{k^2}}{\frac{1}{k^2}}\right)$$

$$= \lim_{k \to \infty} \sec\left(\frac{\frac{3}{k} + \frac{1}{k^2}}{\frac{2}{k^2} + 1} \right)$$

$$= \sec(0) = 1 \neq 0$$

The series diverges by the Divergence Test.

$\boxed{\times}$

(r) Obvious candidate for LCT.

$$\sum_{n=1}^{\infty} \frac{n^4}{(n^2-2n-1)^3} \quad (\#\#)$$

consider $\sum \frac{n^4}{n^6} = \sum \frac{1}{n^2}$

we apply LCT

$$\lim_{n\to\infty} \frac{n^4}{(n^2-2n-1)^3} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^6}{(n^2-2n-1)^3} \cdot \frac{\frac{1}{n^6}}{\frac{1}{n^6}}$$

$$= \lim_{n\to\infty} \frac{1}{(n^2-2n-1)^3 \frac{1}{(n^2)^3}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1 - \frac{2}{n} - \frac{1}{n^2}\right)^3}$$

$$= 1 \text{ (finite \& positive)}$$

$\sum \frac{1}{n^2}$ is a convergent p-series $(p = 2 > 1)$

so, $(\#\#)$ also converges by LCT.

$\boxed{\times}$