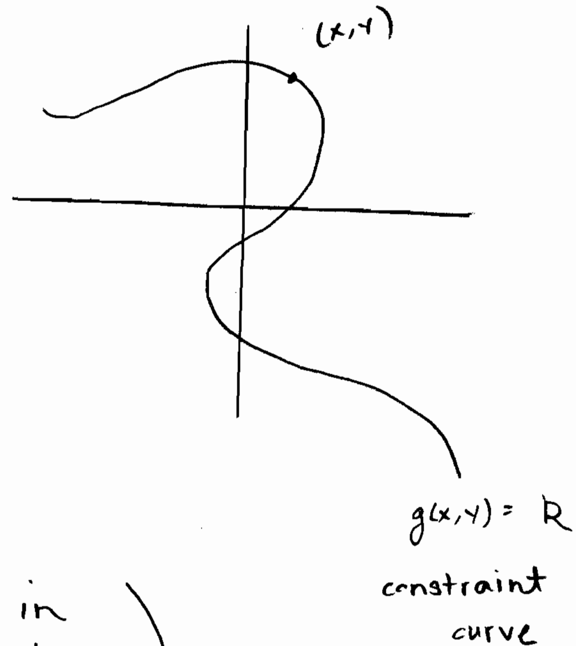
13.10 Lagrange Multipliers

Let $f(x,y)$ be a function.

Consider a curve given by the equation

$g(x,y) = k$ (constraint)



Main Problem

Find the extreme values of $f(x,y)$ when (x,y) is restricted to lie on the constraint curve.

(compare this with the problem in section 13.8 where we were finding relative extrema with no restrictions.)

Like before, we begin by looking for potential candidates, ie critical points.

We solve

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g(x,y) = k \quad \text{(constraint)} \end{cases}$$

Examples

(a) $f(x,y) = x^2 - y^2$; $x^2 + y^2 = 1$

We need to express the constraint as the level curve of some function (we need to rewrite the constraint so that it is in the form $g(x,y) = \text{constant}$).

If we let $g(x,y) = x^2 + y^2$, then our constraint is precisely the level curve $g(x,y) = 1$.

$\underline{\underline{1^0}}$ Find critical points

We solve

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g & (1') \\ \text{constraint} & (2') \end{cases}$$

$$\langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$$

So, our system

$$\begin{cases} 2x = \lambda 2x & (1) \\ -2y = \lambda 2y & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$

we don't want to divide by x in (1) since x could be zero. So, we do the following:

(1) gives $\quad 2x - \lambda 2x = 0$

$$2x(1-\lambda) = 0$$

$x = 0$ or $\lambda = 1$

plug into (3)	plug into (2)
$y^2 = 1$	$-2y = 2y$
$y = 1$ or $y = -1$	$-4y = 0$
$(0,1)$ $(0,-1)$	$y = 0$
crit points	
	plug into (3)
	$x^2 = 1$
	$x = 1$ or $x = -1$
	$(1,0)$ $(-1,0)$ crit points

$\underline{\underline{2^{\circ}}}$ If f has a maximum (or minimum) subject to the constraint, then we are guaranteed that it will occur at one of the critical points.

But without additional information, we don't know whether a max, a min, or neither occurs at a particular critical point, say $(-1,0)$.

In this particular example, the constraint is a circle. I know for a fact (by compactness) that f does have a maximum and a minimum.

So, we know the max and min will occur at the critical points.

$f(0,1) = -1$

$f(0,-1) = -1$

$f(1,0) = 1$

$f(-1,0) = 1$

maximum value of f is 1

minimum value of f is -1.

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(b) $f(x,y) = 3xy$; $\underbrace{4x^2+y^2}_{g(x,y)} = 200$

To find critical points, we solve

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ \text{constraint} \end{cases}$$

$$\langle 3y, 3x \rangle = \lambda \langle 8x, 2y \rangle$$

So, our system is

$$\begin{cases} 3y = \lambda\, 8x & (1) \\ 3x = \lambda\, 2y & (2) \\ 4x^2 + y^2 = 200 & (3) \end{cases}$$

Note that x can't be 0. If $x=0$, (1) would imply that $y=0$. But this would contradict (3). Thus, we can divide by x with no worries.

Solving for λ in (1) gives $\lambda = \frac{3y}{8x}$

Plugging this into (2) yields

$$3x = \frac{6y^2}{8x}$$

$$24x^2 = 6y^2$$

$$4x^2 = y^2$$

using this in (3) we have

$$y^2 + y^2 = 200$$

$$y^2 = 100$$

$y = 10$ or $y = -10$

using eq (3)	
$4x^2 + 100 = 200$	$4x^2 + 100 = 200$
$x^2 = 25$	$x^2 = 25$
$x = 5$ or $x = -5$	$x = 5$ or $x = -5$
$(5, 10)$ & $(-5, 10)$ are crit pts	$(5, -10)$ & $(-5, -10)$ are crit pts.

$\underline{\underline{2^{0}}}$ As before, the constraint is an ellipse. By compactness, I know max and min will occur.

check that maximum value of f is 150

minimum value of f is -150.

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(c) $f(x,y) = x^2 y$; $\underbrace{4x + 3y}_{g(x,y)} = 18$

$\underline{\underline{1^\circ}}$ Find critical points

we solve

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ \text{constraint} \end{cases}$$

$$\langle 2xy , x^2 \rangle = \lambda \langle 4, 3 \rangle$$

our system is

$$\begin{cases} 2xy = \lambda 4 & (1) \\ x^2 = \lambda 3 & (2) \\ 4x + 3y = 18 & (3) \end{cases}$$

$\Longrightarrow$ $\lambda = \frac{1}{2} xy$

(2) becomes

$$x^2 = \frac{3}{2} xy$$

$$x^2 - \frac{3}{2} xy = 0$$

$$x\left(x - \frac{3}{2} y\right) = 0$$

$x = 0$ or $x = \frac{3}{2} y$ (*)

$x = 0$:

$\left((3) \Rightarrow y = 6 \right)$

$(0, 6)$ crit pt

$x = \frac{3}{2} y$:

plug into (3)

$$6y + 3y = 18$$

$$y = 2$$

$\left((*) \Rightarrow x = 3 \right)$

$(3, 2)$ crit pt.

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(d) $f(x,y) = x + 2y - 2$; $\underbrace{x^2 y}_{g(x,y)} = 432$

$\underline{\underline{1}}^{\circ}$ Find critical points

we solve

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ \text{constraint} \end{cases}$$

$$\langle 1, 2 \rangle = \lambda \langle 2xy, x^2 \rangle$$

Our system is

$$\begin{cases} 1 = \lambda (2xy) & (1) \\ 2 = \lambda x^2 & (2) \\ x^2 y = 432 & (3) \end{cases}$$

<u>Note</u> Neither x nor y can be 0. If either one were 0 we would contradict (3). Hence, we can divide by x & y to our heart's content.

Solving for λ in (1) gives

$$\lambda = \frac{1}{2xy}$$

Plug this into (2) gives $2 = \dfrac{x}{2y} \Rightarrow 4y = x$

Now (3) becomes $16y^3 = 432$

$$y^3 = 27$$

$$y = 3$$

$$(x = 12)$$

$\underline{\underline{(12, 3) \text{ crit pt}}}$