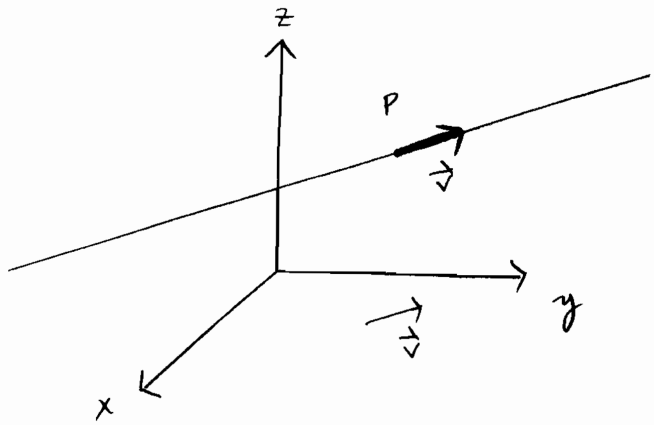
Lines

A line in space is completely determined if you are given a point $P(x_0, y_0, z_0)$ on the line and a direction vector $\vec{v} = \langle a, b, c \rangle$ parallel to the line.



Such a line has

Parametric Equations

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

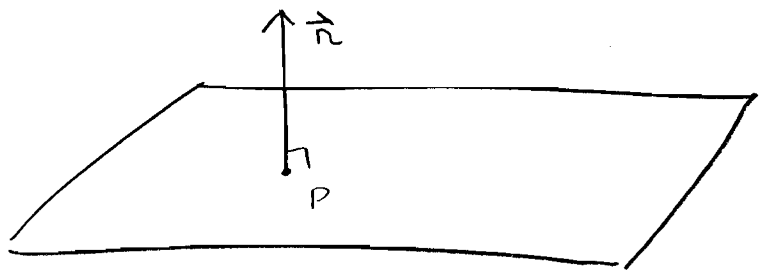
Symmetric Equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Remark: Be sure you understand how to read off the direction vector and a point on the line from the parametric and symmetric equations.

Planes

A plane is completely determined by a point $P(x_0, y_0, z_0)$ on the plane and a vector $\vec{n} = \langle a, b, c \rangle$ normal (perpendicular) to the plane.



Such a plane has equation

$$(*) \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

(Multiply out and group terms to get in the form

$$(**) \quad ax + by + cz + d = 0 \quad \text{or}$$

$$(***) \quad ax + by + cz = D$$
)

Remark: Be sure you understand how to read off the normal vector from $(*)$, $(**)$, $(***)$.

Two planes are perpendicular if their normal vectors are perpendicular, $\vec{n}_1 \perp \vec{n}_2$ (or equivalently $\vec{n}_1 \cdot \vec{n}_2 = 0$)

Two planes are parallel if their normal vectors are parallel, $\vec{n}_1 \parallel \vec{n}_2$ (or equivalently $\vec{n}_1 = c\vec{n}_2$ for some c).

These make sense if we think in terms of the picture.

In general, the angle between two planes is the angle between their normal vectors.

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \, \|\vec{n}_2\|}$$

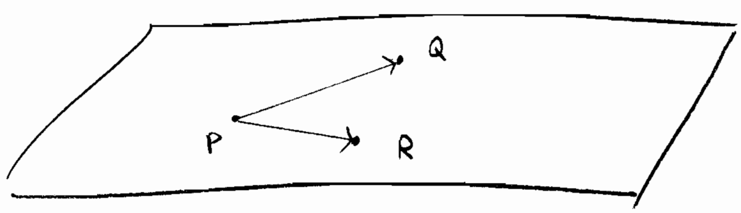
Examples

Find an equation of the plane.

a) The plane through the points

$P(3,-1,2)$ $\quad$ $Q(8,2,4)$ $\quad$ $R(-1,-2,-3)$

soln: Picture of the situation



$\vec{PQ} = \langle 5, 3, 2 \rangle$

$\vec{PR} = \langle -4, -1, -5 \rangle$

we need a vector normal to the plane. Recall that the cross product $\vec{u} \times \vec{v}$ gives a vector perpendicular to both $\vec{u}$ & $\vec{v}$. This and the picture above suggests that we take

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & 2 \\ -4 & -1 & -5 \end{vmatrix} = (-13)\vec{i} - (-17)\vec{j} + (7)\vec{k}$$

$$= \langle -13, 17, 7 \rangle$$

The equation of the plane is (we'll use the pt $(3,-1,2)$)

$$-13(x-3) + 17(y+1) + 7(z-2) = 0$$

$\boxed{\;}$

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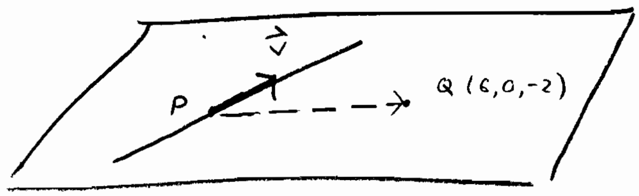
b) The plane through the point $(6,0,-2)$ and containing the line

$$x = 4-2t$$
$$y = 3+5t$$
$$z = 7+4t$$



$\vec{v} = \langle -2, 5, 4 \rangle$

Soln :

we need a point on the line. Letting $t=0$, we get $P(4,3,7)$.

$$\vec{PQ} = \langle 2, -3, -9 \rangle$$

The picture suggests that to get a normal vector, we take

$$\vec{n} = \vec{PQ} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & -9 \\ -2 & 5 & 4 \end{vmatrix} = (33)\vec{i} - (-10)\vec{j} + (4)\vec{k}$$

$$= \langle 33, 10, 4 \rangle$$

The equation of the plane is

$$33(x-6) + 10(y) + 4(z+2) = 0$$

$\boxed{\;}$

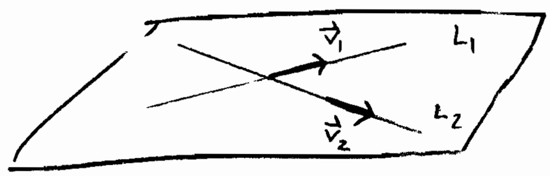
c) The plane containing the lines

$$\frac{x-2}{-3} = \frac{y-2}{6} = z-3 \quad (L_1) \qquad \text{and} \qquad \begin{aligned} x &= 3+2t \\ y &= -5+t \\ z &= -2+4t \end{aligned} \quad (L_2)$$

Soln : Direction vector for L_1 : $\vec{v}_1 = \langle -3, 6, 1 \rangle$

" " L_2 : $\vec{v}_2 = \langle 2, 1, 4 \rangle$

L_1 & L_2 are not parallel. The picture is then



To get a normal vector we take

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 6 & 1 \\ 2 & 1 & 4 \end{vmatrix} = (23)\vec{i} - (-14)\vec{j} + (-15)\vec{k}$$

$$= \langle 23, 14, -15 \rangle$$

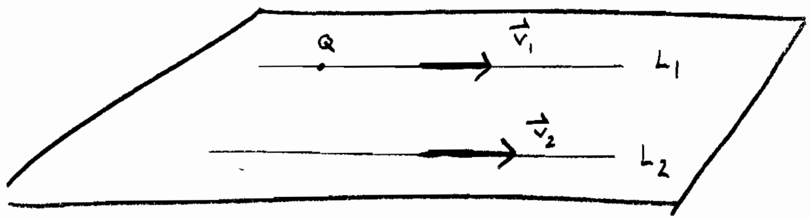
We need a point on the plane. Let's find a point on one of the lines, letting $t=0$ in L_2 yields the point $(3,-5,-2)$.

The equation of the plane is

$$23(x-3) + 14(y+5) - 15(z+2) = 0$$

d) How would you find an equation of the plane containing two distinct parallel lines?

Soln: Draw a picture.



Pick any point Q on L_1. Now, we are in the situation of part b).

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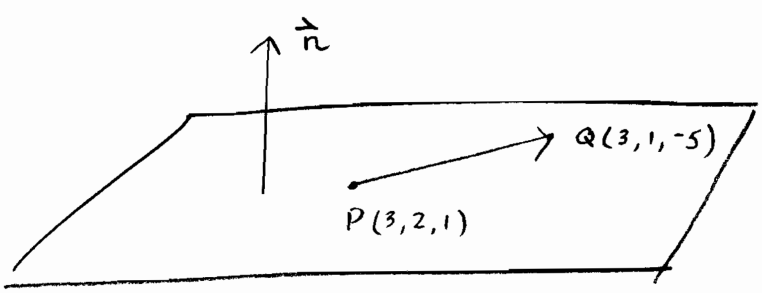
e) The plane that passes through the points $(3, 2, 1)$ and $(3, 1, -5)$ and is perpendicular to the plane $6x + 7y + 2z = 10$.

Soln:



The plane $6x + 7y + 2z = 10$ has normal vector $\vec{n}_2 = \langle 6, 7, 2 \rangle$.

We need a vector $\vec{n}$ normal to our desired plane, what do we know about $\vec{n}$?

$\vec{n} \perp \overrightarrow{PQ}$ since $\vec{n}$ is normal to the plane.

Also, $\vec{n} \perp \vec{n}_2$ since the planes are perpendicular.
So, $\vec{n}$ is perpendicular to both $\vec{PQ}$ & $\vec{n}_2$.
we can pin down $\vec{n}$ by taking cross products.

$$\vec{n} = \vec{PQ} \times \vec{n}_2$$

You finish the rest.

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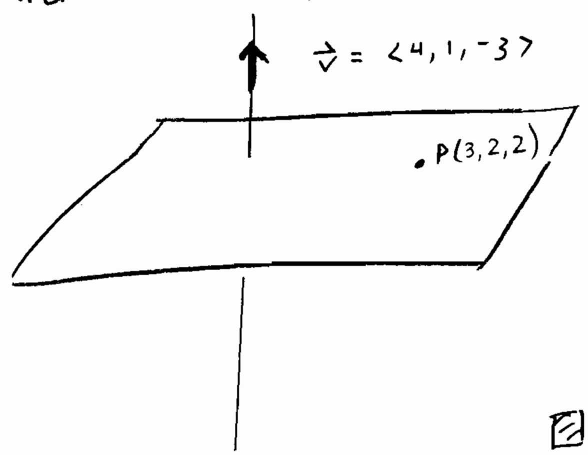
f) The plane through the point $(3,2,2)$ and perpendicular to the line

$$\frac{x-1}{4} = y+2 = \frac{z+3}{-3}$$



Soln: We can take $\vec{n} = \vec{v}$ to be our normal vector. we already have a point.

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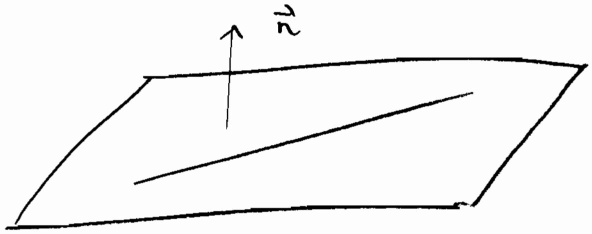
g) The plane that contains the line $\frac{x-1}{4} = y+2 = \frac{z+3}{-3}$ and is parallel to the plane
$2x + 4y + 8z = 17$.



Soln:

The plane $2x+4y+8z=17$ has normal vector $\vec{n}_2 = \langle 2,4,8 \rangle$.

Since our desired plane is parallel to this one, $\vec{n}_2$ should also be perpendicular to our plane. So, we take $\vec{n} = \vec{n}_2$. To get a point on the plane, we take a point on the line. $(1,-2,-3)$ is one such pt.

Two distinct planes are either parallel or intersect in a line.

Example Consider the two planes

$$6x - 3y + z = 5 \qquad (P_1)$$

$$-x + y + 5z = 5 \qquad (P_2)$$

Plane P_1 has normal vector $\vec{n}_1 = \langle 6, -3, 1 \rangle$
Plane P_2 has normal vector $\vec{n}_2 = \langle -1, 1, 5 \rangle$

The two planes are not parallel as you can check. So, they intersect in a line. What is an equation for this line?

We need a point on the line and a direction vector $\vec{v}$.

What do we know about $\vec{v}$?
Well, the line is going to lie in both planes so, $\vec{v}$ is going to lie in both planes (More accurately, $\vec{v}$ is parallel to both planes).

Therefore, $\vec{v} \perp \vec{n}_1$ since $\vec{v}$ lies in plane P_1.
$\vec{v} \perp \vec{n}_2$ since $\vec{v}$ lies in plane P_2.

$\vec{v}$ is perpendicular to both $\vec{n}_1$ & $\vec{n}_2$. To pin down $\vec{v}$, we can take cross products.

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -3 & 1 \\ -1 & 1 & 5 \end{vmatrix} = (-16)\vec{i} - (31)\vec{j} + (3)\vec{k}$$

$$= \langle -16, -31, 3 \rangle$$

Now we need a point $Q(x, y, z)$ on the line. This is the same as looking for a point $Q(x, y, z)$ that lies in both planes. This in turn is equivalent to looking for (x, y, z) that satisfy simultaneously,

$$6x - 3y + z = 5$$
$$-x + y + 5z = 5$$

Solve this system of equations. To make things simpler set $z = 0$ or $y = 0$ or $x = 0$. why?
we'll set $z = 0$

$$6x - 3y = 5$$
$$-x + y = 5 \Rightarrow y = 5 + x$$

$$6x - 3(5+x) = 5$$
$$6x - 15 - 3x = 5$$
$$x = \frac{20}{3}$$
$$y = 5 + \frac{20}{3} = \frac{35}{3}$$
$$z = 0$$

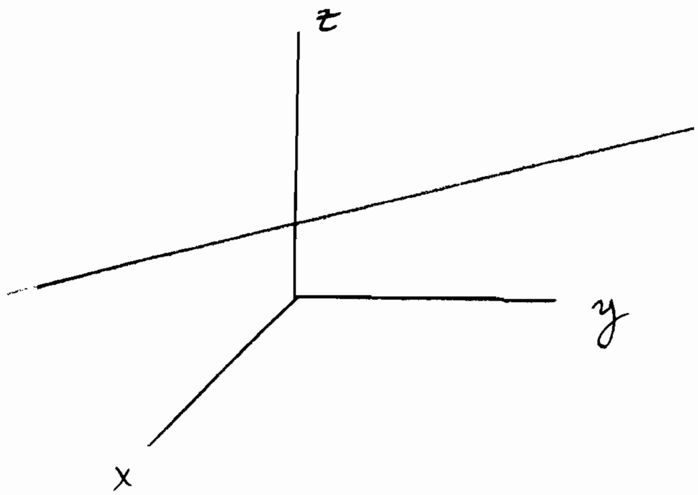
So, $\left(\frac{20}{3}, \frac{35}{3}, 0\right)$ is a point on the line.

Parametric Eq of Line is

$$x = \frac{20}{3} + -16t$$
$$y = \frac{35}{3} + -31t$$
$$z = 0 + 3t$$

Why were we allowed to simplify the problem by setting $z = 0$ or $y = 0$ or $x = 0$?

we know the intersection of two planes is a line. So, the system of equations will have lots of solutions. we just need one.



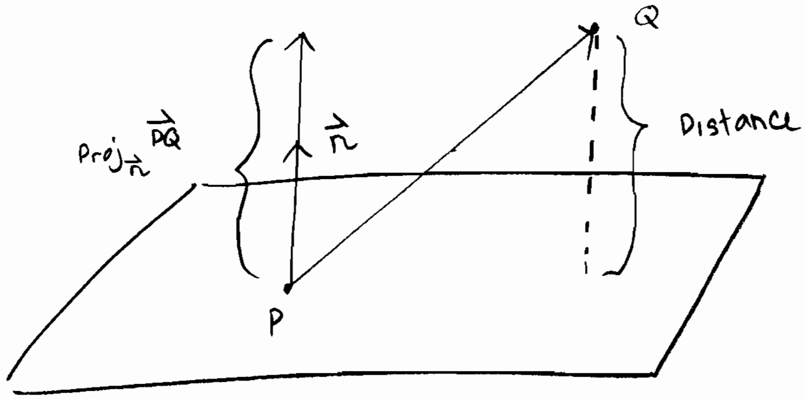
Geometrically, the line is going to intersect one of the xy-plane, yz-plane, xz-plane (may be all three).

When we set $z = 0$, we are looking for the point where the line intersects the xy-plane.

Note: There are lines that don't intersect the xy-plane. So setting $z = 0$ will not always work. But you are guaranteed that it's going to intersect one of the xy, yz, xz-planes. Try setting $y = 0$ or $x = 0$.

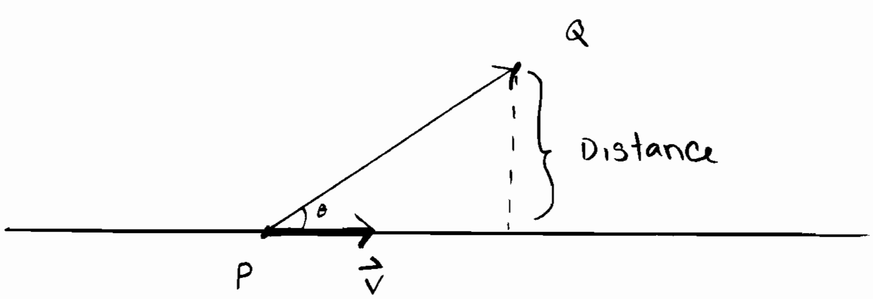
Unrelated, has nothing to do with this class question:
Does the "typical" line intersect all three xy, yz, xz planes?
How would you justify your belief?

Distance between point & plane



$$\text{Distance} = \| \text{Proj}_{\vec{n}} \vec{PQ} \| = \left\| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n} \right\| = \boxed{\frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|}}$$

Distance between point & line

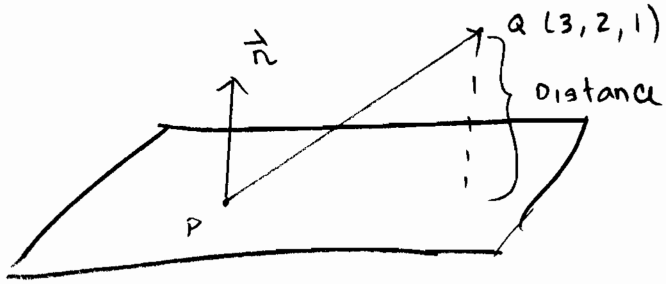


$$\text{Distance} = \|\vec{PQ}\| \sin\theta = \frac{\|\vec{v}\| \, \|\vec{PQ}\| \sin\theta}{\|\vec{v}\|} = \boxed{\frac{\|\vec{PQ} \times \vec{v}\|}{\|\vec{v}\|}}$$

Example

Find the distance between the point and the plane.

$(3,2,1)$ $\qquad$ $x - y + 2z = 4$



$\vec{n} = \langle 1, -1, 2 \rangle$

we'll need a point on the plane. we need (x,y,z) that satisfies $x - y + 2z = 4$. $(0,0,2)$ is one such point. $P(0,0,2)$

$\vec{PQ} = \langle 3, 2, -1 \rangle$

$$\text{Distance} = \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|\langle 3,2,-1\rangle \cdot \langle 1,-1,2\rangle|}{\|\vec{n}\|}$$

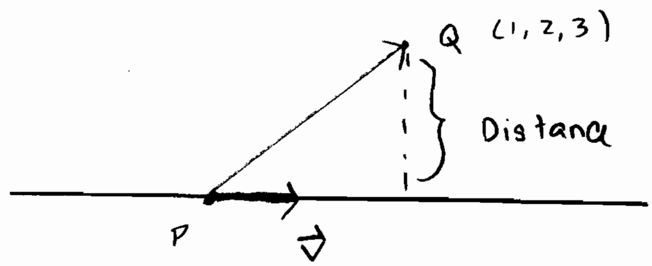
$$= \frac{|3 - 2 - 2|}{\sqrt{1+1+4}}$$

$$= \frac{1}{\sqrt{6}}$$

Example

Find the distance between the point (1, 2, 3) and the line

$$x - 2 = \frac{y-2}{-3} = \frac{z}{5}$$



$\vec{v} = \langle 1, -3, 5 \rangle$

We need a point on the line. P(2, 2, 0) is one such point.

$\vec{PQ} = \langle -1, 0, 3 \rangle$

$$\vec{PQ} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 3 \\ 1 & -3 & 5 \end{vmatrix} = (9)\vec{i} - (-8)\vec{j} + (3)\vec{k}$$

$$= \langle 9, 8, 3 \rangle$$

$$\text{Distance} = \frac{\|\vec{PQ} \times \vec{v}\|}{\|\vec{v}\|} = \frac{\sqrt{81 + 64 + 9}}{\sqrt{1 + 9 + 25}} = \frac{\sqrt{154}}{\sqrt{35}} = \frac{\sqrt{22}}{\sqrt{5}}$$

$\boxed{}$

Question: How would you find the distance between two parallel planes? two parallel lines?