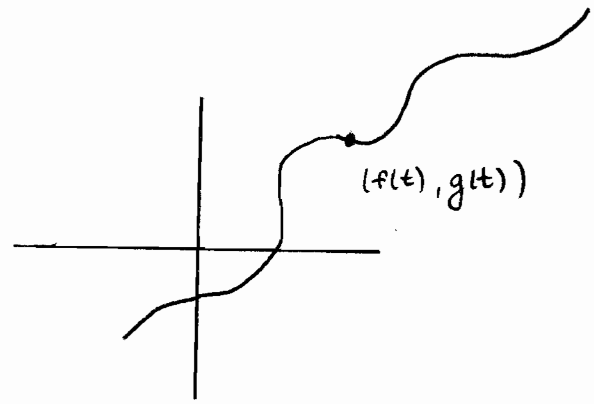
10.2 & 10.3 : Parametric Curves

10.2



$x = f(t)$

$y = g(t)$

are called parametric equations and t is called the parameter. Each value of t determines a point $(x,y) = (f(t), g(t))$ in the xy-plane. As t varies, the point $(x,y) = (f(t), g(t))$ varies and traces out a curve.

Sketching Parametric Curves

The direct way to sketch parametric curves is to pick (many) values of t, plot points, and connect the dots.

However, we will sketch parametric curves in a more indirect and clever way.

Example 1

Sketch the curve represented by the parametric equations (indicate the orientation).

$$x = 2 - \sin^2 t \qquad y = \cos t \qquad 0 \le t \le \pi$$

$\underline{\underline{1^\circ}}$ We begin by finding a relationship between x and y. We do this by <u>eliminating</u> <u>the</u> <u>parameter</u>.

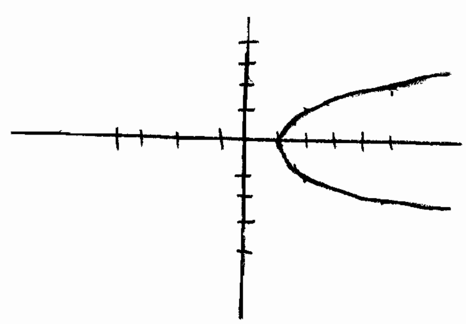
Recall that $\sin^2 t + \cos^2 t = 1$.

So,

$$\begin{aligned} x &= 2 - \sin^2 t \\ &= 2 - (1 - \cos^2 t) \\ &= 1 + \cos^2 t \\ &= 1 + y^2 \end{aligned}$$

$(*) \quad x = 1 + y^2$

$\underline{\underline{2^\circ}}$ This tells us that as t varies, the point (x,y) must satisfy $(*)$ or, equivalently, (x,y) must lie on the curve $x = 1 + y^2$, which we recognize is a parabola.



$\underline{\underline{3^\circ}}$ This however does not tell us anything about where the curve starts, where it ends, and what its orientation is. To get this information, we study the original parametric equations.

when $t = 0$ we are at the point $(2 - \sin^2(0), \cos(0)) = (2, 1)$

when $t = \pi$ we are at the point $(2, -1)$.

So, the curve starts at $(2,1)$ and ends at $(2,-1)$.

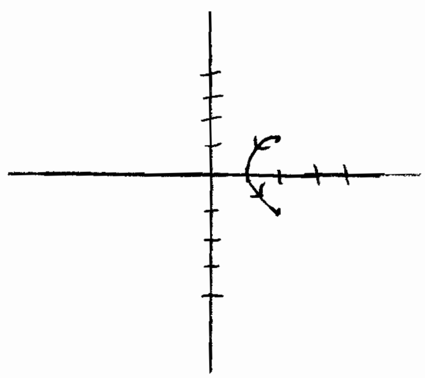
Moreover, as t varies from 0 to π,

$y = \cos t$ varies from 1 (when $t = 0$) to

0 (when $t = \frac{\pi}{2}$) to

-1 (when $t = \pi$)

This gives us enough information to draw our curve.



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Example 2

Sketch the curve represented by the parametric equations.

$x = 2\cos t + 1 \qquad y = 3\sin t - 1 \qquad 0 \leq t \leq \pi$

1° Eliminate the parameter

Recall that $\sin^2 t + \cos^2 t = 1$

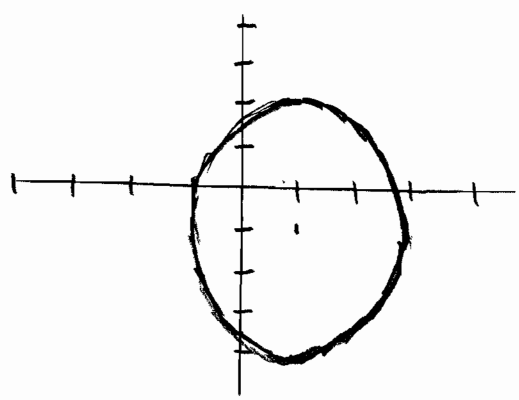
Since $\frac{x-1}{2} = \cos t$

$\frac{y+1}{3} = \sin t$

we have

$$\frac{(x-1)^2}{2^2} + \frac{(y+1)^2}{3^2} = 1$$

$\underline{\underline{2^\circ}}$ We recognize this as the equation of an ellipse whose graph is shown below:

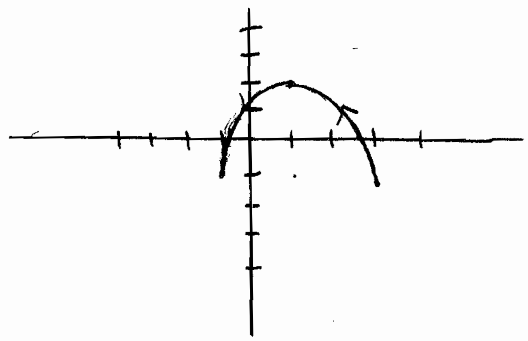


$\underline{\underline{3^\circ}}$ The original parametric equations tell us that the curve starts at $(3,-1)$ (when $t=0$) and ends at $(-1,-1)$ (when $t=\pi$).

As t varies from 0 to π, note that

$y = 3\sin t - 1$ varies from -1 (when $t=0$) to
2 (when $t=\frac{\pi}{2}$) to
-1 (when $t=\pi$).

So, our curve is:



Example 3

Sketch the curve represented by the parametric equations

$$x = \ln t \qquad y = \sqrt{t} \qquad t \geq 1$$

1° Eliminate the parameter

Since $y^2 = t$, we have

$$\begin{aligned} x &= \ln t \\ &= \ln y^2 \\ &= 2 \ln y \end{aligned}$$

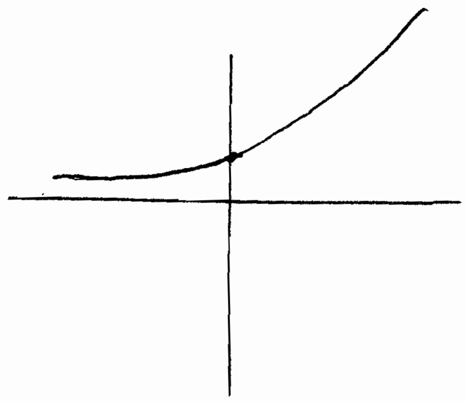
$$x = 2 \ln y.$$

You may not be comfortable graphing this equation, as is. If so, we can solve for y to get

$$\frac{x}{2} = \ln y$$

$$y = e^{x/2}$$

$\underline{\underline{2^\circ}}$. The graph of $y = e^{x/2}$ is shown below

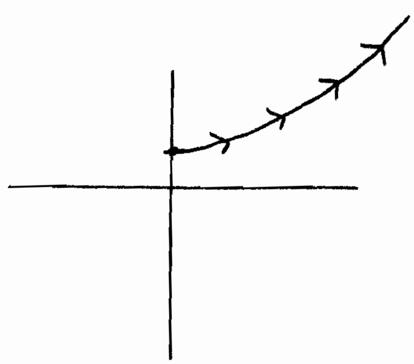


$\underline{\underline{3^\circ}}$. When $t = 1$, we are at the point $(0,1)$.

As t increases, $x = \ln t$ increases.

Moreover, $x = \ln t$ increases without bound.

Hence, our curve is



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Example 4

$x = 4\sec\theta \qquad y = 3\tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

1° Eliminate the parameter

Recall the Trig identity $\tan^2\theta + 1 = \sec^2\theta$

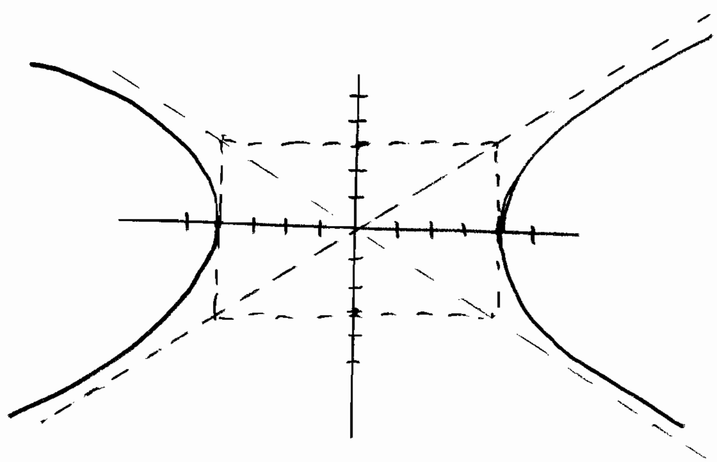
Since $\frac{x}{4} = \sec\theta$

$\frac{y}{3} = \tan\theta$

we have

$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$

2°. We recognize this to be the equation of a hyperbola.



3° Since the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ is open, there is no "start" and "end."

When $\theta = 0$, we are at the point $(4, 0)$.

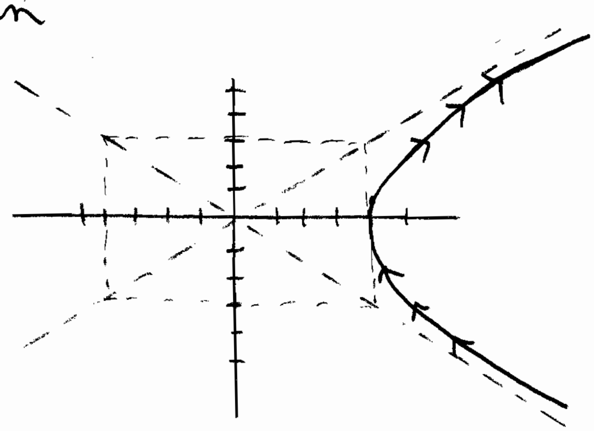
(As θ increases from 0 to $\frac{\pi}{2}$, $\cos\theta$ decreases from 1 and approaches 0.)

Thus, as θ increases from 0 to $\frac{\pi}{2}$, $x = 4\sec\theta$ increases from 4 and approaches ∞.

As θ increases from 0 to $\frac{\pi}{2}$, $y = 3\tan\theta$ increases from 0 and approaches ∞.

We can also examine what happens when we "go backwards" from $\theta = 0$ towards $\theta = -\frac{\pi}{2}$. I'll leave that to you.

Our curve is then



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Example 5

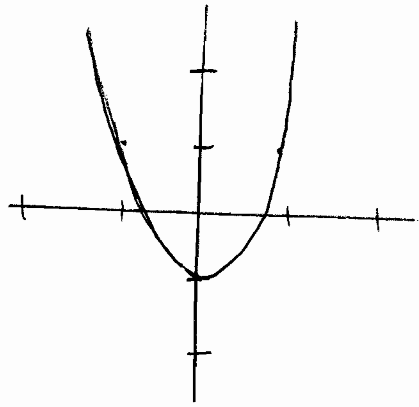
$x = \cos t \qquad y = \cos 2t \qquad -\pi \leq t \leq \frac{-\pi}{2}$

$\underline{\underline{1^\circ}}$ We eliminate the parameter by recalling the trig identity $\cos 2t = 2\cos^2 t - 1$.

This gives

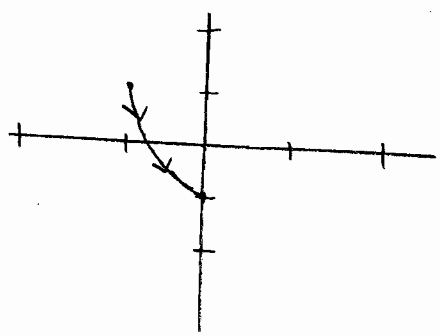
$$y = 2x^2 - 1$$

$\underline{\underline{2^\circ}}$. This is the equation of the parabola



$\underline{\underline{3^\circ}}$. The curve starts at $(-1, 1)$ (when $t = -\pi$) and ends at $(0, -1)$ (when $t = -\frac{\pi}{2}$).

This alone gives good indication that the curve is the following



To make sure that this is all there is observe that as t varies from $-\pi$ to $-\frac{\pi}{2}$,

$x = \cos t$ starts at -1 and decreases to 0

$y = \cos 2t$ starts at 1, decreases to 0 (when $t = -\frac{3\pi}{4}$) and decreases further to -1 (when $t = -\frac{\pi}{2}$).

$\boxed{E}$

10.3

If a curve is given by the parametric equations $x = f(t)$, $y = g(t)$, then its slope (first derivative) is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Its second derivative is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

Horizontal tangents occur when

$$\frac{dy}{dt} = 0 \quad \underline{\underline{\text{and}}} \quad \frac{dx}{dt} \neq 0$$

Vertical tangents occur when

$$\frac{dx}{dt} = 0 \quad \underline{\underline{\text{and}}} \quad \frac{dy}{dt} \neq 0$$

Example 1

Find all points (if any) of horizontal and vertical tangency to the curve.

$$x = 2t^3 - 3t^2 - 36t + 1 \qquad y = t^4 - 8t^2 - 5$$

Soln :

$$\frac{dy}{dt} = 4t^3 - 16t = 4t(t^2 - 4) = 4t(t+2)(t-2)$$

$$\therefore \quad \frac{dy}{dt} = 0 \quad \text{when} \quad t = 0, -2, 2$$

$$\frac{dx}{dt} = 6t^2 - 6t - 36 = 6(t^2 - t - 6) = 6(t-3)(t+2)$$

$$\therefore \quad \frac{dx}{dt} = 0 \quad \text{when} \quad t = 3, -2$$

Hence horizontal tangents occur when

$t = 0$ at the point $(1, -5)$

$t = 2$ at the point $(-67, -21)$

Vertical tangents occur when

$t = 3$ at the point $(-80, 4)$.

Why do we not have horizontal or vertical tangents when $t = -2$?

Example 2

Find the slope and concavity of the curve at the indicated value of the parameter

$$x = \sqrt{3+t^2} \qquad y = 1 - t^2 \qquad t = 1$$

Soln

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2t}{\frac{1}{2}(3+t^2)^{-1/2}(2t)}$$

$$= -2(3+t^2)^{1/2}$$

When $t = 1$, the slope of the curve is -4.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[-2(3+t^2)^{1/2}\right]}{\frac{1}{2}(3+t^2)^{-1/2}(2t)}$$

$$= \frac{-(3+t^2)^{-1/2}(2t)}{\frac{1}{2}(3+t^2)^{-1/2}(2t)}$$

$$= -2$$

When $t = 1$, $\frac{d^2y}{dx^2} = -2$. The curve is concave down.

$\boxed{=}$

Arc Length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The difficulty in these problems is that the integrand involves a square root. If you're lucky, after some algebraic manipulation, you will end up with a square under the radical $\left(\sqrt{(\ \ )^2}\right)$. In this case, the radical will go away. In other cases you may need to do a u-substitution. If you're really unlucky, you may need to do a trig substitution.

Example 3

Find the arc length of the curve described by the parametric equations

$$x = (4\sqrt{2})\, t^{1/2} \qquad y = t - \ln(t^2) \qquad 1 \le t \le 3.$$

Soln

$$\frac{dx}{dt} = 2\sqrt{2}\, t^{-1/2} = \frac{2\sqrt{2}}{t^{1/2}}$$

$$\frac{dy}{dt} = 1 - \frac{1}{t^2}(2t) = 1 - \frac{2}{t}$$

$$L = \int_1^3 \sqrt{\left(\frac{2\sqrt{2}}{t^{1/2}}\right)^2 + \left(1 - \frac{2}{t}\right)^2}\ dt$$

$$= \int_1^3 \sqrt{\frac{8}{t} + \left(1 - \frac{4}{t} + \frac{4}{t^2}\right)}\ dt$$

$$= \int_1^3 \sqrt{1 + \frac{4}{t} + \frac{4}{t^2}}\ dt$$

$$= \int_1^3 \sqrt{\left(1 + \frac{2}{t}\right)^2}\ dt$$

$$= \int_1^3 \left|1 + \frac{2}{t}\right|\ dt$$

$$= \int_1^3 \left(1 + \frac{2}{t}\right) dt \qquad \left(\begin{array}{l}\text{On } 1 \leq t \leq 3, \quad 1 + \frac{2}{t} \geq 0. \\ \text{so, } \left|1 + \frac{2}{t}\right| = 1 + \frac{2}{t}\end{array}\right)$$

You should know how to integrate this. $\boxed{\diagup}$

Example 4

$x = t^2 + 1 \qquad y = 4t^3 + 3 \qquad -1 \leq t \leq 0.$

Soln

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 12t^2$$

$$L = \int_{-1}^{0} \sqrt{(2t)^2 + (12t^2)^2}\ dt$$

$$= \int_{-1}^{0} \sqrt{4t^2 + 144t^4}\, dt$$

$$= \int_{-1}^{0} \sqrt{4t^2(1 + 36t^2)}\, dt$$

$$= \int_{-1}^{0} |2t| \sqrt{1 + 36t^2}\, dt$$

$$= \int_{-1}^{0} -(2t) \sqrt{1+36t^2}\, dt \qquad \left(\begin{array}{l} \text{on } -1 \leq t \leq 0, \ 2t \leq 0. \\ \text{so } |2t| = -(2t) \end{array} \right)$$

$$u = 1 + 36t^2$$

$$du = 72t\, dt$$

$$-\frac{2}{72}\, du = -2t\, dt$$

$$= \int_{37}^{1} -\frac{2}{72} u^{1/2}\, du$$

$$= \left[-\frac{2}{72} \frac{u^{3/2}}{3/2} \right]_{37}^{1} = \cdots$$

$\boxed{E}$

Other Problems

1. $x = t + 5 \qquad y = \frac{2}{3} t^{3/2} \qquad 1 \leq t \leq 3$

2. $x = 2t \qquad y = t^4 + \frac{1}{8t^2} \qquad 1 \leq t \leq 2$

3. $x = e^t + e^{-t} \qquad y = 5 - 2t \qquad 0 \leq t \leq 3$

4. $x = \cos\theta + \theta \sin\theta \qquad y = \sin\theta - \theta\cos\theta \qquad 0 \leq \theta \leq \pi$

5. $x = e^t - t \qquad y = 4e^{t/2} \qquad -8 \leq t \leq 3$

6. $x = \cos^3\theta \qquad y = \sin^3\theta \qquad 0 \leq \theta \leq 2\pi$