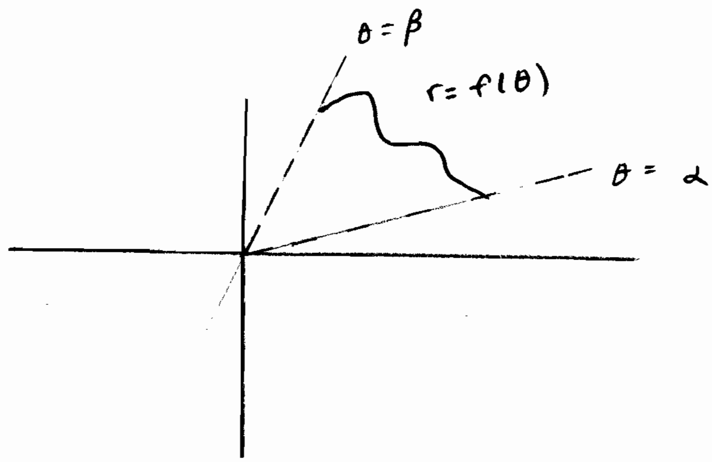
Polar Area & Arc Length



$$\text{Area} = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

$$\text{Arc Length} = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Useful Trig Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

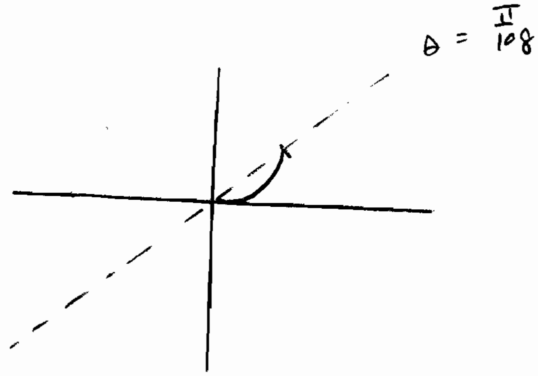
The difficulty in these problems is finding the correct limits of integration. Polar graphs are often traced out in non-obvious ways. Always sketch a picture & use as much of the symmetry as possible.

Example 1

Find the area enclosed by one loop (petal) of $r = \sin 54\theta$.

Soln: This is a rose with 108 petals.

When $\theta = 0$, $r = 0$. As θ increases $\sin 54\theta$ will increase until it is 1 at $\theta = \frac{\pi}{108}$. So, as θ increases from 0 to $\frac{\pi}{108}$, $r = \sin 54\theta$ increases from 0 to 1. This portion of the curve is shown



$$\text{Area of half a petal} = \int_0^{\frac{\pi}{108}} \frac{1}{2} (\sin 54\theta)^2 \, d\theta$$

Hence,

$$\text{Area of one petal} = 2\left[\int_0^{\frac{\pi}{108}} \frac{1}{2} (\sin 54\theta)^2 \, d\theta \right] \qquad (*)$$

Note. If half of a petal is traced out as θ ranges from 0 to $\frac{\pi}{108}$, then by symmetry, the other half will be traced out as θ ranges from $\frac{\pi}{108}$ to $2\frac{\pi}{108}$. Thus, we also have

$$\text{Area of one petal} = \int_0^{2\frac{\pi}{108}} \frac{1}{2} (\sin 54\theta)^2 \, d\theta$$

Let us compute using (*)

$$\text{Area of one petal} = \int_0^{\frac{\pi}{108}} (\sin 54\theta)^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{108}} \frac{1 - \cos(108\theta)}{2} \, d\theta \qquad \text{(Power Reduction)}$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{108}} (1 - \cos(108\theta)) \, d\theta$$

$$= \frac{1}{2} \left[\theta - \frac{\sin(108\theta)}{108} \right]_0^{\frac{\pi}{108}}$$

$$= \frac{1}{2} \left[\left(\frac{\pi}{108} - 0 \right) - (0 - 0) \right]$$

$$= \frac{\pi}{216}$$

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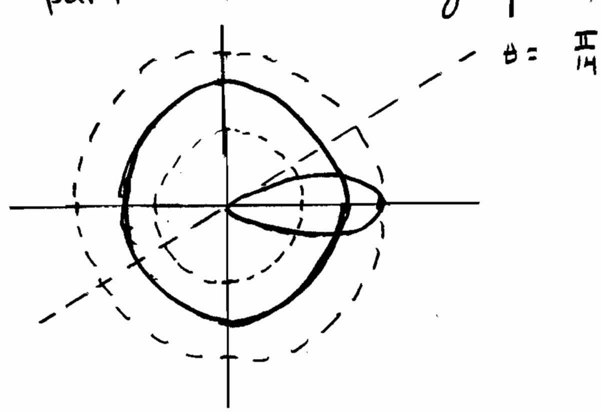
Example 2a

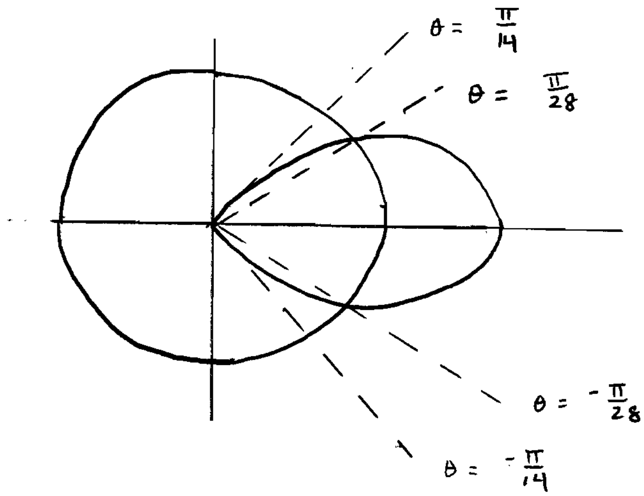
Find the area inside $r = 2\cos 7\theta$ and outside $r = \sqrt{2}$.

Soln : we sketch part of the graph.



There are six other petals which are not shown. The tips are $\frac{2\pi}{7}$ rad apart. The next tip occurs at $\theta = \frac{2\pi}{7}$ and the next one occurs at $\theta = \frac{4\pi}{7}$, etc.

Exaggerated drawing



To find the points of intersection, we solve

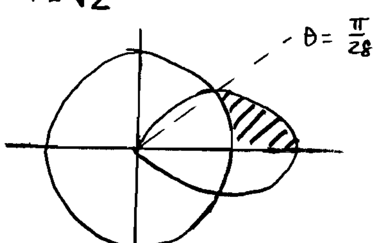
$$2\cos 7\theta = \sqrt{2}$$

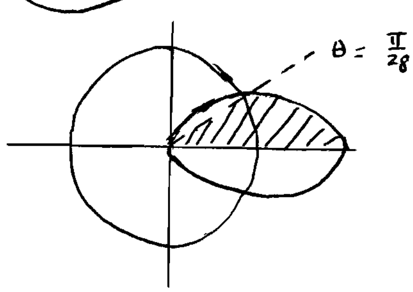
$$\cos 7\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

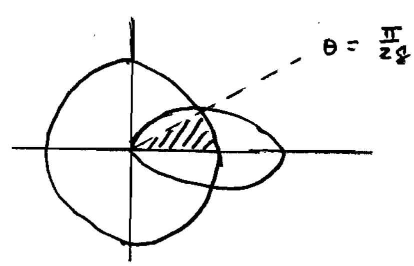
$$7\theta = \frac{\pi}{4},\ \frac{7\pi}{4},\ \ldots$$

$$\theta = \frac{\pi}{28},\ \frac{\pi}{4},\ \ldots$$

Area inside half petal and outside $r = \sqrt{2}$ $= \underbrace{\int_0^{\frac{\pi}{28}} \frac{1}{2}(2\cos 7\theta)^2\, d\theta}\ -\ \underbrace{\int_0^{\frac{\pi}{28}} \frac{1}{2}(\sqrt{2})^2\, d\theta} = (**)$





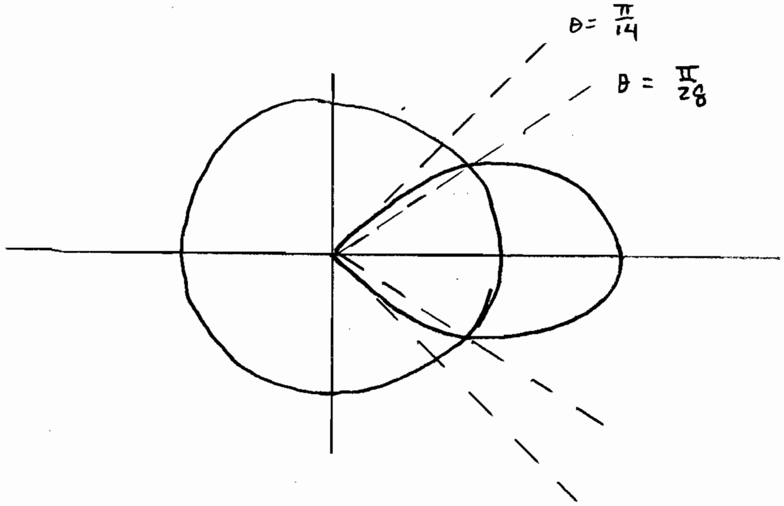


Area inside $r = 2\cos 7\theta$ and outside $r = \sqrt{2}$ $= 14\ (**)$

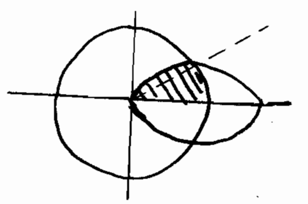
Example 2b

Find the area of the common interior of $r = \sqrt{2}$ and one loop of $r = 2\cos 7\theta$.

Soln: We have the same picture as in Example 2a.



Area of



$$= \underbrace{\int_0^{\frac{\pi}{28}} \frac{1}{2}(\sqrt{2})^2 \, d\theta}_{} + \underbrace{\int_{\frac{\pi}{28}}^{\frac{\pi}{14}} \frac{1}{2}(2\cos 7\theta)^2 \, d\theta}_{}$$

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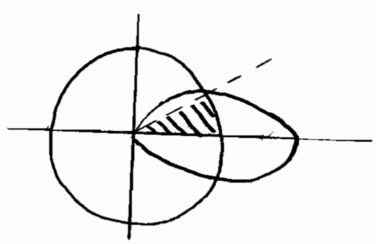
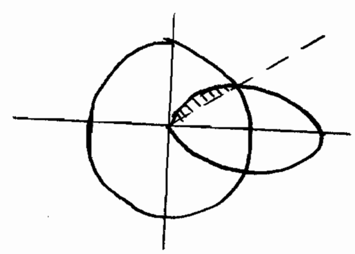




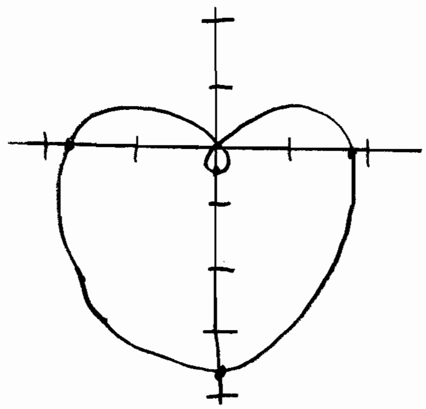
Thus, Area common to $r = \sqrt{2}$ and one petal of $r = 2\cos 7\theta$ $= 2 \cdot$ ☺

Example 3a

Find the area inside the inner loop of $r = \sqrt{3} - 2\sin\theta$.

Soln We begin by drawing a rough sketch.

θ	r
0	$\sqrt{3}$
$\frac{\pi}{2}$	$\sqrt{3}-2$
π	$\sqrt{3}$
$\frac{3\pi}{2}$	$\sqrt{3}+2$



The question now is: when is the inner loop traced out?

The curve starts at $(r, \theta) = (\sqrt{3}, 0)$. If you follow the way the curve is traced out as θ increases, you find that the curve will hit the origin, trace out the inner loop, hit the origin again, and then trace out the rest of the outer loop.

Hence, we solve

$$\sqrt{3} - 2\sin\theta = 0$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

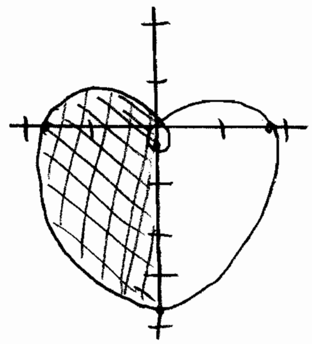
$$\theta = \frac{\pi}{3}, \quad \frac{2\pi}{3}$$

$$\text{Area of inner loop} = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{2}\left(\sqrt{3} - 2\sin\theta\right)^2 \, d\theta$$

Example 3b

Find the area of the region lying between the inner and outer loops of $r = \sqrt{3} - 2\sin\theta$.

Soln



Area of shaded region $= \displaystyle\int_{\frac{2\pi}{3}}^{\frac{3\pi}{2}} \frac{1}{2}\left(\sqrt{3} - 2\sin\theta\right)^2 d\theta$

Area inside outer loop $= 2 \cdot \displaystyle\int_{\frac{2\pi}{3}}^{\frac{3\pi}{2}} \frac{1}{2}\left(\sqrt{3} - 2\sin\theta\right)^2 d\theta$

Area between loops $= 2 \cdot \displaystyle\int_{\frac{2\pi}{3}}^{\frac{3\pi}{2}} \frac{1}{2}\left(\sqrt{3} - 2\sin\theta\right)^2 d\theta$ $-$ Area of inner loop (Example 3a)

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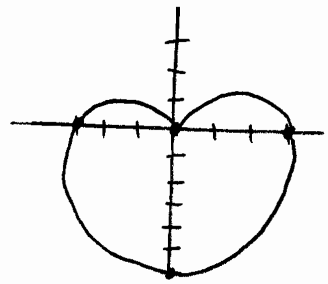
Example 4

Find the arc length of the cardiod $r = 3 - 3\sin\theta$.

Soln :

θ	r
0	3
$\frac{\pi}{2}$	0
π	3
$\frac{3\pi}{2}$	6



As θ ranges from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, the left half of the curve is traced out.

Arc length of cardiod $= 2\left[\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{(3-3\sin\theta)^2 + (-3\cos\theta)^2}\, d\theta\right]$

$$= 2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{(9 - 18\sin\theta + 9\sin^2\theta) + 9\cos^2\theta}\, d\theta$$

$$= 2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{9 - 18\sin\theta + 9}\, d\theta$$

$$= 2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{18(1-\sin\theta)}\, d\theta$$

$$= 2\sqrt{18}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{1-\sin\theta}\, d\theta$$

$$= 2\sqrt{18}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{1-\sin\theta}\ \frac{\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta}}\, d\theta$$

$$= 6\sqrt{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1+\sin\theta}}$$

$$= 6\sqrt{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{-\cos\theta}{\sqrt{1+\sin\theta}}\, d\theta \qquad \left(\begin{array}{l} \sqrt{\cos^2\theta} = |\cos\theta| \\ \text{on } \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \quad \cos\theta \leq 0, \\ \text{so } |\cos\theta| = -\cos\theta \end{array} \right)$$

$u = 1 + \sin\theta$

$du = \cos\theta\, d\theta$

$-du = -\cos\theta\, d\theta$

$$= 6\sqrt{2} \int_2^0 -u^{-1/2}\, du$$

$$= -6\sqrt{2} \left[\frac{u^{1/2}}{1/2} \right]_2^0$$

$$= -6\sqrt{2}\left[0 - 2\sqrt{2}\right]$$

$$= \underline{\underline{24}}$$

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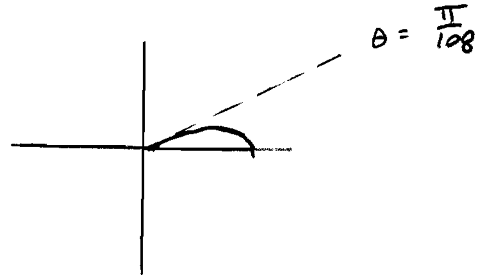
Example 5

Find the arc length of one loop of the rose

$r = \cos 54\theta$

Soln :

when $\theta = 0$ $r = 1$. As θ increases $\cos 54\theta$ decreases until it is 0 at $\theta = \frac{\pi}{108}$. This portion of the curve from $\theta = 0$ to $\theta = \frac{\pi}{108}$ is shown.



$$\text{Arc Length of one petal} = 2\left[\int_0^{\pi/108} \sqrt{(\cos 54\theta)^2 + (-54 \sin 54\theta)^2}\, d\theta\right]$$

I don't know of a nice way to evaluate this integral. In this case, you would probably only be asked to set up the integral. $\boxtimes$