A REMARK ON POWER SERIES

MATH 2019 – SPRING 2008

1. Introduction

Up until now, we have been considering series of numbers $\sum a_n$.

Example 1.1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

The whole point of developing series (from the point of view of this class) is to get to *power series.* A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad (\dagger)$$

Here, $a_0, a_1, a_2, \ldots$ are numbers, and x is a variable. More generally, we will study power series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots \qquad (\ddagger)$$

This is said to be a *power series centered at c.* Notice that the series in ($\dagger$) is a power series centered at 0.

Example 1.2.

$$\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n} = 1 + \frac{x+3}{2} + \frac{(x+3)^2}{4} + \frac{(x+3)^3}{8} + \cdots$$

is a power series centered at -3.

2. Radius of Convergence and Interval of Convergence

If we plug in $x = 0$, the series in Example 1.2 becomes $\sum \frac{3^n}{2^n}$ which we know is a divergent geometric series ($r = \frac{3}{2}$). If we plug in $x = -4$, the series becomes $\sum \frac{(-1)^n}{2^n}$ which we know converges ($r = -\frac{1}{2}$). So, for some values of x, the series will converge, and, for other values of

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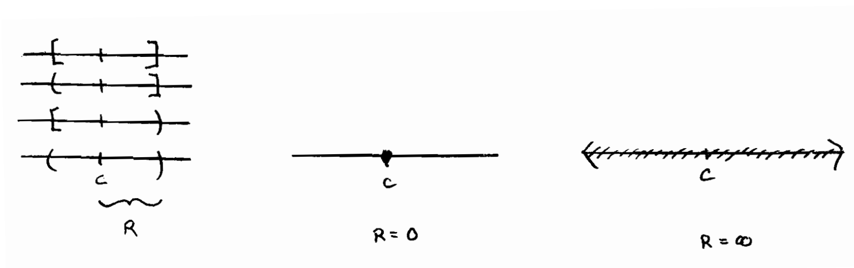
x, the series will diverge. Given a general power series (‡), the first question we will ask is

Question. For what values of x does the power series converge?

Notice that there is always an obvious value of x that makes the power series converge, namely the center ($x = 0$ in (†), $x = c$ in (‡), $x = -3$ in Example 1.2).

Fact. The set of values of x for which the power series converges will always be an interval of some radius R around the center c. We allow the cases $R = 0$ and $R = \infty$. The number R is called the *radius of convergence*. Pictorially, we have the following possibilities.



Example 2.1. Find the radius of convergence and the interval of convergence for the following power series

$$\sum_{k=0}^{\infty} \frac{(-1)^{k-1}\,(x-5)^k}{6^k\,\sqrt{k+4}}$$

Solution. This is a power series centered at 5. Our interval of convergence should be an interval of some radius R around 5. Let

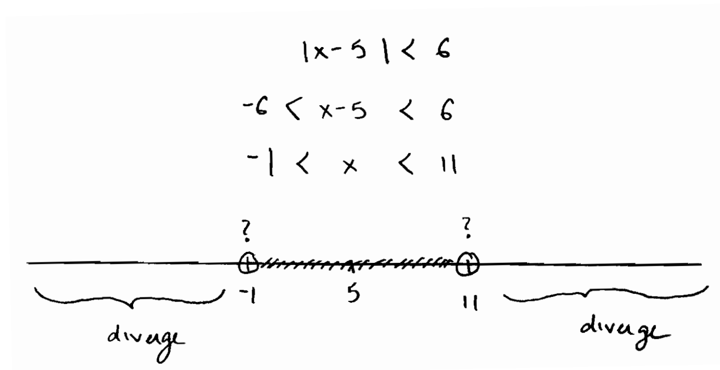
$$a_k = \frac{(-1)^{k-1}\,(x-5)^k}{6^k\,\sqrt{k+4}}$$

We'll apply the Ratio Test.

$$\begin{aligned}
\left|\frac{a_{k+1}}{a_k}\right| &= \left|\frac{(x-5)^{k+1}}{6^{k+1}\,\sqrt{(k+1)+4}} \cdot \frac{6^k\,\sqrt{k+4}}{(x-5)^k}\right| \\
&= \left|\frac{6^k}{6^k\,6} \cdot \frac{\sqrt{k+4}}{\sqrt{k+5}} \cdot \frac{(x-5)^k(x-5)}{(x-5)^k}\right| \\
&= \left|\frac{1}{6}\,\frac{\sqrt{k+4}}{\sqrt{k+5}}\,(x-5)\right| \\
&= \frac{1}{6}\,\frac{\sqrt{k+4}}{\sqrt{k+5}}\,|x-5|
\end{aligned}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{1}{6}\frac{\sqrt{k+4}}{\sqrt{k+5}}\,|x-5| = \frac{1}{6}\,|x-5|$$

By the Ratio Test, the series converges if $\frac{1}{6}|x-5| < 1$ (and diverges if $\frac{1}{6}|x-5| > 1$). Equivalently, the series converges if $|x-5| < 6$ (and diverges if $|x-5| > 6$). This tells us that the *radius of convergence* is $R = 6$.



To find the *interval of convergence*, it remains to check the endpoints separately (these correspond to the case $\frac{1}{6}|x-5| = 1$ for which the Ratio Test gives no information).

For $x = -1$, the power series becomes

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^{k-1}(-6)^k}{6^k\sqrt{k+4}} &= \sum_{k=0}^{\infty}(-1)^{k-1}(-1)^k\frac{1}{\sqrt{k+4}}\\ &= \sum_{k=0}^{\infty}(-1)^{2k-1}\frac{1}{\sqrt{k+4}}\\ &= \sum_{k=0}^{\infty}\left((-1)^2\right)^k(-1)^{-1}\frac{1}{\sqrt{k+4}}\\ &= -\underbrace{\sum_{k=0}^{\infty}\frac{1}{\sqrt{k+4}}}_{(*)}\end{aligned}$$

We'll apply LCT to $(*)$. We compare with $\sum\frac{1}{k^{1/2}}$.

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{\sqrt{k+4}}\cdot\frac{k^{1/2}}{1} &= \lim_{k\to\infty}\frac{k^{1/2}}{\sqrt{k+4}}\\ &= 1 \quad \text{(finite and positive)}\end{aligned}$$

$\sum \frac{1}{k^{1/2}}$ is a divergent p-series. So $(*)$ also diverges by LCT.

For $x = 11$, the power series becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^{k-1}\, 6^k}{6^k \sqrt{k+4}} = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{1}{\sqrt{k+4}} \qquad (**)$$

This is an alternating series. Clearly

(1) $\lim_{k\to\infty} \frac{1}{\sqrt{k+4}} = 0$

(2) $\frac{1}{\sqrt{k+4}}$ is decreasing

It follows that $(**)$ converges by AST.

Our final answer is: the *radius of convergence* is $R = 6$, and the *interval of convergence* is $(-1, 11]$.

$\square$

3. Power Series as Functions

Why do we care about the interval of convergence? We can think of a power series as a function of x.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-c)^n \\ &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots \qquad (⧉) \end{aligned}$$

This function is, of course, only defined at values of x for which the series converges, i.e. the domain is the interval of convergence of the power series.

We would like to differentiate and integrate f. Notice that f looks like an "infinite" polynomial, especially as written in (⧉). If there is any justice in the world, we should be able to differentiate and integrate f term-by-term as if it were a polynomial. Indeed, we can.

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-2)^2 + \cdots$$
$$= \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

$$\int f(x)\,dx = C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$$
$$= C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$$

Fact. If $f(x)$ has radius of convergence R, $f'(x)$ and $\int f(x)\,dx$ also have radius of convergence R.

Careful. Although $f(x)$, $f'(x)$, $\int f(x)\,dx$ all have the same radius of convergence, their *interval of convergence* may differ. In other words, behavior at the endpoints may differ.

4. Representing Functions by Power Series

Now, we go in the reverse direction. Given a function $f(x)$, we would like to express it as a power series. One reason for this is that if f has a power series representation, then, f is essentially a polynomial, albeit an "infinite" polynomial. We saw in the previous section that these "infinite" polynomials were ridiculously easy to differentiate and integrate.

We begin by combining something old with something new.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Notice that this is our familiar geometric series (compare $\sum ar^n$). We know that this series converges if and only if $|x| < 1$, and, moreover, it converges to $\frac{1}{1-x}$. In the language of power series, this is a power series centered at 0. The radius of convergence is $R = 1$, and the interval of convergence is $(-1, 1)$. To summarize, we have

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1} \qquad (\triangle)$$

We dub this the geometric power series. In words, what we have done is we have expressed the function $f(x) = \frac{1}{1-x}$ as a power series centered at 0. This representation is valid for $|x| < 1$.

We can obtain power series expansions for other functions by using this geometric power series combined with differentiation and integration.

Example 4.1. Find a power series centered at $c = -1$ for the function $f(x) = \dfrac{2}{3x+9}$ and identify the interval of convergence.

Solution. We are looking for a power series centered at $c = -1$. So, we need a power series of the form $\sum a_n(x+1)^n$. The idea is to use the geometric series in ($\triangle$). But, we have to somehow "force" $(x+1)$ into ($\triangle$). To do this, we will manipulate f until we get something that looks like $\frac{1}{1-(K(x+1))}$. Then, we can just replace x with $K(x+1)$ in ($\triangle$).

$$\begin{aligned}
\frac{2}{3x+9} &= \frac{2}{3(x+1-1)+9} \\
&= \frac{2}{3(x+1)+6} \\
&= \frac{2}{6+3(x+1)} \\
&= \frac{2}{6\left(1+\frac{1}{2}(x+1)\right)} \\
&= \frac{1}{3}\,\frac{1}{1-\left(-\frac{1}{2}(x+1)\right)}
\end{aligned}$$

We can use ($\triangle$) now to get

$$\begin{aligned}
f(x) &= \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{1}{2}(x+1)\right)^n && (\diamondsuit) \\
&= \sum_{n=0}^{\infty}\frac{1}{3}\left(-\frac{1}{2}\right)^n (x+1)^n
\end{aligned}$$

To find the radius of convergence and the interval of convergence, it is not necessary to use the Ratio Test and check the endpoints like in Example 2.1. We know the original geometric series converges if and only if $|x| < 1$. We obtained ($\diamondsuit$) by replacing x with $-\frac{1}{2}(x+1)$. It follows that our series converges if and only if

$$\begin{aligned} \left| -\frac{1}{2}(x+1) \right| &< 1 \\ \frac{1}{2}|x+1| &< 1 \\ |x+1| &< 2 \end{aligned}$$

This tells us that the radius of convergence is $R = 2$.

$$\begin{array}{ccccc} & & |x+1| & < & 2 \\ -2 & < & x+1 & < & 2 \\ -3 & < & x & < & 1 \end{array}$$

The interval of convergence is $(-3, 1)$.

□

Example 4.2. Find a power series centered at $c = -1$ for the function $g(x) = \ln(3x + 9)$ and identify the interval of convergence.

Solution. Observe that $\ln(3x + 9)$ is related to $\frac{2}{3x+9}$ in Example 4.1. Indeed,

$$\int \frac{2}{3x+9}\,dx = \frac{2}{3}\ln(3x+9) \qquad (u = 3x + 9)$$

We know from Example 4.1 that

$$\frac{2}{3x+9} = \sum_{n=0}^{\infty} \underbrace{\frac{1}{3}\left(-\frac{1}{2}\right)^n}_{a_n} (x+1)^n \qquad (❶)$$

$$= a_0 + a_1(x+1) + a_2(x+1)^2 + \cdots \qquad (❷)$$

Integrating term-by-term yields

$$\begin{aligned} \frac{2}{3}\ln(3x+9) &= \int \frac{2}{3x+9}\,dx \\ &= C + a_0(x+1) + a_1\frac{(x+1)^2}{2} + a_2\frac{(x+1)^3}{3} + \cdots \qquad (\spadesuit) \\ &= C + \sum_{n=0}^{\infty} a_n \frac{(x+1)^{n+1}}{n+1} \\ &= C + \sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{1}{2}\right)^n \frac{(x+1)^{n+1}}{n+1} \end{aligned}$$

To recover the constant C, we can plug in $x = -1$ into (♠)

$$\frac{2}{3}\ln(3(-1)+9) = C + 0 + 0 + 0 + \cdots$$

Thus,

$$C = \frac{2}{3}\ln 6$$

This gives us

$$\frac{2}{3}\ln(3x+9) = \frac{2}{3}\ln 6 + \underbrace{\sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{1}{2}\right)^n \frac{(x+1)^{n+1}}{n+1}}_{(\#)} \qquad (✿)$$

We obtained this series by integrating the series in Example 4.1. So, the radius of convergence will stay the same, $R = 2$. However, the *interval of convergence* may differ. We need to check the endpoints $x = -3$ and $x = 1$. We can just deal with (#).

For $x = -3$, (#) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{1}{2}\right)^n \frac{(-2)^{n+1}}{n+1} &= \sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{1}{2}\right)^n \frac{(-2)^n\,(-2)}{n+1} \\ &= \sum_{n=0}^{\infty} -\frac{2}{3}\,(1)^n\,\frac{1}{n+1} \\ &= -\frac{2}{3}\underbrace{\sum_{n=0}^{\infty}\frac{1}{n+1}}_{(♣)}\end{aligned}$$

We can use the Limit Comparison Test on (♣). We compare with $\sum \frac{1}{n}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n+1}\cdot\frac{n}{1} &= \lim_{n\to\infty}\frac{n}{n+1} \\ &= 1 \quad \text{(finite and positive)}\end{aligned}$$

$\sum \frac{1}{n}$ is a divergent p-series ($p = 1$). So, (♣) diverges by LCT.

For $x = 1$, (#) becomes

$$\sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{1}{2}\right)^n \frac{2^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{1}{2}\right)^n \frac{2^n\, 2}{n+1}$$
$$= \sum_{n=0}^{\infty} \frac{2}{3}(-1)^n \frac{1}{n+1}$$
$$= \frac{2}{3}\underbrace{\sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}}_{(\blacklozenge)}$$

($\blacklozenge$) is an alternating series. It is clear that

(1) $\lim_{n\to\infty} \frac{1}{n+1} = 0$

(2) $\frac{1}{n+1}$ is decreasing

So, ($\blacklozenge$) converges by AST.

Our final answer (after multiplying (✿) through by $\frac{3}{2}$) is

$$g(x) = \ln(3x+9) = \ln 6 + \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{1}{2}\right)^n \frac{(x+1)^{n+1}}{n+1}$$

and the interval of convergence is $(-3, 1]$.

□

Example 4.3. Find a power series centered at $c = -1$ for the function $h(x) = \frac{1}{(3x+9)^2}$ and state the radius of convergence.

Solution. Observe that

$$f'(x) = \frac{-6}{(3x+9)^2}$$

We know from Example 4.1 that

$$\frac{2}{3x+9} = \sum_{n=0}^{\infty} \underbrace{\frac{1}{3}\left(-\frac{1}{2}\right)^n}_{a_n}(x+1)^n \qquad (❶)$$

$$= a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3 + \cdots \qquad (❷)$$

Differentiating term-by-term yields

$$\begin{aligned}\frac{-6}{(3x+9)^2} &= \frac{d}{dx}\left(\frac{2}{3x+9}\right)\\ &= 0 + a_1 + 2a_2(x+1) + 3a_3(x+1)^2 + \cdots\\ &= \sum_{n=1}^{\infty} na_n(x+1)^{n-1}\\ &= \sum_{n=1}^{\infty} n\,\frac{1}{3}\left(-\frac{1}{2}\right)^n (x+1)^{n-1} \qquad (❄)\end{aligned}$$

Dividing through by -6, we get the final answer

$$h(x) = \frac{1}{(3x+9)^2} = \sum_{n=1}^{\infty} -n\,\frac{1}{18}\left(-\frac{1}{2}\right)^n (x+1)^{n-1}$$

The radius of convergence remains the same, $R = 2$.

□

Remark. When differentiating and integrating power series, it is often helpful to work with both the summation notation (❶) and the "writing out the first few terms" notation (❷). Notice that the series in (❄) starts at $n = 1$. You may miss this if you worked exclusively with (❶).

5. Taylor and Maclaurin Series

We, now, discuss a more general method for finding power series expansions of functions.

Definition. The *n^{th}-degree Taylor polynomial for f at c* is

$$P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

The special case $c = 0$ is called the *n^{th}-degree Maclaurin polynomial for f.*

For most reasonable functions, $P_n(x)$ will be a good approximation to f near the point c. Indeed, observe that $P_n(c) = f(c)$. For most reasonable functions, we can obtain a power series expansion of f centered at c by extending the Taylor polynomial indefinitely.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots \end{aligned}$$

The series above is called the *Taylor series for f at c (or centered at c or about c)*. The special case where $c = 0$ is called the *Maclaurin series for f*.

Notice that computing the power series representation for f using this method requires that we know *all* of the derivatives of f at c. This is, of course, impractical, unless the derivatives follow some pattern. In lecture and in your book, the Maclaurin series for $\sin x$, $\cos x$, and e^x were computed in this way, since the derivatives of $\sin x$, $\cos x$, and e^x at 0 followed a nice pattern.