A NOTE ON SERIES

MATH 2019 – SPRING 2008

1. Introduction

1.1. **Sequences.** A sequence is a list of numbers written in a definite order

$$a_1, a_2, a_3, a_4, \ldots$$

Example 1.1. $a_n = \frac{1}{2^n}, \quad n = 1, 2, 3, \ldots$ is the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

1.2. **Series.** Given a sequence $\{a_n\}$, we would like to add up the terms, i.e. we would like to give meaning to the expression

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Example 1.2. What should the expression

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

mean?

How are we going to add an infinite number of terms? Well, we just try. We add the first term, we add the first two terms, we add the first three terms, the first four terms, etc., and see what happens. More precisely, we form the partial sums

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$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &\vdots \end{aligned} \tag{$\dagger$}$$

and, then, we look and see if these partial sums $\{S_n\}$ approach anything.

Definition 1.3 (Convergence/Divergence of a Series)**.**

(i) If $\lim_{n\to\infty} S_n = S$ exists as a finite number, we say the series $\sum a_n$ is *convergent* and write $\sum a_n = S$ to indicate that S is the *sum* of the series.
(ii) If $\lim_{n\to\infty} S_n$ does not exist, we say the series $\sum a_n$ is *divergent.*

Remark. There are two sequences associated to a series $\sum_{n=1}^{\infty} a_n$.

(1) the sequence of terms $\{a_n\}$
(2) the sequence of partial sums $\{S_n\}$ as defined in ($\dagger$)

By definition, convergence/divergence of the series $\sum a_n$ is convergence/divergence of the sequence of partial sums S_n. So, in principle, to determine whether a series converges or diverges, we should form the partial sums S_n and see whether or not $\lim_{n\to\infty} S_n$ exists. However, in practice, calculating $\lim_{n\to\infty} S_n$ requires that we have a "nice" expression for S_n. Finding such an expression for S_n is usually difficult.

Execeptions.

(1) Telescoping series – finding a "nice" expression for the partial sums S_n is easy.
(2) Geometric series – finding a "nice" expression for the partial sums S_n is kind of easy.

You will not be responsible for the partial sums of a geometric series. So, except for telescoping series, we will not compute $\lim_{n\to\infty} S_n$.

Whatever shall we do then? There will be tests that will allow us to conclude convergence/divergence of the series $\sum a_n$ from the behavior of $\{a_n\}$ the sequence of terms. This allows us to avoid the sequence of partial sums $\{S_n\}$.

In the examples that follow, unless otherwise stated, the main problem we are concerned with is

Main Problem. Determine whether the given series converges or diverges. If the series converges, find its sum (when possible).

2. Special Series

In this section, we discuss three special series. Telescoping series are nice examples of series where the Main Problem can by answered by directly applying the definition of convergence/divergence. Geometric series are arguably the most important series in all of mathematics, followed closely by the p-series.

2.1. **Telescoping Series.** As remarked earlier, these are the only series for which we will deal directly with the sequence of partial sums S_n.

Example 2.1. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2k+3}-\frac{1}{2k+7}\right)$

Solution. We'll write out the partial sums until we see a pattern

$$S_1 = \left(\frac{1}{5}-\frac{1}{9}\right) \qquad {\scriptstyle k=2}$$

$$S_2 = \left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{7}-\frac{1}{11}\right) \qquad {\scriptstyle k=3}$$

$$S_3 = \left(\frac{1}{5}-\cancel{\frac{1}{9}}\right)+\left(\frac{1}{7}-\frac{1}{11}\right)+\left(\cancel{\frac{1}{9}}-\frac{1}{13}\right) \qquad {\scriptstyle k=4}$$

$$S_4 = \left(\frac{1}{5}-\cancel{\frac{1}{9}}\right)+\left(\frac{1}{7}-\cancel{\frac{1}{11}}\right)+\left(\cancel{\frac{1}{9}}-\frac{1}{13}\right)+\left(\cancel{\frac{1}{11}}-\frac{1}{15}\right) \qquad {\scriptstyle k=5}$$

$$S_5 = \left(\frac{1}{5}-\cancel{\frac{1}{9}}\right)+\left(\frac{1}{7}-\cancel{\frac{1}{11}}\right)+\left(\cancel{\frac{1}{9}}-\cancel{\frac{1}{13}}\right)+\left(\cancel{\frac{1}{11}}-\frac{1}{15}\right)+\left(\cancel{\frac{1}{13}}-\frac{1}{17}\right)$$

$$\vdots$$

$$S_k = \frac{1}{5}+\frac{1}{7}-\frac{1}{2(k-1)+7}-\frac{1}{2k+7}$$

$$\vdots$$

We see that

$$\lim_{k\to\infty} S_k = \frac{1}{5} + \frac{1}{7} = \frac{12}{35}$$

Thus, the series converges and the sum is $\frac{12}{35}$. In short,

$$\sum_{k=1}^{\infty} \left(\frac{1}{2k+3} - \frac{1}{2k+7} \right) = \frac{12}{35}$$

$\square$

How do we recognize that a series is telescoping? Look for two similar looking terms joined by a subtraction sign.

Example 2.2. $\displaystyle\sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k+1} \right)$ is a telescoping series.

Example 2.3. $\displaystyle\sum_{k=2}^{\infty} \ln \left(\frac{k}{k+1} \right)$ is a telescoping series.
To see this, we must first notice that

$$\sum_{k=2}^{\infty} \ln \left(\frac{k}{k+1} \right) = \sum_{k=2}^{\infty} (\ln(k) - \ln(k+1))$$

We see similar looking terms joined by a subtraction sign. This is probably telescoping.

Notice that for telescoping series, finding a "nice" expression for the partial sum S_n was easy, because except for some terms at the beginning and some terms at the end, everything else cancels. So, if things don't cancel, its probably not a telescoping series.

2.2. **Geometric Series.**

A geometric series is a series of the form

$$\sum_{n=0}^{\infty} ar^n \quad = \quad a \underset{\text{mult } r}{\underset{\curvearrowright}{+}} ar \underset{r}{\underset{\curvearrowright}{+}} ar^2 \underset{r}{\underset{\curvearrowright}{+}} ar^3 \ + \ \cdots$$

Notice that the next term of the series is obtained by multiplying the previous term by the *common ratio* r. Equivalently, observe that the ratio of consecutive terms is r (hence the name *common ratio*).

Example 2.4. $\sum_{n=0}^{\infty} \frac{1}{\pi^n}$ is a geometric series. Indeed,

$$\sum_{n=0}^{\infty} \frac{1}{\pi^n} = \sum_{n=0}^{\infty} \left(\frac{1}{\pi}\right)^n$$

and the common ratio is $\frac{1}{\pi}$.

Theorem (Geometric Series)**.** *For the geometric series* $\sum ar^n$

(i) *If* $|r| \geq 1$*, the series diverges.*
(ii) *If* $|r| < 1$*, the series converges, and the sum is*

$$\sum ar^n = \frac{(\textit{first term})}{1 - (\textit{common ratio})}$$

Example 2.5. $\sum_{k=1}^{\infty} \frac{(-2)^{3k-1}}{(-3)^{2k+1}}$

Solution.

- Quick and dirty

$$\sum_{k=1}^{\infty} \frac{(-2)^{3k-1}}{(-3)^{2k+1}} = \overset{k=1}{\frac{(-2)^2}{(-3)^3}} \underset{\overset{\curvearrowright}{\text{mult } \frac{(-2)^3}{(-3)^2}}}{+} \overset{k=2}{\frac{(-2)^5}{(-3)^5}} \underset{\overset{\curvearrowright}{\frac{(-2)^3}{(-3)^2}}}{+} \overset{k=3}{\frac{(-2)^8}{(-3)^7}} + \cdots$$

This looks like a geometric series with common ratio $-\frac{8}{9}$ and first term $-\frac{4}{27}$.

- Proper way

We do some algebraic manipulation.

$$\frac{(-2)^{3k-1}}{(-3)^{2k+1}} = \frac{(-2)^{3k}(-2)^{-1}}{(-3)^{2k}(-3)} = \frac{1}{6}\frac{((-2)^3)^k}{((-3)^2)^k} = \frac{1}{6}\left(-\frac{8}{9}\right)^k$$

So,

$$\sum_{k=1}^{\infty} \frac{(-2)^{3k-1}}{(-3)^{2k+1}} = \sum_{k=1}^{\infty} \frac{1}{6}\left(-\frac{8}{9}\right)^k$$

This is a geometric series with common ratio $-\frac{8}{9}$. Since $\left|-\frac{8}{9}\right| < 1$, the series converges and the sum is

$$\frac{\text{(first term)}}{1-\text{(common ratio)}} = \frac{-\frac{8}{54}}{1-\left(-\frac{8}{9}\right)} = \frac{-\frac{8}{54}}{\frac{17}{9}} = -\frac{4}{51}$$

□

Example 2.6. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{5^{2k+1}}{3^{3k+3}} \qquad (\triangle)$

Solution.

- Quick and dirty

$$\begin{tikzcd}
(\triangle) \; = & \overset{k=1}{\frac{5^3}{3^6}} \arrow[rr, bend right, "-\frac{5^2}{3^3}"'] & + & \overset{k=2}{-\frac{5^5}{3^9}} \arrow[rr, bend right, "-\frac{5^2}{3^3}"'] & + & \overset{k=3}{\frac{5^7}{3^{12}}} \arrow[rr, bend right, "-\frac{5^2}{3^3}"'] & + & \overset{k=4}{-\frac{5^9}{3^{15}}} & + \; \cdots
\end{tikzcd}$$

  This looks like a geometric series with common ratio $-\frac{25}{27}$ and first term $\frac{5^3}{3^6}$.

- Proper way

  We do some algebraic manipulation.

$$(-1)^{k+1}\,\frac{5^{2k+1}}{3^{3k+3}} = (-1)^k(-1)\,\frac{5^{2k}\,5}{3^{3k}\,3^3} = -\frac{5}{27}\,\frac{(-1)^k(5^2)^k}{(3^3)^k} = -\frac{5}{27}\left(-\frac{25}{27}\right)^k$$

So,

$$\sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{5^{2k+1}}{3^{3k+3}} = \sum_{k=1}^{\infty}-\frac{5}{27}\left(-\frac{25}{27}\right)^k$$

This is a geometric series with common ratio $-\frac{25}{27}$. Since $\left|-\frac{25}{27}\right| < 1$, the series converges and the sum is

$$\frac{\text{(first term)}}{1-\text{(common ratio)}} = \frac{\frac{125}{729}}{1-\left(-\frac{25}{27}\right)}$$

□

Example 2.7. $\displaystyle\sum_{n=0}^{\infty}\frac{(-5)^{n+2}}{2^{2n+1}}$

Solution.
$$\frac{(-5)^{n+2}}{2^{2n+1}} = \frac{(-5)^n(-5)^2}{(2^2)^n\,2} = \frac{25}{2}\left(-\frac{5}{4}\right)^n$$

So,

$$\sum_{n=0}^{\infty} \frac{(-5)^{n+2}}{2^{2n+1}} = \sum_{n=0}^{\infty} \frac{25}{2}\left(-\frac{5}{4}\right)^n$$

This is a geometric series with common ratio $-\frac{5}{4}$. Since $\left|-\frac{5}{4}\right| > 1$, the series diverges.

$\square$

Remark. You need to do these problems the proper way. You can use the quick and dirty way to easily make sure that you have the correct first term and common ratio.

Remark. If the terms of the series have stuff like $(\text{constant})^{cn+d}$ (and nothing else) in the numerator and denominator, then the series is probably a geometric series.

Remark. Telescoping series and geometric series are the only series for which we can easily find the actual sum (assuming the series converges). If a question asks for the sum of the series, you are probably dealing with a telescoping series or a geometric series.

2.3. ***p*-series.**

A p-series is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

Theorem (p-series). *The p-series* $\sum \frac{1}{n^p}$ *behaves as follows:*

(i) *If* $p > 1$*, the series converges.*
(ii) *If* $p \leq 1$*, the series diverges.*

Example 2.8. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ is a p-series with $p = \pi > 1$. So, the series converges.

Example 2.9. $\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=5}^{\infty} \frac{1}{n^{1/2}}$ $\qquad p = \frac{1}{2} \leq 1 \Rightarrow$ series diverges

Example 2.10. $\sum_{n=1}^{\infty} n^{-3/2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $\qquad p = \frac{3}{2} > 1 \Rightarrow$ series converges

3. Divergence Test

Consider a series $\sum a_n$.

Theorem (Divergence Test)**.** *If* $\lim_{n\to\infty} a_n \neq 0$*, then* $\sum a_n$ *diverges.*

Careful. If $\lim_{n\to\infty} a_n = 0$, the test gives no information.

Remark. The Divergence Test is the easiest and quickest test to apply. So, it should be the first test that you try. It should take no more than 30 seconds. If it takes any longer than that, you should probably move on to another test. For instance, trying the Divergence Test in Examples 2.6 and 8.1 is not so easy. But, in Example 2.6, the terms have stuff like $(\text{constant})^{cn+d}$ (and nothing else) in the numerator and denominator. Geometric series. And, after seeing the factorial in Example 8.1, you should already be reaching for the Ratio Test.

Example 3.1. $\displaystyle\sum_{n=2}^{\infty} \ln\left(\frac{3n^2-1}{2+2n^2}\right)$

Solution. Observe that

$$\begin{aligned}\lim_{n\to\infty} \ln\left(\frac{3n^2-1}{2+2n^2}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) &= \lim_{n\to\infty} \ln\left(\frac{3-\frac{1}{n^2}}{\frac{2}{n^2}+2}\right)\\ &= \ln\frac{3}{2} \neq 0\end{aligned}$$

By the Divergence Test, the series diverges.

□

Example 3.2. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+1}}$

Solution. You may be tempted to do the Integral Test here, but this over and done with by the Divergence Test. Observe that

$$\begin{aligned}\lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}}\cdot\frac{\frac{1}{k}}{\frac{1}{k}} &= \lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}}\cdot\frac{\frac{1}{k}}{\frac{1}{\sqrt{k^2}}}\\ &= \lim_{k\to\infty} \frac{1}{\sqrt{1+\frac{1}{k^2}}} = 1 \neq 0\end{aligned}$$

The series diverges by the Divergence Test.

□

All of the following series diverge by the Divergence Test.

Example 3.3. $\displaystyle\sum_{k=1}^{\infty} \cos\frac{1}{k}$

Example 3.4. $\displaystyle\sum_{k=1}^{\infty} \arctan k$

Example 3.5. $\displaystyle\sum_{n=1}^{\infty} \frac{7n^5+8n^3+10}{9n^5+6n+1}$

Example 3.6. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{9n^5-6n^3+10}{7n^5+8n+1}$

4. Integral Test

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a positive term series.

Let $f(x)$ be the function associated to a_n (i.e. $f(x)$ is the function that we get when we replace n by x in the formula for a_n).

If

1. f is continuous for $x \geq 1$
2. f is positive for $x \geq 1$
3. f is decreasing for $x \geq 1$

Then, $\displaystyle\int_1^{\infty} f(x)\,dx$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ behave the same, i.e.

if the integral converges, the series converges.
if the integral diverges, the series diverges.

Example 4.1. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Solution. Consider $f(x) = x^2 e^{-x^3}$.

1. f is clearly continuous for $x \geq 1$
2. f is clearly positive for $x \geq 1$

(3) Is f decreasing?

$$\begin{aligned} f'(x) &= 2xe^{-x^3} + x^2e^{-x^3}(-3x^2) \\ &= \frac{2x-3x^4}{e^{x^3}} \\ &= \frac{x(2-3x^3)}{e^{x^3}} \end{aligned}$$

$f'(x) < 0$ if $2 - 3x^3 < 0$ or equivalently if $x^3 > \frac{2}{3}$. Surely, this condition is satisfied for $x \geq 1$. So, f is decreasing for $x \geq 1$.

The conditions of the Integral Test are verified.

$$\begin{aligned} \int_1^\infty x^2e^{-x^3}\,dx &= \lim_{b\to\infty}\int_1^b x^2e^{-x^3}\,dx \\ &= \lim_{b\to\infty}\left[-\frac{1}{3}e^{-x^3}\right]_1^b \qquad (u=-x^3) \\ &= \lim_{b\to\infty}\left[-\frac{1}{3}e^{-b^3}+\frac{1}{3}e^{-1}\right] \\ &= \lim_{b\to\infty}\left[-\frac{1}{3e^{b^3}}+\frac{1}{3e}\right] \\ &= \frac{1}{3e} \end{aligned}$$

The integral converges. So, the series converges by the Integral Test. □

Remark. When should you use the Integral Test? Look at the associated function $f(x)$. Can you integrate $f(x)$ (easily)? If yes, the Integral Test might be useful. If no, there's no point in trying the Integral Test.

Remark. If you use the Integral Test, you need to verify the 3 conditions.

5. Comparison Tests

Like the Integral Test, these tests are for positive term series.

5.1. **Direct Comparison Test.**

Roughly speaking, the Direct Comparison Test says the following:

Suppose you have a series $\sum a_n$ (positive term)

(1) If there is a series $\sum b_n$ <u>above</u> your series ($a_n \leq b_n$) that <u>converges</u>, then the bigger convergent series will "push your series down" forcing it to also converge.
(2) If there is a series $\sum c_n$ <u>below</u> your series ($c_n \leq a_n$) that <u>diverges</u>, then the smaller divergent series will "push your series up" so that it also diverges.

Example 5.1. $\displaystyle\sum_{n=1}^{\infty} \frac{3+\cos n}{3^n}$

Solution. Observe that the numerator is bounded

$$2 \leq 3 + \cos n \leq 4$$

(since $-1 \leq \cos n \leq 1$).

So, that is under control. What is left looks like $\frac{1}{3^n}$ which is geometric. We know $\sum \frac{1}{3^n}$ converges. So, we think $\sum \frac{3+\cos n}{3^n}$ also conveges. To use the Direct Comparison Test, we need to find a convergent series above our series. $\sum \frac{1}{3^n}$ is a natural candidate, but it is not above our series.

$$\frac{3+\cos n}{3^n} \leq \frac{1}{3^n} \quad \text{is not true}$$

We tweak things a little bit and consider $\sum \frac{4}{3^n}$ which is still a convergent geometric series.

$$\text{Since} \quad 3+\cos n \leq 4 \quad \text{we have}$$
$$\frac{3+\cos n}{3^n} \leq \frac{4}{3^n}$$

By DCT, $\sum \frac{3+\cos n}{3^n}$ converges.

□

Example 5.2. $\displaystyle\sum_{n=1}^{\infty} \frac{3+\sin n}{\sqrt{n}}$

Solution. Observe that the numerator is bounded

$$2 \leq 3 + \sin n \leq 4$$

What is left looks like $\frac{1}{\sqrt{n}}$. We know $\sum \frac{1}{n^{1/2}}$ is a divergent p-series ($p = \frac{1}{2}$). So, we think our series also diverges. To use the Direct Comparison Test, we need a divergent series below our series. $\sum \frac{1}{n^{1/2}}$

is a natural candidate. Is it below our series? Yes. Since

$$1 \leq 3 + \sin n$$
$$\frac{1}{\sqrt{n}} \leq \frac{3 + \sin n}{\sqrt{n}}$$

By DCT, $\sum \frac{3+\sin n}{\sqrt{n}}$ diverges.

□

5.2. **Limit Comparison Test.** Let $\sum a_n$ and $\sum b_n$ be positive term series.

Theorem (Limit Comparison Test)**.** *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ *is finite and positive, then* $\sum a_n$ *and* $\sum b_n$ *behave the same.*

Remark. If $\sum a_n$ involves only powers (including roots) of n, LCT will always work with

$$b_n = \frac{\text{highest power of } n \text{ in numerator}}{\text{highest power of } n \text{ in denominator}}$$

Example 5.3. $\displaystyle\sum_{n=1}^{\infty} \frac{13n^5 - 11n^3 + 7}{23n^6 - 19n^4 + 17} \quad (*)$

Solution. The Divergence Test gives us nothing. Consider

$$\sum_{n=1}^{\infty} \frac{n^5}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n}$$

Both $(*)$ and $\sum \frac{1}{n}$ are positive term series. We apply LCT.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{13n^5-11n^3+7}{23n^6-19n^4+17}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{13n^6 - 11n^4 + 7n}{23n^6 - 19n^4 + 17} \quad \cdot \quad \frac{\frac{1}{n^6}}{\frac{1}{n^6}} \\
&= \lim_{n\to\infty} \frac{13 - \frac{11}{n^2} + \frac{7}{n^5}}{23 - \frac{19}{n^2} + \frac{17}{n^6}} \\
&= \frac{13}{23} \quad \text{(finite and positive)}
\end{aligned}$$

By LCT both series behave the same. $\sum \frac{1}{n}$ diverges, since it is a p-series with $p = 1$. So, $(*)$ diverges also.

□

Example 5.4. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{3n^2 - 1}}{n^3 + 2n^2 + 5} \quad (**)$

Solution. The Divergence Test gives us nothing. Consider

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Both $\sum \frac{1}{n^2}$ and $(**)$ are positive term series. We apply LCT.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{\sqrt{3n^2-1}}{n^3+2n^2+5}}{\frac{1}{n^2}} &= \lim_{n\to\infty} \frac{\sqrt{3n^2-1}\cdot n^2}{n^3+2n^2+5} \\
&= \lim_{n\to\infty} \frac{\sqrt{3n^2-1}\cdot \sqrt{n^4}}{n^3+2n^2+5} \\
&= \lim_{n\to\infty} \frac{\sqrt{3n^6-n^4}}{n^3+2n^2+5} \quad \cdot \quad \frac{\frac{1}{\sqrt{n^6}}}{\frac{1}{n^3}} \\
&= \lim_{n\to\infty} \frac{\sqrt{3-\frac{1}{n^2}}}{1+\frac{2}{n}+\frac{5}{n^3}} = \sqrt{3} \quad \text{(finite and positive)}
\end{aligned}$$

Since $\sum \frac{1}{n^2}$ is a convergent p-series ($p = 2$), $(**)$ also converges by LCT.

□

6. Alternating Series Test

An alternating series is a series whose terms are alternately positive and negative. Consider an alternating series $\sum (-1)^n a_n$.

Theorem (Alternating Series Test)**.**
If

(i) $\lim_{n\to\infty} a_n = 0$, *and*
(ii) $\{a_n\}$ *is decreasing, i.e.* $a_{n+1} \le a_n$

Then, $\sum (-1)^n a_n$ *converges.*

Remark. If you use the Alternating Series Test (AST), you must verify the two conditions. If you can't verify the conditions, this may be a sign that you should use another test. Examples 2.6 and 8.2 are both alternating series, but verifying condition (i) of AST is difficult in both cases.

Example 6.1. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n-1}$

Solution. The Divergence Test gives us nothing. This is an alternating series.

(1) Observe that $\lim_{n\to\infty} \frac{n-1}{n^2+n-1} = 0$

(2) Is $\frac{n-1}{n^2+n-1}$ decreasing?

Consider $f(x) = \frac{x-1}{x^2+x-1}$

$$\begin{aligned} f'(x) &= \frac{(x^2+x-1)1-(x-1)(2x+1)}{(x^2+x-1)^2} \\ &= \frac{(x^2+x-1)-(2x^2-x-1)}{(x^2+x-1)^2} \\ &= \frac{2x-x^2}{(x^2+x-1)^2} \\ &= \frac{x(2-x^2)}{(x^2+x-1)^2} \end{aligned}$$

$f'(x) < 0$ if $2 - x^2 < 0$ or equivalently if $x^2 > 2$.
Surely, this is true for $x > 2$.

So, $\frac{n-1}{n^2+n-1}$ is decreasing for $n > 2$.

By the Alternating Series Test (AST), the series converges.

□

7. ABSOLUTE CONVERGENCE AND CONDITIONAL CONVERGENCE

Consider the series $\sum a_n$. If $\sum a_n$ converges, we would like to know, in addition, whether the series of absolute values $\sum |a_n|$ converges. If $\sum a_n$ converges and $\sum |a_n|$ also converges, then we say $\sum a_n$ *converges absolutely*. It is a fact that if $\sum |a_n|$ converges, $\sum a_n$ automatically converges. So we are lead to the following definitions.

Definition 7.1. If $\sum |a_n|$ converges, we say $\sum a_n$ *converges absolutely.*

Definition 7.2. If $\sum a_n$ converges, but $\sum |a_n|$ diverges, then we say $\sum a_n$ *converges conditionally.*

In both cases, the series converges in the sense of Definition 1.3. So, why do we need to distinguish between the two? Although a conditionally convergent series converges, the convergence is in some sense "unstable." For a conditionally convergent series, it is a remarkable fact that we can rearrange the series to get whatever sum we want, including ±∞. Conditionally convergent series are very sensitive to the order in which the terms are added. This is not true for absolutely convergent series. An absolutely convergent series converges, and every rearrangement converges to the same sum.

Example 7.3. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,3k}{\sqrt{k^3+4}} \tag{☺}$$

Solution.

(1) Test for absolute convergence

Does $\displaystyle\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}\,3k}{\sqrt{k^3+4}}\right| = \sum_{k=1}^{\infty} \frac{3k}{\sqrt{k^3+4}}$ converge?

We use the limit comparison test. We compare with

$$\sum_{k=4}^{\infty} \frac{k}{k^{3/2}} = \sum_{k=4}^{\infty} \frac{1}{k^{1/2}}$$

Observe that

$$\begin{aligned}
\lim_{k\to\infty} \frac{\frac{3k}{\sqrt{k^3+4}}}{\frac{1}{k^{1/2}}} &= \lim_{k\to\infty} \frac{3k^{3/2}}{\sqrt{k^3+4}} \quad \cdot \quad \frac{\frac{1}{k^{3/2}}}{\frac{1}{k^{3/2}}} \\
&= \lim_{k\to\infty} \frac{3}{\frac{\sqrt{k^3+4}}{\sqrt{k^3}}} \\
&= \lim_{k\to\infty} \frac{3}{\sqrt{1+\frac{4}{k^3}}} = 3 \quad \text{(finite and positive)}
\end{aligned}$$

By LCT both series behave the same. $\sum \frac{1}{k^{1/2}}$ is a divergent p-series ($p = \frac{1}{2}$).

Hence, $\displaystyle\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}\,3k}{\sqrt{k^3+4}}\right|$ also diverges.

(2) Does (☺) converge?

(☺) is an alternating series. Observe that

(a)

$$\begin{aligned}\lim_{k\to\infty}\frac{3k}{\sqrt{k^3+4}}\cdot\frac{\frac{1}{k^{3/2}}}{\frac{1}{k^{3/2}}} &= \lim_{k\to\infty}\frac{\frac{3}{k^{1/2}}}{\frac{\sqrt{k^3+4}}{\sqrt{k^3}}}\\ &= \lim_{k\to\infty}\frac{\frac{3}{k^{1/2}}}{\sqrt{1+\frac{4}{k^3}}}\\ &= 0\end{aligned}$$

(b) Is $\dfrac{3k}{\sqrt{k^3+4}}$ decreasing?

Consider $f(x) = \dfrac{3x}{\sqrt{x^3+4}} = 3x(x^3+4)^{-1/2}$

$$\begin{aligned}f'(x) &= 3(x^3+4)^{-1/2} + 3x\left(-\frac{1}{2}\right)(x^3+4)^{-3/2}(3x^2)\\ &= \frac{3}{(x^3+4)^{1/2}} - \frac{9x^3}{2(x^3+4)^{3/2}}\\ &= \frac{3\cdot 2(x^3+4) - 9x^3}{2(x^3+4)^{3/2}}\\ &= \frac{24-3x^3}{2(x^3+4)^{3/2}} < 0 \quad \text{for} \quad x > 2\end{aligned}$$

So, $\dfrac{3k}{\sqrt{k^3+4}}$ is decreasing for $k > 2$

By the Alternating Series Test (AST), (☺) converges.

(3) Thus (☺) converges conditionally.

□

Example 7.4. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n} \qquad (\heartsuit)$$

Solution.

(1) Test for absolute convergence

Does $\displaystyle\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{n\ln n}\right| = \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ converge?

We use the integral test.

$$f(x) = \frac{1}{x\ln x}$$

Check the conditions for the integral test.

(a) f is clearly continuous for $x \geq 2$
(b) f is clearly positive for $x \geq 2$
(c) Is f decreasing?

$$\begin{aligned} f(x) &= x^{-1}(\ln x)^{-1} \\ f'(x) &= -1x^{-2}(\ln x)^{-1} + x^{-1}(-1)(\ln x)^{-2}\frac{1}{x} \\ &= \frac{-1}{x^2\ln x} + \frac{-1}{x^2(\ln x)^2} < 0 \quad \text{for} \quad x \geq 2 \end{aligned}$$

So f is decreasing for $x \geq 2$

The conditions for the Integral Test are verified.

$$\begin{aligned} \int_2^{\infty}\frac{1}{x\ln x}\,dx &= \lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}\,dx \\ &= \lim_{b\to\infty}\Big[\ln|\ln x|\Big]_2^b \qquad (u = \ln x) \\ &= \lim_{b\to\infty}\left(\ln(\ln b) - \ln(\ln 2)\right) \\ &= \infty \end{aligned}$$

By the Integral Test, $\displaystyle\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{n\ln n}\right|$ diverges.

(2) Does ($\heartsuit$) converge?

This is an alternating series. Observe that

(a) $\displaystyle\lim_{n\to\infty}\frac{1}{n\ln n} = 0$

(b) When we used the Integral Test, we showed that $f(x) = \dfrac{1}{x \ln x}$ is decreasing for $x \geq 2$. Thus, $\dfrac{1}{n \ln n}$ is decreasing.

By the Alternating Series Test (AST), ($\heartsuit$) converges.

(3) Therefore, ($\heartsuit$) <u>converges conditionally</u>.

□

8. Ratio Test and Root Test

8.1. **Ratio Test.** Consider a series $\sum a_n$

(i) If $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$, then $\sum a_n$ converges absolutely.

(ii) If $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L > 1$ or $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty$, then $\sum a_n$ diverges.

(iii) If $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$, then the test gives no information.

Remark. If the series involves factorials and/or $(\text{constant})^n$, use the Ratio Test.

Example 8.1. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(2n+1)!}{5^n}$$

Solution. We use the Ratio Test.

$$\begin{aligned}
\frac{|a_{n+1}|}{|a_n|} &= \frac{\frac{(2(n+1)+1)!}{5^{n+1}}}{\frac{(2n+1)!}{5^n}} \\
&= \frac{(2n+3)!}{5^{n+1}} \cdot \frac{5^n}{(2n+1)!} \\
&= \frac{(2n+3)(2n+2)(2n+1)!}{5^n\,5} \cdot \frac{5^n}{(2n+1)!} \\
&= \frac{(2n+3)(2n+2)}{5}
\end{aligned}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(2n+3)(2n+2)}{5} = \infty$$

The series diverges by the Ratio Test.

$\square$

Example 8.2. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{1\cdot 3\cdot 5\cdots(2n-1)}{5^{2n-3}\,n!}$$

Solution. We use the Ratio Test.

$$\begin{aligned}
\frac{|a_{n+1}|}{|a_n|} &= \frac{\frac{1\cdot 3\cdot 5\cdots(2(n+1)-1)}{5^{2(n+1)-3}\,(n+1)!}}{\frac{1\cdot 3\cdot 5\cdots(2n-1)}{5^{2n-3}\,n!}}\\
&= \frac{1\cdot 3\cdot 5\cdots(2n+1)}{5^{2n-1}\,(n+1)!}\cdot\frac{5^{2n-3}\,n!}{1\cdot 3\cdot 5\cdots(2n-1)}\\
&= \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{5^{2n}\,5^{-1}\,(n+1)\,n!}\cdot\frac{5^{2n}\,5^{-3}\,n!}{1\cdot 3\cdot 5\cdots(2n-1)}\\
&= \frac{5(2n+1)}{5^{3}(n+1)}
\end{aligned}$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{2n+1}{25n+25} = \frac{2}{25} < 1$$

The series converges absolutely by the Ratio Test.

$\square$

8.2. **Root Test.** The Root Test is exactly the same as the Ratio Test, except that we replace

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} \qquad \text{with} \qquad \lim_{n\to\infty}|a_n|^{1/n}$$

You should use the Root Test in situations where taking the n^{th} root of the terms simplify things.