Vector - Valued Functions

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$
$$= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

is a vector - valued function. It takes real numbers as input, and it outputs vectors.

Calculus of vector-valued functions

Take limits component by component
Differentiate component by component
Integrate component by component

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$
$$= \left[\lim_{t\to a} f(t)\right]\vec{i} + \left[\lim_{t\to a} g(t)\right]\vec{j} + \left[\lim_{t\to a} h(t)\right]\vec{k}$$

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$
$$= f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

$$\int \vec{r}(t)\,dt = \langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \rangle$$
$$= \left[\int f(t)\,dt\right]\vec{i} + \left[\int g(t)\,dt\right]\vec{j} + \left[\int h(t)\,dt\right]\vec{k}$$

Remember: These operations result in vectors not scalars.

Example:

$$\lim_{t\to 0}\left(\frac{e^t-1}{t}\vec{i}+\frac{\sqrt{1+t}-1}{t}\vec{j}+\frac{3}{2+t}\vec{k}\right)$$

Soln: From Calculus I, you have essentially two ways to take limits. One is direct substitution. If that doesn't work (ie you get $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$ or some other indeterminate form) then we use L'hôpital's Rule.

$$\lim_{t\to 0}\left\langle \frac{e^t-1}{t},\ \frac{\sqrt{1+t}-1}{t},\ \frac{3}{2+t}\right\rangle$$

$$=\left\langle \lim_{t\to 0}\frac{e^t-1}{t},\ \lim_{t\to 0}\frac{\sqrt{1+t}-1}{t},\ \lim_{t\to 0}\frac{3}{2+t}\right\rangle$$

$$=\left\langle \lim_{t\to 0}\frac{e^t}{1},\ \lim_{t\to 0}\frac{\frac{1}{2}(1+t)^{-1/2}(1)}{1},\ \frac{3}{2+0}\right\rangle$$

$$=\left\langle 1,\ \frac{1}{2},\ \frac{3}{2}\right\rangle$$

Example Find $\vec{r}(t)$ given

$$\vec{r}\,''(t) = 6t\vec{j} - e^{-5t}\vec{k}$$

$$\vec{r}\,'(0) = 3\vec{i} + 2\vec{j} - 5\vec{k}$$

$$\vec{r}(0) = 2\vec{i} - 3\vec{j} + 4\vec{k}$$

Soln: Differentiation and Integration are inverse operations. From $\vec{r}(t)$, we differentiate twice to get $\vec{r}\,''(t)$. So, if we are given $\vec{r}\,''(t)$, to recover $\vec{r}(t)$, we have to integrate twice.

$$\vec{r}\,'(t) = \int \vec{r}\,''(t)\,dt$$

$$= \left\langle \int 0\,dt,\ \int 6t\,dt,\ \int -e^{-5t}\,dt \right\rangle$$

$$= \left\langle c_1,\ 3t^2 + c_2,\ \tfrac{1}{5}e^{-5t} + c_3 \right\rangle$$

We use the initial conditions to recover the constants.

$$\vec{r}\,'(0) = \left\langle c_1,\ c_2,\ \tfrac{1}{5} + c_3 \right\rangle$$

$$= \langle 3, 2, -5 \rangle$$

$c_1 = 3$

$c_2 = 2$

$\frac{1}{5} + c_3 = -5 \Rightarrow c_3 = \frac{-26}{5}$

So, $\vec{r}\,'(t) = \langle 3,\ 3t^2 + 2,\ \frac{1}{5}e^{-5t} - \frac{26}{5} \rangle$

$\vec{r}(t) = \int \vec{r}\,'(t)\,dt$

$= \langle \int 3\,dt,\ \int (3t^2+2)\,dt,\ \int (\frac{1}{5}e^{-5t} - \frac{26}{5})\,dt \rangle$

$= \langle 3t + D_1,\ t^3 + 2t + D_2,\ \frac{-1}{25}e^{-5t} - \frac{26}{5}t + D_3 \rangle$

$\vec{r}(0) = \langle D_1,\ D_2,\ \frac{-1}{25} + D_3 \rangle$

$= \langle 2,\ -3,\ 4 \rangle$

$D_1 = 2$

$D_2 = -3$

$\frac{-1}{25} + D_3 = 4 \implies D_3 = \frac{101}{25}$

So,

$\vec{r}(t) = \langle 3t + 2,\ t^3 + 2t - 3,\ \frac{-1}{25}e^{-5t} - \frac{26}{5}t + \frac{101}{25} \rangle$

$\boxed{/}$

I personally prefer the component notation $\langle\ ,\ ,\ \rangle$ rather than the $\vec{i}, \vec{j}, \vec{k}$ notation. I think its cleaner and it helps me avoid mistakes. But that's just me.

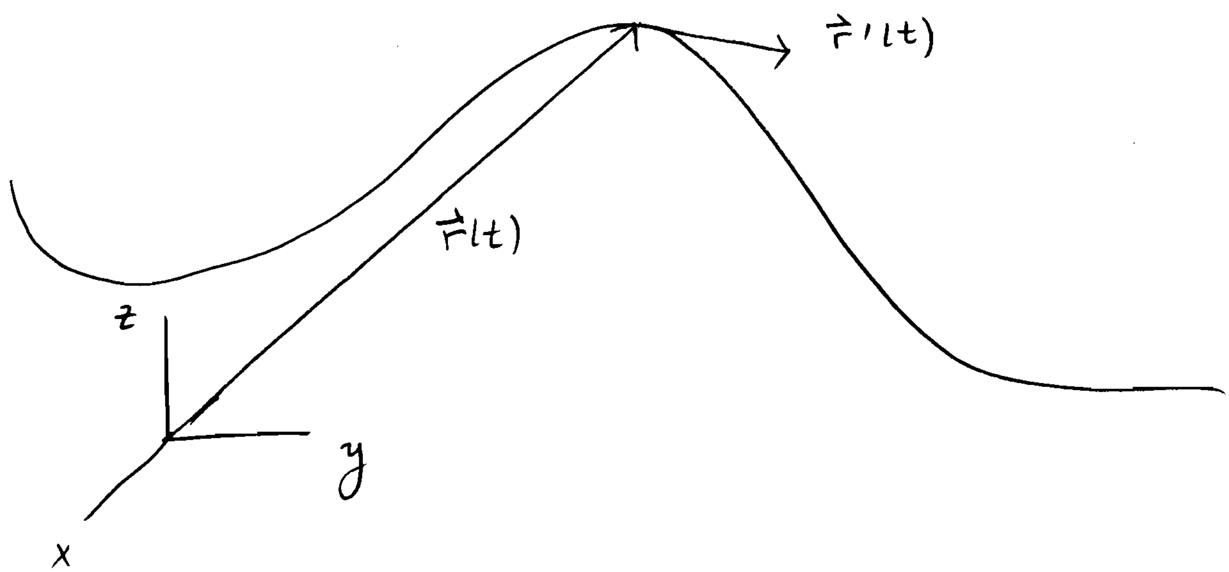
Physics Interpretation

If $\vec{r}(t)$ gives the position of a moving object at time t then

velocity $\vec{v}(t) = \vec{r}\,'(t)$

acceleration $\vec{a}(t) = \vec{r}\,''(t)$

speed $\|\vec{v}(t)\|$



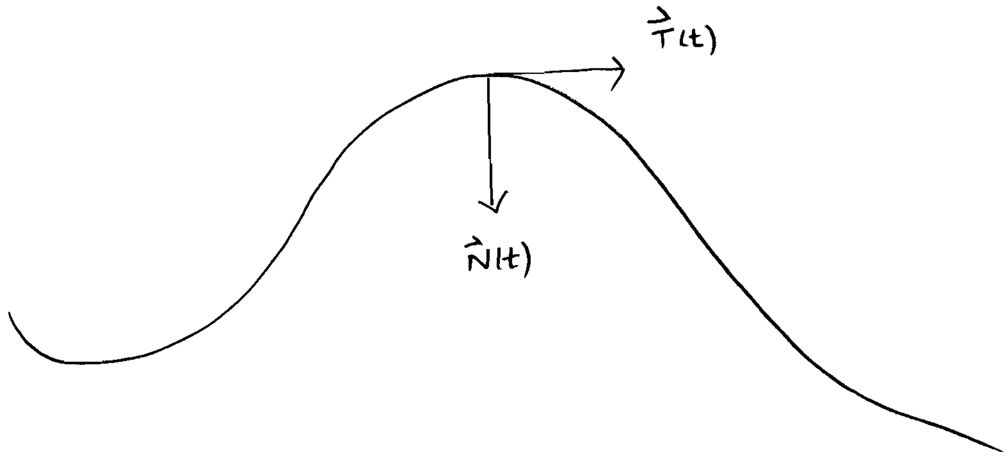
The terminal point of the vector $\vec{r}(t)$ traces out a curve. $\vec{r}\,'(t)$ is a vector tangent to the curve at t.

Unit Tangent vector $\vec{T}(t)$ at t

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

Principal Unit Normal Vector $\vec{N}(t)$ at t

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$



$\vec{T}(t)$ & $\vec{N}(t)$ are unit vectors. They are perpendicular.
$\vec{N}(t)$ points in the direction the object is turning.

Some remarks

$\vec{T}(t)$ is easy to compute. First compute $\vec{r}\,'(t)$, find its length $\|\vec{r}\,'(t)\|$, and then use the formula.

What does $\vec{T}(t)$ look like?
Well, in general $\|\vec{r}\,'(t)\|$ will look like the square root of some expression involving t.
$\sqrt{\text{expression involving } t}$. Each component of $\vec{T}(t)$ will then have this in its denominator.

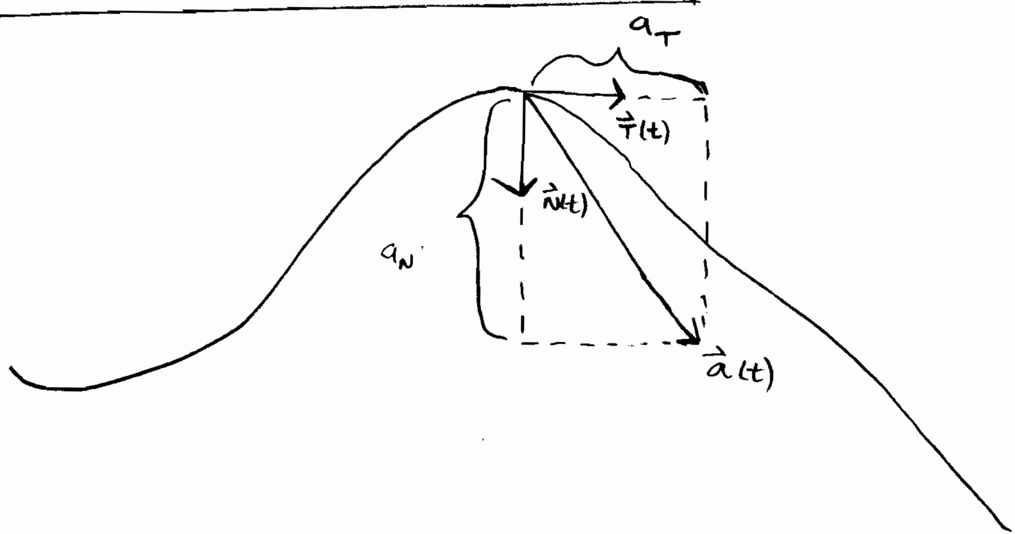
So, in general, calculating $\vec{T}\,'(t)$ will involve quotient rule and chain rule at a minimum. $\vec{T}\,'(t)$ is hideous to calculate in most cases. Naturally then $\vec{N}(t)$ is even more hideous to calculate.

Moral: Avoid calculating $\vec{T}\,'(t)$ & $\vec{N}(t)$ if possible. If $\|\vec{r}\,'(t)\|$ is nice, say a constant, then $\vec{T}\,'(t)$ and $\vec{N}(t)$ are very easy to calculate.

Tangential and Normal Components of Acceleration

We want to decompose the acceleration vector into the part that points in the tangential direction and the part that points in the normal direction. We want to write $\vec{a}(t)$ as

$$\boxed{\vec{a}(t) = a_T \vec{T}(t) + a_N \vec{N}(t)}$$



Tangential component

$$a_T(t) = \vec{a}(t) \cdot \vec{T}(t) = \frac{\vec{v}(t) \cdot \vec{a}(t)}{\|\vec{v}(t)\|} = \frac{d}{dt}\left[\, \|\vec{v}(t)\| \,\right]$$

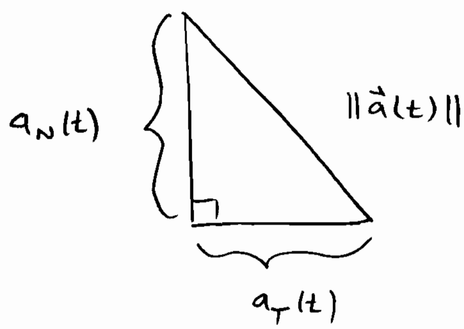
Normal component

$$a_N(t) = \underbrace{\vec{a}(t) \cdot \vec{N}(t)}_{\text{Not very useful because of remarks on previous page.}} = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|} = \sqrt{\|\vec{a}(t)\|^2 - (a_T(t))^2}$$

<u>Note</u>: $a_T(t)$ & $a_N(t)$ are <u>scalars</u>.

Note : $a_N(t) = \sqrt{\|\vec{a}(t)\|^2 - (a_T(t))^2}$

is just the Pythagorean Thm. Look at the picture on the previous page



Example

Find $\vec{T}(t)$, $\vec{N}(t)$, a_T, a_N at the given time t for the plane curve $\vec{r}(t)$.

$$\vec{r}(t) = t\vec{i} + \ln t\,\vec{j} + t^2\vec{k} \qquad t = 1.$$

Soln :

$\underline{\underline{1^\circ}}$ $\underline{\vec{T}(t) \text{ at } t = 1}$

$$\vec{r}\,'(t) = \langle 1, \tfrac{1}{t}, 2t \rangle$$

we could compute $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$, then plug in $t=1$ at the end. But this can get messy. we are only looking for $\vec{T}(t)$ at $t=1$. So, we just want

$$\vec{T}(1) = \frac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|}$$

This suggests that we find $\vec{r}'(1)$ and $\|\vec{r}'(1)\|$ first.

$$\vec{r}'(1) = \langle 1, 1, 2 \rangle$$

$$\|\vec{r}'(1)\| = \sqrt{1+1+4} = \sqrt{6}$$

So, $\underline{\underline{\vec{T}(1) = \langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \rangle}}$

2°. $a_T(t)$ at $t = 1$.

$$\vec{a}(t) = \vec{r}''(t) = \langle 0, \frac{-1}{t^2}, 2 \rangle$$

$$a_T(1) = \vec{a}(1) \cdot \vec{T}(1) = \langle 0, -1, 2 \rangle \cdot \langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \rangle$$

$$= 0 - \frac{1}{\sqrt{6}} + \frac{4}{\sqrt{6}}$$

$$= \underline{\underline{\frac{3}{\sqrt{6}}}}$$

we could have used $a_T(t) = \dfrac{\vec{v}(t) \cdot \vec{a}(t)}{\|\vec{v}(t)\|}$.

Computing this, then plugging in $t = 1$ at the end will get messy. Remember we are only after

$$a_T(1) = \frac{\vec{v}(1) \cdot \vec{a}(1)}{\|\vec{v}(1)\|} = \frac{\langle 1, 1, 2 \rangle \cdot \langle 0, -1, 2 \rangle}{\|\langle 1, 1, 2 \rangle\|}$$

So, find $\vec{v}(1)$, $\vec{a}(1)$ first, then use the formula.

$\underline{\underline{3^\circ}}$ $\underline{a_N(t) \text{ at } t=1}$.

$$a_N(1) = \sqrt{\|\vec{a}(1)\|^2 - (a_T(1))^2}$$

$\vec{a}(1) = \langle 0, -1, 2 \rangle$

$\|\vec{a}(1)\| = \sqrt{0+1+4} = \sqrt{5}$

$a_T(1) = \frac{3}{\sqrt{6}}$

So, $a_N(1) = \sqrt{5 - \frac{9}{6}} = \sqrt{\frac{21}{6}}$

we could have used $a_N(t) = \dfrac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|}$.

At $t=1$, this becomes

$$a_N(1) = \frac{\|\vec{v}(1) \times \vec{a}(1)\|}{\|\vec{v}(1)\|}$$

Avoid messiness by finding $\vec{v}(1)$, $\vec{a}(1)$, then using formula.

$\underline{\underline{4^\circ}}$ $\underline{\vec{N}(t) \text{ at } t=1}$

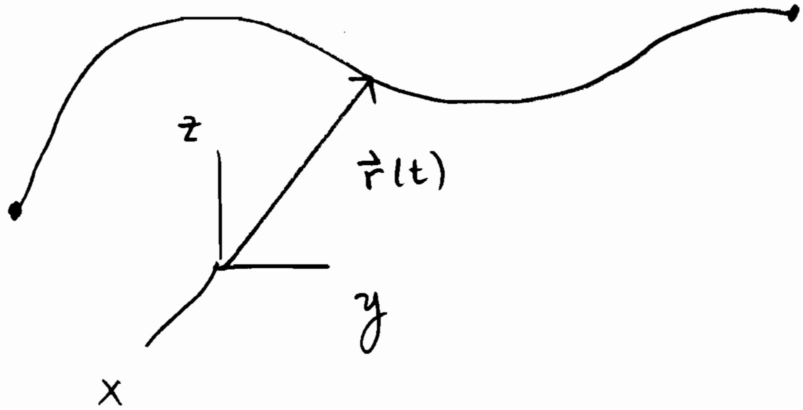
$$\vec{a}(1) = a_T(1)\,\vec{T}(1) + a_N(1)\,\vec{N}(1)$$

$$\langle 0, -1, 2 \rangle = \frac{3}{\sqrt{6}} \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle + \sqrt{\frac{21}{6}}\,\vec{N}(1)$$

Solve for $\vec{N}(1)$.

$\boxed{\text{E}}$

$\vec{r}(t)$ on interval $[a, b]$



Arc Length $\quad s = \int_a^b \|\vec{r}\,'(t)\|\, dt$

Curvature at t

$$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

Example

Find the curvature at the given value of t.

$$\vec{r}(t) = (3t - 4\cos t)\,\vec{i} + (4t + 3\cos t)\,\vec{j} + 5\sin t\,\vec{k}$$

$t = \pi$

Soln : we'll use $\kappa(t) = \dfrac{\| \vec{r}'(t) \times \vec{r}''(t) \|}{\| \vec{r}'(t) \|^3}$

$\vec{r}'(t) = \langle 3 + 4\sin t ,\ 4 - 3\sin t ,\ 5\cos t \rangle$

$\vec{r}''(t) = \langle 4\cos t ,\ -3\cos t ,\ -5\sin t \rangle$

we are after

$$\kappa(\pi) = \frac{\| \vec{r}'(\pi) \times \vec{r}''(\pi) \|}{\| \vec{r}'(\pi) \|^3}$$

$\vec{r}'(\pi) = \langle 3, 4, -5 \rangle$

$\vec{r}''(\pi) = \langle -4, 3, 0 \rangle$

$$\vec{r}'(\pi) \times \vec{r}''(\pi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & -5 \\ -4 & 3 & 0 \end{vmatrix} = (15)\vec{i} - (-20)\vec{j} + (25)\vec{k}$$

$$= \langle 15, 20, 25 \rangle$$

$\| \vec{r}'(\pi) \times \vec{r}''(\pi) \| = \sqrt{15^2 + 20^2 + 25^2} = \sqrt{1250}$

$\| \vec{r}'(\pi) \| = \sqrt{9 + 16 + 25} = \sqrt{50}$

$$\kappa(\pi) = \frac{\sqrt{1250}}{(\sqrt{50})^3} = \frac{1}{10}.$$

$\boxed{E}$

Recall that $\vec{T}'(t)$ is in general hideous to compute. Most of the time you will avoid

$$K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$ for this reason.

But if $\|\vec{r}'(t)\|$ is nice, say a constant, $\vec{T}'(t)$ is easy to compute. Need proof? see below.

<u>Example</u>

$$\vec{r}(t) = 4t\vec{i} + 3\cos t\,\vec{j} + 3\sin t\,\vec{k}$$

Find the curvature $K(t)$

<u>Soln</u>: which formula to use? Both require $\vec{r}'(t)$. So let's compute that first.

$$\vec{r}'(t) = \langle 4, -3\sin t, 3\cos t \rangle$$

Let's see if $\|\vec{r}'(t)\|$ is nice.

$$\|\vec{r}'(t)\| = \sqrt{16 + 9\sin^2 t + 9\cos^2 t} = \sqrt{25} = 5$$ Nice!

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \langle \tfrac{4}{5}, \tfrac{-3}{5}\sin t, \tfrac{3}{5}\cos t \rangle$$

$$\vec{T}'(t) = \langle 0, -\tfrac{3}{5}\cos t, -\tfrac{3}{5}\sin t \rangle$$ ← Easy to compute, right?

$$\|\vec{T}'(t)\| = \sqrt{0 + \tfrac{9}{25}\cos^2 t + \tfrac{9}{25}\sin^2 t} = \tfrac{3}{5}$$

So, $\kappa(t) = \dfrac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \dfrac{\frac{3}{5}}{5} = \dfrac{3}{25}$.

$\boxed{E}$

Example

$\vec{r}(t) = 2t^2\,\vec{i} + t\,\vec{j} + \frac{1}{2}t^2\,\vec{k}$

Find the curvature $\kappa(t)$.

Soln Both formulas require $\vec{r}'(t)$. So,

$\vec{r}'(t) = \langle 4t, 1, t \rangle$

$\|\vec{r}'(t)\| = \sqrt{16t^2 + 1 + t^2} = \sqrt{17t^2 + 1}$

$\|\vec{r}'(t)\|$ is not a nice constant. So, we'll use

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

$\vec{r}''(t) = \langle 4, 0, 1 \rangle$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4t & 1 & t \\ 4 & 0 & 1 \end{vmatrix} = (1)\vec{i} - (0)\vec{j} + (-4)\vec{k}$$

$= \langle 1, 0, -4 \rangle$

$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{17}$

$$K(t) = \frac{\sqrt{17}}{\left(\sqrt{17t^2+1}\right)^3}$$

E

Arc Length Example

Find the length of the given curve.

$$\vec{r}(t) = t^2\vec{i} + 2t\vec{j} + \ln t\,\vec{k} \qquad 1 \leq t \leq e$$

Soln:

$$\vec{r}\,'(t) = \langle 2t, 2, \tfrac{1}{t} \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 4 + \frac{1}{t^2}}$$

$$S = \int_1^e \sqrt{4t^2+4+\frac{1}{t^2}}\,dt$$

$$= \int_1^e \sqrt{\frac{4t^4+4t^2+1}{t^2}}\,dt$$

$$= \int_1^e \sqrt{\frac{(2t^2+1)^2}{t^2}}\,dt$$

$$= \int_1^e \frac{2t^2+1}{t}\,dt$$

(Actually $\sqrt{\frac{(2t^2+1)^2}{t^2}} = \frac{|2t^2+1|}{|t|}$

But on $[1,e]$ everything is ≥ 0,

So $\frac{|2t^2+1|}{|t|} = \frac{2t^2+1}{t}$ on $[1,e]$

$$= \int_1^e \left(2t + \frac{1}{t} \right) dt$$

$$= \left[t^2 + \ln|t| \right]_1^e$$

$$= (e^2 + 1) - (1 + 0)$$

$$= e^2$$

The difficulty in these problems is that the integrand involves a square root. We hope, after some algebraic manipulation, to get a square under the radical $\left(\sqrt{(\ \ )^2}\right)$. In this case the radical will go away. The next best thing we could hope for is that we end up with something that we can integrate by u-substitution. Worst case scenario, we need a trig substitution.