Neural Networks

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Given a collection of handwritten digits and their corresponding labels, we’d like to be able to correctly classify handwritten digits.

- A simple algorithmic technique can solve this problem with 95% accuracy.

This seems surprising, in fact, state-of-the-art methods can achieve near 99% accuracy (you’ve probably seen these in action if you’ve deposited a check recently).

Digits from the MNIST data set.
Neural Networks

• The basis of neural networks was developed in the 1940s -1960s
  — The idea was to build mathematical models that might “compute” in the same way that neurons in the brain do
  — As a result, neural networks are biologically inspired, though many of the algorithms that are used to work with them are not biologically plausible
  — Perform surprisingly well for the handwritten digit recognition task
Neural Networks

• Neural networks consist of a collection of artificial neurons

• There are different types of neuron models that are commonly studied
  – The perceptron (one of the first studied)
  – The sigmoid neuron (most common)

• A neural network is typically a directed graph consisting of a collection of neurons (the nodes in the graph), directed edges (each with an associated weight), and a collection of fixed binary inputs
The Perceptron

- A perceptron is an artificial neuron that takes a collection of binary inputs and produces a binary output.
  - The output of the perceptron is determined by summing up the weighted inputs and thresholding the result: if the weighted sum is larger than the threshold, the output is one (and zero otherwise).

\[
y = \begin{cases} 
1 & w_1x_1 + w_2x_2 + w_3x_3 > \text{threshold} \\
0 & \text{otherwise}
\end{cases}
\]
The Perceptron

\[ y = \begin{cases} 
1 & w_1 x_1 + w_2 x_2 + w_3 x_3 > \text{threshold} \\
0 & \text{otherwise}
\end{cases} \]

- The weights can be both positive and negative
- Many simple decisions can be modeled using perceptrons
  - Example: AND, OR, NOT
Perceptron for NOT

• Choose $w = -1$, threshold = $-0.5$

• $y = \begin{cases} 
1 & -x > -0.5 \\
0 & -x \leq -0.5 
\end{cases}$
Perceptron for OR
Perceptron for OR

- Choose $w_1 = w_2 = 1$, threshold = 0

- $y = \begin{cases} 
1 & x_1 + x_2 > 0 \\
0 & x_1 + x_2 \leq 0 
\end{cases}$
Perceptron for AND
Perceptron for AND

- Choose $w_1 = w_2 = 1$, threshold $= 1.5$

- $y = \begin{cases} 
1 & x_1 + x_2 > 1.5 \\
0 & x_1 + x_2 \leq 1.5 
\end{cases}$
Perceptron for XOR
Perceptron for XOR

• Need more than one perceptron!

\[ x_1 \quad \lor \quad \land \quad \neg \quad y \]

\[ x_2 \]

• Weights for incoming edges are chosen as before
  – Networks of perceptrons can encode any circuit!
Perceptrons

- Perceptrons are usually expressed in terms of a collection of input weights and a bias $b$ (which is the negative threshold)

\[
y = \begin{cases} 
1 & \text{if } w_1 x_1 + w_2 x_2 + w_3 x_3 + b > 0 \\
0 & \text{otherwise}
\end{cases}
\]
Neural Networks

- Gluing a bunch of perceptrons together gives us a neural network
- In general, neural nets have a collection of binary inputs and a collection of binary outputs
Beyond Perceptrons

- Given a collection of input-output pairs, we’d like to learn the weights of the neural network so that we can correctly predict the output of an unseen input
  - We could try learning via gradient descent (e.g., by minimizing the Hamming loss)
    - This approach doesn’t work so well: small changes in the weights can cause dramatic changes in the output
    - This is a consequence of the discontinuity of sharp thresholding (same problem we saw in SVMs)
The Sigmoid Neuron

- A sigmoid neuron is an artificial neuron that takes a collection of inputs in the interval $[0,1]$ and produces an output in the interval $[0,1]$

  - The output is determined by summing up the weighted inputs plus the bias and applying the sigmoid function to the result

\[
y = \sigma(w_1 x_1 + w_2 x_2 + w_3 x_3 + b)
\]

where $\sigma$ is the sigmoid function
The Sigmoid Function

- The sigmoid function is a continuous function that approximates a step function

\[ \sigma(z) = \frac{1}{1 + e^{-z}} \]
Multilayer Neural Networks

from Neural Networks and Deep Learning by Michael Nielson
Multilayer Neural Networks

NO intralayer connections

from Neural Networks and Deep Learning by Michael Nielson
Neural Network for Digit Classification

from Neural Networks and Deep Learning by Michael Nielson
Neural Network for Digit Classification

Why 10 instead of 4?

from Neural Networks and Deep Learning by Michael Nielson
Expressiveness of NNs

• Boolean functions
  • Every Boolean function can be represented by a network with a single hidden layer consisting of possibly exponentially many hidden units

• Continuous functions
  • Every bounded continuous function can be approximated up to arbitrarily small error by a network with one hidden layer
  • Any function can be approximated to arbitrary accuracy with two hidden layers
Training Neural Networks

- To do the learning, we first need to define a loss function to minimize

$$C(w, b) = \frac{1}{2M} \sum_{m} \| y^m - a(x^m, w, b) \|^2$$

- The training data consists of input output pairs

$$(x^1, y^1), \ldots, (x^M, y^M)$$

- $a(x^m, w, b)$ is the output of the neural network for the $m^{th}$ sample

- $w$ and $b$ are the weights and biases
Gradient of the Loss

- The derivative of the loss function is relatively straightforward to calculate

\[ \frac{\partial C(w, b)}{\partial w_k} = \frac{1}{M} \sum_m \left[ y^m - \frac{\partial a(x^m, w, b)}{\partial w_k} \right] \]

- To compute the derivative of \( a \), use the chain rule and the derivative of the sigmoid function

\[ \frac{d\sigma(z)}{dz} = \sigma(z) \cdot (1 - \sigma(z)) \]

- This gets complicated quickly with lots of layers of neurons
Stochastic Gradient Descent

• To make the training more practical, stochastic gradient descent is used instead of standard gradient descent

• The idea of stochastic gradient descent is to approximate the gradient of a sum by sampling a few indices and averaging

$$\nabla_x \sum_{i=1}^{n} f_i(x) \approx \frac{1}{K} \sum_{k=1}^{K} \nabla_x f_{i^k}(x)$$

here, for example, each $i^k$ is sampled uniformly at random from $\{1, ..., n\}$
Computing the Gradient

- We’ll compute the gradient for a single sample
  \[ C(w, b) = \|y - a(x, w, b)\|^2 \]

- Some definitions:
  - \( L \) is the number of layers
  - \( a^l_j \) is the output of the \( j^{th} \) neuron on the \( l^{th} \) layer
  - \( z^l_j \) is the input of the \( j^{th} \) neuron on the \( l^{th} \) layer
    \[
    z^l_j = \sum_k w^l_{jk} a^{l-1}_k + b^l_j
    \]
  - \( \delta^l_j \) is defined to be \( \frac{\partial C}{\partial z^l_j} \)
Computing the Gradient

For the output layer, we have the following partial derivative

\[
\frac{\partial C}{\partial z_j^L} = -(y_j - a_j^L) \frac{\partial a_j^L}{\partial z_j^L}
\]

\[
= -(y_j - a_j^L) \frac{\partial \sigma(z_j^L)}{\partial z_j^L}
\]

\[
= -(y_j - a_j^L) \sigma(z_j^L) \left( 1 - \sigma(z_j^L) \right)
\]

\[\equiv \delta_j^L\]

- For simplicity, we will denote the vector of all such partials for each node in the \(l^{th}\) layer as \(\delta^l\)
Computing the Gradient

For the $L - 1$ layer, we have the following partial derivative

$$\frac{\partial C}{\partial z_{k}^{L-1}} = \sum_{j} (a_{j}^{L} - y_{j}) \frac{\partial a_{j}^{L}}{\partial z_{k}^{L-1}}$$

$$= \sum_{j} (a_{j}^{L} - y_{j}) \frac{\partial \sigma(z_{j}^{L})}{\partial z_{k}^{L-1}}$$

$$= \sum_{j} (a_{j}^{L} - y_{j}) \sigma(z_{j}^{L}) \left(1 - \sigma(z_{j}^{L})\right) \frac{\partial z_{j}^{L}}{\partial z_{k}^{L-1}}$$

$$= \sum_{j} (a_{j}^{L} - y_{j}) \sigma(z_{j}^{L}) \left(1 - \sigma(z_{j}^{L})\right) \frac{\partial \sum_{k'} w_{jk'} a_{k'}^{L-1} + b_{j}^{L}}{\partial z_{k}^{L-1}}$$

$$= \sum_{j} (a_{j}^{L} - y_{j}) \sigma(z_{j}^{L}) \left(1 - \sigma(z_{j}^{L})\right) \sigma(z_{k}^{L-1}) \left(1 - \sigma(z_{k}^{L-1})\right) w_{jk}$$

$$= \left( (\delta^{L})^{T} w_{*k}^{L} \right) \left(1 - \sigma(z_{k}^{L-1})\right) \sigma(z_{k}^{L-1})$$
Computing the Gradient

• We can think of $w^l$ as a matrix

• This allows us to write

$$\delta^{L-1} = ((\delta^L)^T w^L)(1 - \sigma(z^{L-1}))\sigma(z^{L-1})$$

where $\sigma(z^{L-1})$ is the vector whose $k^{th}$ component is $\sigma(z_k^{L-1})$

• Applying the same strategy, for $l < L$

$$\delta^l = ((\delta^{l+1})^T w^{l+1})(1 - \sigma(z^l))\sigma(z^l)$$
Computing the Gradient

- Now, for the partial derivatives that we care about

\[
\frac{\partial C}{\partial b_j^l} = \frac{\partial C}{\partial z_j^l} \cdot \frac{\partial z_j^l}{\partial b_j^l} = \delta_j^l
\]

\[
\frac{\partial C}{\partial w_{jk}^l} = \frac{\partial C}{\partial z_j^l} \cdot \frac{\partial z_j^l}{\partial w_{jk}^l} = \delta_j^l a_k^{l-1}
\]

- We can compute these derivatives one layer at a time!
Backpropagation: Putting it all together

- Compute the inputs/outputs for each layer by starting at the input layer and applying the sigmoid functions

- Compute $\delta^L$ for the output layer

\[
\delta^L = - (y_j - a_j^L) \sigma(z_j^L) \left(1 - \sigma(z_j^L)\right)
\]

- Starting from $l = L - 1$ and working backwards, compute

\[
\delta^l = \left((\delta^{l+1})^T w^{l+1}\right) \sigma(z^l) \left(1 - \sigma(z^l)\right)
\]

- Perform gradient descent

\[
b_j^l = b_j^l - \gamma \cdot \delta_j^l
\]

\[
w_{jk}^l = w_{jk}^l - \gamma \cdot \delta_j^l a_k^{l-1}
\]
Backpropagation

- Backpropagation converges to a local minimum (loss is not convex in the weights and biases)
  - Like EM, can just run it several times with different initializations
  - Training can take a very long time (even with stochastic gradient descent)
  - Prediction after learning is fast
  - Sometimes include a momentum term $\alpha$ in the gradient update

$$w_{jk}^l(t) = w_{jk}^l(t-1) - \gamma \cdot \nabla_w C(t-1) + \alpha(-\gamma \cdot \nabla_w C(t-2))$$
Overfitting

Error versus weight updates (example 1)

Training set error
Validation set error

Number of weight updates

Error

0.01

0.009

0.008

0.007

0.006

0.005

0.004

0.003

0.002

0 5000 10000 15000 20000

34
Overfitting

Error versus weight updates (example 2)

Training set error
Validation set error

Number of weight updates
Neural Networks in Practice

• Many ways to improve weight learning in NNs
  – Use a regularizer! (better generalization)
  – Try other loss functions
  – Initialize the weights of the network more cleverly
    • Random initializations are likely to be far from optimal
  – etc.

• The learning procedure can have numerical difficulties if there are a large number of layers
Regularized Loss

- Penalize learning large weights

\[ C'(w, b) = \frac{1}{2M} \sum_{m} \|y^m - a(x^m, w, b)\|^2 + \frac{\lambda}{2} \|w\|^2 \]

- Can still use the backpropagation algorithm in this setting

- \( \ell_1 \) regularization can also be useful

- Regularization can significantly help with overfitting, but \( \lambda \) will often need to be quite large as the size of the training set is typically much larger than what we have been working with

  - How to choose \( \lambda \)?
• A heuristic bagging-style approach applied to neural networks to counteract overfitting

  – Randomly remove a certain percentage of the neurons from the network and then train only on the remaining neurons

  – The networks are recombined using an approximate averaging technique (keeping around too many networks and doing proper bagging is too costly in practice)
Other Techniques

• Early stopping
  – Stop the learning early in the hopes that this prevents overfitting

• Parameter tying
  – Assume some of the weights in the model are the same to reduce the dimensionality of the learning problem
  – Also a way to learn “simpler” models
Other Ideas

• Convolutional neural networks
  – Instead of the output of every neuron at layer $l$ being used as an input to every neuron at layer $l + 1$, the edges between layers are chosen more locally
  – Many tied weights and biases (i.e., convolution nets apply the same process to many different local chunks of neurons)
  – Often combined with pooling layers (i.e., layers that, say, half the number of neurons by replacing small regions of neurons with their maximum output)
  – Used extensively in neural nets for image classification tasks