GPS GDOP metric

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Abstract: The authors present a review of the GDOP metric as used in GPS. Their goal is to review this metric and many of its known bounds as well as to report some new results. They use a formal linear algebraic framework to aid further study and insight.

1 Introduction

With the advent of the Global Positioning System (GPS), it has been necessary to select a calculus for determining what is a good selected satellite geometry and what constitutes a poor choice. The most popular metric today is a dimensionless single number termed the geometric dilution of precision (GDOP).

This paper is a review of the GDOP metric as used in GPS. Our goal is to review this metric and many of its known bounds as well as to report some new results. We use a formal linear algebraic framework to aid further study and insight.

While this paper addresses the GPS issue, it should be noted that much valuable work has been done on this and similar metrics for geolocation systems predating GPS. The authors especially wish to cite fundamental work done by H. B. Lee [1–3]. His papers imbue the reader with the fundamental insights and provide many useful approximations and bounds on accuracy. His third paper [3] also provides insight regarding the incorporation of ancillary sensor measurements such as barometric altitude estimates.

One final note of introduction concerns terminology. The GPS question involves pseudoranges exclusively. The GDOP concept which had come into being before GPS, did not, of course, include an estimate of time offset error. In the non-exclusive GPS applications, the term GDOP should be replaced by the term PDOP, (position dilution of precision).

2 Dilution of precision

Consider the user position vector representation where the vector $\boldsymbol{s}$ represents the vector from the Earth's centre to a satellite, $\boldsymbol{u}$ represents the vector from the Earth's centre to a user's position, and $\boldsymbol{r}$ represents vector offset from the user to the satellite, and is related to $\boldsymbol{s}$ and $\boldsymbol{u}$ by

$$\boldsymbol{r} = \boldsymbol{s} - \boldsymbol{u} \tag{1}$$

The distance $\|\boldsymbol{r}\|$ is computed by measuring the propagation time from the transmitting satellite to the receiver. The pseudorange, ρ_j, is defined for the jth satellite by

$$\rho_j = \|\boldsymbol{s}_j - \boldsymbol{u}\| + ct_b \tag{2}$$

where c is the speed of light and t_b is the receiver clock offset or bias from system time.

The user position in three dimensions is denoted by (x_u, y_u, z_u) and the time offset is denoted by t_b. Pseudorange measurements are made to four satellites resulting in

$$\rho_j = \sqrt{(x_j - x_u)^2 + (y_j - y_u)^2 + (z_j - z_u)^2} + ct_b, \quad j = 1, 2, 3, 4 \tag{3}$$

where (x_j, y_j, z_j) denotes the jth satellite's position in three dimensions. Eqns. 3 are linearised by using Taylor's series around the approximate user position $(\widehat{x_u}, \widehat{y_u}, \widehat{z_u})$ and neglecting the higher order terms. Defining $\widehat{\rho_j}$ as ρ_j at $(\widehat{x_u}, \widehat{y_u}, \widehat{z_u})$, we can write

$$\begin{aligned}\Delta\rho_j &= \rho_j - \widehat{\rho_j} \\ &= a_{x_j}\Delta x_u + a_{y_j}\Delta y_u + a_{z_j}\Delta z_u - ct_b\end{aligned} \tag{4}$$

where

$$\begin{aligned} a_{\xi_j} &= \frac{\xi_j - \hat{\xi}_j}{\hat{r}_j}, \ \xi = x, y, z \\ \hat{r}_j &= \sqrt{(x_j - \widehat{x_u})^2 + (y_j - \widehat{y_u})^2 + (z_j - \hat{z}_u)^2} \\ \Delta\xi_u &= \xi_u - \widehat{\xi_u}, \ \xi = x, y, z \end{aligned} \tag{5}$$

We can write the above in a compact matrix formulation as

$$\Delta\rho = \begin{pmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \Delta\rho_3 \\ \Delta\rho_4 \end{pmatrix} \approx \begin{pmatrix} a_{x_1} & a_{y_1} & a_{z_1} & 1 \\ a_{x_2} & a_{y_2} & a_{z_2} & 1 \\ a_{x_3} & a_{y_3} & a_{z_3} & 1 \\ a_{x_4} & a_{y_4} & a_{z_4} & 1 \end{pmatrix} \begin{pmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ -c\Delta t_b \end{pmatrix} \tag{6}$$

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or

$$\Delta\boldsymbol{\rho} \approx \boldsymbol{H}\Delta\boldsymbol{x} \tag{7}$$

The matrix $\boldsymbol{H}$, sometimes referred to as the 'visibility matrix' [4], is in general an $n \times 4$ matrix with $n \geq 4$. The pseudorange errors are generally considered random as are $\boldsymbol{dx}$ and $\boldsymbol{d\rho}$. These latter random variables are functionally related as

$$\boldsymbol{d\rho} = \boldsymbol{Hdx} + \boldsymbol{e} \tag{8}$$

where $\boldsymbol{e}$ is an error vector. It is generally assumed that $E[\boldsymbol{e}] = 0$.

Eqn. 8 can be solved by minimising the least squared error $\|\boldsymbol{e}\|^2$. The minimisation of the least squared error assumes that the error is Gaussian. It has been shown [5] that measurements made by GPS P-code receivers are not Gaussian and other possible density functions based on kurtosis have been discussed [6]. For users near the Earth, the solutions are claimed as likely nearly Gaussian. As developed in [7], the least squares solution for general $n \geq 4$ is given by:

$$\boldsymbol{dx} = \boldsymbol{H}^{-}\boldsymbol{d\rho} \tag{9}$$

where $\boldsymbol{H}^{-}$ is called the pseudoinverse and defined as

$$\boldsymbol{H}^{-} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T \tag{10}$$

The reviewer points out that this least squares solution is not the optimal solution if the $cov(\boldsymbol{d\rho})$ is not a constant times an identity matrix. The reviewer notes that in that case a weighted least squares solution will be better, if the weight matrix is chosen as the inverse of the measurement error covariance matrix.

For the underdetermined system, i.e. where $n < 4$, we designate $\boldsymbol{H}_r = \boldsymbol{H}$ and, from [8], we can write

$$\boldsymbol{H}_r^{-} = \boldsymbol{H}_r^T(\boldsymbol{H}_r\boldsymbol{H}_r^T)^{-1} \tag{11}$$

Clearly fewer than four satellites will not provide enough information for an acceptable position fix, in general, but other sensor data may be fused with available pseudorange data such as altimeter information or accurate clock information.

The covariance of the vector $\boldsymbol{dx}$ can be obtained as

$$\begin{aligned} cov(\boldsymbol{dx}) &= E[\boldsymbol{dxdx}^T] \\ &= E[\boldsymbol{H}^{-}(\boldsymbol{d\rho d\rho}^T)(\boldsymbol{H}^{-})^T] \\ &= \boldsymbol{H}^{-}cov(\boldsymbol{d\rho})(\boldsymbol{H}^{-})^T \end{aligned} \tag{12}$$

where we assumed that the user–satellite geometry is fixed. This is a reasonable assumption for short time intervals.

For i.i.d. range error of variance σ_{uere}^2, the covariance may be expressed in a convenient matrix form as

$$cov(\boldsymbol{d\rho}) = \boldsymbol{K}_n\sigma_{uere}^2 \tag{13}$$

where $\boldsymbol{K}_n$ is a symmetric positive definite matrix and σ_{uere}^2 is the user equivalent range error variance. Let

$$cov(\boldsymbol{dx}) = \begin{pmatrix} \sigma_{x_u}^2 & \cdot & \cdot & \cdot \\ \cdot & \sigma_{y_u}^2 & \cdot & \cdot \\ \cdot & \cdot & \sigma_{z_u}^2 & \cdot \\ \cdot & \cdot & \cdot & \sigma_{ct_b}^2 \end{pmatrix} \tag{14}$$

where the off-diagonal entries are not critical to the following discussion. The GDOP is given by

$$GDOP = \sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2 + \sigma_{ct_b}^2}/\sigma_{uere} \tag{15}$$

which represents the amplification of the standard deviation of the measurement errors onto the position solution. Note that

$$GDOP = GDOP(cov(dx)) \tag{16}$$

Other dilution of precision parameters include:

- PDOP (position DOP)

$$PDOP = \sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2}/\sigma_{uere} \tag{17}$$

- HDOP (horizontal DOP)

$$HDOP = \sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2}/\sigma_{uere} \tag{18}$$

- VDOP (vertical DOP)

$$VDOP = \sigma_{z_u}/\sigma_{uere} \tag{19}$$

- TDOP (time DOP)

$$TDOP = \sigma_{ct_b}/\sigma_{uere} \tag{20}$$

For a thorough discussion of these parameters, see [9].

The GDOP is often associated with a baseline or 'viewing volume'. We may define a volume for a four-satellite case [10]. This volume, V, is that of the tetrahedron formed by connecting the three satellite positions with the fourth satellite's position at the user's zenith. The PDOP is inversely proportional to this volume (See also [8, 9] for further discussion), i.e.

$$PDOP \propto \frac{1}{V} \tag{21}$$

We can generalise the dimensionality and visualise this volume using an elegant multidimensional integral [11],

$$\begin{aligned} J_n &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x}^T} \boldsymbol{dx} \\ &= \sqrt{\frac{\pi^n}{|\boldsymbol{R}|}} \end{aligned} \tag{22}$$

where $\boldsymbol{x}$ is an n-dimensional vector and $\boldsymbol{R}$ is a positive definite matrix of order n.

In our present case, the viewing volume is proportional to J_n and therefore

$$PDOP \propto \sqrt{|(\boldsymbol{H}^T\boldsymbol{H})^{-1}|} \tag{23}$$

It follows, therefore, that to minimise PDOP, or, in general GDOP, we need to maximise $|\boldsymbol{H}^T\boldsymbol{H}|$.

3 GDOP bounds

We first consider the ideal case wherein

$$cov(\boldsymbol{d\rho}) = \boldsymbol{I}_n\sigma_{uere}^2 \tag{24}$$

where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix. For $n \geq 4$ we have

$$\begin{aligned} cov(\boldsymbol{dx}) &= (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{H}(\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{I}_n\sigma_{uere}^2 \\ &= (\boldsymbol{H}^T\boldsymbol{H})^{-1}\sigma_{uere}^2 \end{aligned} \tag{25}$$

Let

$$\begin{aligned} \boldsymbol{B} &= Diag(\boldsymbol{H}^T\boldsymbol{H})^{-1} \\ &= diag(b_{11}, b_{22}, b_{33}, b_{44}) \end{aligned} \tag{26}$$

then we can write

$$GDOP = \sqrt{b_{11} + b_{22} + b_{33} + b_{44}} = \sqrt{Tr(\boldsymbol{H}^T\boldsymbol{H})^{-1}},$$

- $PDOP = \sqrt{b_{11} + b_{22} + b_{33}}$,
- $HDOP = \sqrt{b_{11} + b_{22}}$,
- $VDOP = \sqrt{b_{11}}$,
- $TDOP = \sqrt{b_{44}}/c$,

where for the four satellite case

$$\boldsymbol{H}_4 = (a_x \ a_y \ a_z \ \mathbf{1}) \tag{27}$$

and where a_x, a_y, and a_z are unit vectors pointing from the linearisation point to the location of the ith satellite.

If the number of satellites considered is beyond four, the $\boldsymbol{H}$ matrix can be successively augmented by adding row vectors. For example, if $n=5$,

$$\boldsymbol{H}_5 = \begin{pmatrix} \boldsymbol{H}_4 \\ \boldsymbol{h} \end{pmatrix} \tag{28}$$

in which we are assuming that $\boldsymbol{H}_4$ is nonsingular and $\boldsymbol{h}$ is a nonzero vector.

It seems reasonable to expect that using more than four satellites can only work to diminish the GDOP and not increase it. We can straightforwardly investigate this. First, clearly

$$\boldsymbol{H}_5^T\boldsymbol{H}_5 = (\boldsymbol{H}_4^T\boldsymbol{H}_4) + \boldsymbol{h}^T\boldsymbol{h} \tag{29}$$

Now let

$$\boldsymbol{H}_4^T\boldsymbol{H}_4 = \boldsymbol{P}^T\Lambda\boldsymbol{P} \tag{30}$$

where $\boldsymbol{P}$ is an orthogonal matrix and Λ is a diagonal matrix. It follows that

$$\begin{aligned}\boldsymbol{P}^T(\boldsymbol{H}_5^T\boldsymbol{H}_5)\boldsymbol{P} &= \boldsymbol{P}^T(\boldsymbol{H}_4^T\boldsymbol{H}_4)\boldsymbol{P} + (\boldsymbol{P}^T\boldsymbol{h}^T)(\boldsymbol{h}\boldsymbol{P}) \\ &= \Lambda + \boldsymbol{\alpha}^T\boldsymbol{\alpha} \\ &= diag(\lambda_{11}, \lambda_{22}, \lambda_{33}, \lambda_{44}) + \boldsymbol{\alpha}^T\boldsymbol{\alpha}\end{aligned} \tag{31}$$

where

$$\boldsymbol{\alpha} = \boldsymbol{h}\boldsymbol{P} \tag{32}$$

Consider the form

$$(\boldsymbol{D} + \boldsymbol{\alpha}^T\boldsymbol{\alpha})^{-1} = \boldsymbol{D}^{-1} - v(\boldsymbol{\alpha}^*)^T(\boldsymbol{\alpha}^*) \tag{33}$$

with

$$\boldsymbol{\alpha} = \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{pmatrix} \tag{34}$$

$$(\boldsymbol{\alpha}^*) = \begin{pmatrix}\alpha_1/\lambda_{11} & \alpha_2/\lambda_{22} & \alpha_3/\lambda_{33} & \alpha_4/\lambda_{44}\end{pmatrix} \tag{35}$$

$$v = \frac{1}{1 + \sum_{i=1}^{4} \frac{\alpha_i^2}{\lambda_{ii}}} \tag{36}$$

It is clear that v is positive (and assumed to be finite), $\boldsymbol{D}$ is a positive definite diagonal matrix, $(\boldsymbol{\alpha}^*)^T(\boldsymbol{\alpha}^*)$ is a positive semidefinite matrix and $\boldsymbol{H}_5^T\boldsymbol{H}_5$ is a positive definite matrix. From this it follows that $\boldsymbol{D}^{-1} - v(\boldsymbol{\alpha}^*)^T(\boldsymbol{\alpha}^*)$ is positive definite and that

$$Tr(\boldsymbol{D} + \boldsymbol{\alpha}\boldsymbol{\alpha}^T)^{-1} = Tr(\boldsymbol{D}^{-1})\left[v(\boldsymbol{\alpha}^*)^T(\boldsymbol{\alpha}^*)\right]. \tag{37}$$

Using the material in the Appendix (Section 6.1), we can write

$$\begin{aligned}Tr(\boldsymbol{H}_5^T\boldsymbol{H}_5)^{-1} &= Tr(\boldsymbol{D} + \boldsymbol{\alpha}\boldsymbol{\alpha}^T)^{-1} \\ &= Tr(\boldsymbol{D}^{-1}) - Tr\left[v(\boldsymbol{\alpha}^*)^T(\boldsymbol{\alpha}^*)\right]\end{aligned} \tag{38}$$

where

$$Tr(\boldsymbol{D}^{-1}) = Tr\left[(\boldsymbol{H}_4^T\boldsymbol{H}_4)^{-1}\right] \tag{39}$$

and

$$Tr\left[v(\boldsymbol{\alpha}^*)^T(\boldsymbol{\alpha}^*)\right] = v\sum_{i=1}^{4}\left(\frac{\alpha_i}{\lambda_{ii}}\right)^2 = \frac{\sum_{i=1}^{4}\left(\frac{\boldsymbol{\alpha}_i}{\lambda_{ii}}\right)^2}{1 + \sum_{i=1}^{4}\left(\frac{\alpha_i}{\lambda_{ii}}\right)^2\lambda_{ii}} \tag{40}$$

which proves the assertion that increasing the number of satellites will only reduce the GDOP. From eqn. 38 it can be seen that the trace of $(\boldsymbol{H}_n^T\boldsymbol{H}_n)^{-1}$ decreases. See [6] for additional discussion.

The GDOP is obtained from the trace of $(\boldsymbol{H}^T\boldsymbol{H})^{-1}$. Let

$$\begin{aligned}\boldsymbol{A} &= (\boldsymbol{H}^T\boldsymbol{H}) \\ &= \begin{pmatrix} a_{x_1} & a_{x_2} & a_{x_3} & a_{x_4} \\ a_{y_1} & a_{y_2} & a_{y_3} & a_{y_4} \\ a_{z_1} & a_{z_2} & a_{z_3} & a_{z_4} \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_{x_1} & a_{y_1} & a_{z_1} & 1 \\ a_{x_2} & a_{y_2} & a_{z_2} & 1 \\ a_{x_3} & a_{y_3} & a_{z_3} & 1 \\ a_{x_4} & a_{y_4} & a_{z_4} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^{4} a_{x_i}^2 & \cdot & \cdot & \cdot \\ \cdot & \sum_{i=1}^{4} a_{y_i}^2 & \cdot & \cdot \\ \cdot & \cdot & \sum_{i=1}^{4} a_{z_i}^2 & \cdot \\ \cdot & \cdot & \cdot & 4 \end{pmatrix} = \begin{pmatrix} a_{11} & \cdot & \cdot & \cdot \\ \cdot & a_{22} & \cdot & \cdot \\ \cdot & \cdot & a_{33} & \cdot \\ \cdot & \cdot & \cdot & a_{44} \end{pmatrix}\end{aligned} \tag{41}$$

Note that the $\{a_{\xi_i}\}$, $\xi = x, y, z$, are direction cosines and therefore $a_{ii} \le 4$. To find some coarse bounds on the diagonal entries of $(\boldsymbol{H}^T\boldsymbol{H})^{-1}$, we consider a theorem from [7]: Let A be an $n \times n$ symmetric positive definite matrix and let $\boldsymbol{B} = \boldsymbol{A}^{-1}$. Then the diagonal entries of $\boldsymbol{A}$ and $\boldsymbol{B}$, $\{a_{ii}\}$ and $\{b_{ii}\}$ respectively, satisfy the inequality $a_{ii}b_{ii} \ge 1$. Note that this theorem is not valid if $\boldsymbol{A}$ is a positive semidefinite matrix. For example, if

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

then

$$\boldsymbol{B} = \boldsymbol{A}^{-} = \begin{pmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{pmatrix}$$

and $a_{ii}\, b_{ii} \not\ge 1$.

From this theorem, it follows that since

$$GDOP = \sqrt{b_{11} + b_{22} + b_{33} + b_{44}}, \tag{42}$$

then

$$GDOP \geq \sqrt{\frac{1}{a_{11}} + \frac{1}{a_{22}} + \frac{1}{a_{33}} + \frac{1}{a_{44}}} \tag{43}$$

$$PDOP \geq \sqrt{\frac{1}{a_{11}} + \frac{1}{a_{22}} + \frac{1}{a_{33}}} \tag{44}$$

$$HDOP \geq \sqrt{\frac{1}{a_{11}} + \frac{1}{a_{22}}} \tag{45}$$

$$VDOP \geq \sqrt{\frac{1}{a_{11}}} \tag{46}$$

and

$$TDOP \geq \frac{1}{2c} \tag{47}$$

Clearly the $\{a_{ii}\}$ are functions of the direction cosines and $a_{ii} \leq 4$. The bounds require the knowledge of the direction cosines. We will now consider a sharper bound for the GDOP without the knowledge of exact values of the direction cosines.

If $\boldsymbol{H}$ and $\boldsymbol{H}^T$ are nonsingular for the four satellite case, then

$$(\boldsymbol{H}^T\boldsymbol{H})^{-1} = \boldsymbol{H}^{-1}(\boldsymbol{H}^T)^{-1} \tag{48}$$

From this, as detailed in the Appendix (Section 6.2), we can obtain the following,

$$Tr(\boldsymbol{H}\boldsymbol{H}^T)^{-1} = Tr(\boldsymbol{H}^T\boldsymbol{H})^{-1} \tag{49}$$

Now,

$$\boldsymbol{H}\boldsymbol{H}^T = \begin{pmatrix} a_{x_1} & a_{y_1} & a_{z_1} & 1 \\ a_{x_2} & a_{y_2} & a_{z_2} & 1 \\ a_{x_3} & a_{y_3} & a_{z_3} & 1 \\ a_{x_4} & a_{y_4} & a_{z_4} & 1 \end{pmatrix} \begin{pmatrix} a_{x_1} & a_{x_2} & a_{x_3} & a_{x_4} \\ a_{y_1} & a_{y_2} & a_{y_3} & a_{y_4} \\ a_{z_1} & a_{z_2} & a_{z_3} & a_{z_4} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \end{pmatrix} \tag{50}$$

where we have used the fact that $a_{x_i}^2 + a_{y_i}^2 + a_{z_i}^2 = 1$, i.e. they are direction cosines. Now let

$$(\boldsymbol{H}\boldsymbol{H}^T)^{-1} = \begin{pmatrix} v_{11} & \cdot & \cdot & \cdot \\ \cdot & v_{22} & \cdot & \cdot \\ \cdot & \cdot & v_{33} & \cdot \\ \cdot & \cdot & \cdot & v_{44} \end{pmatrix} \tag{51}$$

Using the previously cited theorem from [7], it follows that

$$v_{ii} \geq \tfrac{1}{2}, i = 1, 2, 3, 4. \tag{52}$$

Using eqn. 49, we then have

$$GDOP \geq \sqrt{2} \tag{53}$$

for four satellites. With more satellites, of course, the GDOP would have a lower bound.

Another lower bound on the GDOP can be derived using the fact that the geometric average of a set of real numbers is always upper bounded by their arithmetic average.

We proceed by denoting the four eigenvalues of $\boldsymbol{H}^T\boldsymbol{H}$ by λ_i, $i = 1, 2, 3, 4$. Then,

$$\begin{aligned} GDOP &= \sqrt{Tr(\boldsymbol{H}^T\boldsymbol{H})^{-1}} \\ &= \sqrt{\sum_i^4 \frac{1}{\lambda_i}} \\ &= 2\sqrt{\frac{1}{4}\sum_{i=1}^4 \frac{1}{\lambda_i}} \\ &\geq 2\sqrt{\left[\prod_{i=1}^4 \frac{1}{\lambda_i}\right]^{1/4}} \\ &= \frac{2}{|\boldsymbol{H}^T\boldsymbol{H}|^{1/8}} \end{aligned} \tag{54}$$

with equality achieved if and only if the condition number of $\boldsymbol{H}^T\boldsymbol{H}$ is unity.

Using eqns. 44–47 with eqn. 52, we may evaluate bounds on the other DOPs for $n = 4$. For example,

$$GDOP \geq \sqrt{\beta + \frac{1}{4}}\quad , \quad \beta = \sum_{i=1}^3 \frac{1}{a_{ii}}$$
$$\beta \leq \left[(GDOP)^2 - \frac{1}{4}\right] = \frac{7}{4} \tag{55}$$

Since $PDOP \geq \sqrt{\beta} \leq 7/4$, we can conclude that $\min(PDOP) \approx 1.323$.

As the calculation of the GDOP requires knowledge of only a diagonal elements of $(\boldsymbol{H}^T\boldsymbol{H})^{-1}$, the GDOP computational complexity is significantly reduced by avoiding a full computation of $(\boldsymbol{H}^T\boldsymbol{H})^{-1}$. We achieve this objective by the following formulation.

Again, we let $\boldsymbol{A} = \boldsymbol{H}^T\boldsymbol{H}$ and, for $n = 4$, we write

$$\begin{aligned} GDOP &= \sqrt{Tr\boldsymbol{A}^{-1}} \\ &= \sqrt{\sum_{i=1}^4 \boldsymbol{A}_{ii}^{-1}} \\ &= \sqrt{\frac{1}{|\boldsymbol{A}|}\sum_{i=1}^4 [Adj(\boldsymbol{A})]_{ii}} \\ &= \sqrt{\frac{1}{|\boldsymbol{A}|}\sum_{i=1}^4 |\overline{\boldsymbol{A}}^{ii}|} \end{aligned} \tag{56}$$

where Adj($\boldsymbol{A}$) is the adjoint of $\boldsymbol{A}$ whose (i, i) element is given by the determinant of the 3×3 matrix $\overline{\boldsymbol{A}}^{ii}$ which is obtained by deleting the ith row and column of $\boldsymbol{A}$ (also called the minor of $\boldsymbol{A}$). Since the determinant of the 4×4 matrix $\boldsymbol{A}$ can also be expressed as a weighted sum of 3×3 determinants, the whole GDOP calculation reduces to evaluating 3×3 determinants—a simple closed form or recursively computable in terms of 2×2 determinants.

We now move on to a more general case wherein

$$GDOP = \sqrt{Tr[\boldsymbol{H}^{-}\boldsymbol{K}_n(\boldsymbol{H}^T)^{-}]} \tag{57}$$

Using the results in Section 6.1, we have

$$GDOP = \sqrt{Tr[\boldsymbol{K}_n(\boldsymbol{H}\boldsymbol{H}^T)^{-}]} \tag{58}$$

where the diagonal entries of $\boldsymbol{K}_n$ are unequal and perhaps a function of a parameter such as signal to noise, i.e.

$$(\boldsymbol{K}_n)_{ii} = f(\zeta_i). \tag{59}$$

Clearly the larger $(\boldsymbol{K}_n)_{ii}$ is, the larger the GDOP becomes. Let us consider a simple model for $\boldsymbol{K}_n$, i.e.

$$\boldsymbol{K}_n = \Lambda_n + \epsilon \mathbf{1}_n \mathbf{1}_n^T \tag{60}$$

where Λ_n is a diagonal matrix formed from the values in eqn. 59, ϵ is a small number, and $\mathbf{1}_n$ is an n-dimensional column vector of ones.

Now,

$$Tr[\boldsymbol{K}_n(\boldsymbol{HH}^T)^-] = Tr[\Lambda_n(\boldsymbol{HH}^T)^-] + \epsilon Tr[(\mathbf{1}_n\mathbf{1}_n^T)(\boldsymbol{HH}^T)^-] \tag{61}$$

Using results in Section 6.3 the second part of the equation above can be simplified and we obtain

$$Tr[(\mathbf{1}_n\mathbf{1}_n^T)(\boldsymbol{HH}^T)^-] = 1 \tag{62}$$

Therefore

$$\begin{aligned} GDOP &= \sqrt{Tr[\Lambda_n(\boldsymbol{HH}^T)^- + \epsilon]} \\ &\approx \sqrt{Tr[\Lambda_n(\boldsymbol{HH}^T)^-]} \end{aligned} \tag{63}$$

for sufficiently small ϵ. Using eqn. 70, we may write

$$GDOP = \sqrt{Tr(\Lambda_n)Tr(\boldsymbol{HH}^T)^-} \tag{64}$$

Finally, noting that

$$Tr[\Lambda_n(\boldsymbol{HH}^T)^-] = \sum_{i=1}^{n}(\Lambda_n)_{ii}(\boldsymbol{HH}^T)^-_{ii} \tag{65}$$

we see that the contribution of the ith satellite to the GDOP is weighted by $(\Lambda_n)_{ii}$.

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6 Appendix

6.1 Traces of matrix products

Most of the following results are well known or can be easily derived.

A1: Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ be an $n \times m$ matrix. Then

$$\begin{aligned} \boldsymbol{T}_{AB} &= Tr(\boldsymbol{AB}) \\ &= Tr(\boldsymbol{BA}) \\ &= \boldsymbol{T}_{BA} \end{aligned} \tag{66}$$

which can be shown by noting

$$\begin{aligned} \boldsymbol{T}_{AB} &= (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) \\ &\quad + (a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2}) \\ &\quad \vdots \\ &\quad + (a_{m1}b_{1m} + a_{m2}b_{2m} + \cdots + a_{mn}b_{nm}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji} \end{aligned} \tag{67}$$

$$\begin{aligned} \boldsymbol{T}_{BA} &= (b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1m}a_{m1}) \\ &\quad + (b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2m}a_{m2}) \\ &\quad \vdots \\ &\quad + (b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nm}a_{mn}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}a_{ji} \end{aligned} \tag{68}$$

A2: If $\boldsymbol{P}$ is an orthogonal matrix, then

$$Tr(\boldsymbol{P}^T\boldsymbol{SP}) = Tr(\boldsymbol{S}) \tag{69}$$

A3: Let $\boldsymbol{D}$ be an nth order diagonal matrix with positive entries and let $\boldsymbol{A}$ be an $m \times n$ positive definite matrix. Then

$$Tr(\boldsymbol{DA}) < Tr(\boldsymbol{D})Tr(\boldsymbol{A}) \tag{70}$$

This is proved by induction. The result is obviously true for $n = 1$. Assume the result is true for $n - 1$ and prove the result for n. Let

$$\boldsymbol{DA} = \begin{pmatrix} \boldsymbol{D}_{11} & 0 \\ 0 & d_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & a_{22} \end{pmatrix} \tag{71}$$

where $\boldsymbol{D}_{11}$ and $\boldsymbol{A}_{11}$ are both of dimension $(n-1) \times (n-1)$.

In the induction process we have

$$Tr(\boldsymbol{D}_{11})Tr(\boldsymbol{A}_{11}) > Tr(\boldsymbol{D}_{11}\boldsymbol{A}_{11}) \tag{72}$$

Now

$$\begin{aligned} Tr(\boldsymbol{DA}) &= Tr(\boldsymbol{D}_{11}\boldsymbol{A}_{11}) + Tr(d_{22}a_{22}) \\ Tr(\boldsymbol{D}) &= Tr(\boldsymbol{D}_{11}) + d_{22} \\ Tr(\boldsymbol{A}) &= Tr(\boldsymbol{A}_{11}) + a_{22} \end{aligned} \tag{73}$$

$$\begin{aligned} Tr(\boldsymbol{D})Tr(\boldsymbol{A}) &= (Tr(\boldsymbol{D}_{11}) + d_{22})(Tr(\boldsymbol{A}_{11}) + a_{22}) \\ &= [Tr(\boldsymbol{D}_{11})Tr(\boldsymbol{A}_{11}) + d_{22}a_{22}] \\ &\quad + [Tr(\boldsymbol{D}_{11})a_{22} + Tr(\boldsymbol{A}_{11})d_{22}] \\ &> [Tr(\boldsymbol{D}_{11}\boldsymbol{A}_{11}) + d_{22}a_{22}] \end{aligned} \tag{74}$$

This is true since the second term $[Tr\ (\boldsymbol{D}_{11})a_{22} + Tr(\boldsymbol{A}_{11})\ d_{22}]$ is always positive due to the fact that $\boldsymbol{D}$ and $\boldsymbol{A}$ are positive definite.

A4: Let $\boldsymbol{S}$ and $\boldsymbol{B}$ be $n \times n$ symmetric positive definite matrices, then

$$Tr(SB) < Tr(S)Tr(B) \tag{75}$$

This follows by writing $\boldsymbol{S} = \boldsymbol{P}^T\boldsymbol{D}\boldsymbol{P}$ where $\boldsymbol{P}$ is an orthogonal matrix. Then

$$\begin{aligned} Tr(\boldsymbol{SB}) &= Tr(\boldsymbol{P}^T\boldsymbol{D}\ \boldsymbol{P}\ \boldsymbol{B}) \\ &= Tr(\boldsymbol{D}\ \boldsymbol{P}\ \boldsymbol{B}\ \boldsymbol{P}^T) \\ &= Tr(\boldsymbol{DA}) \end{aligned} \tag{76}$$

$$Tr(\boldsymbol{DA}) < Tr(\boldsymbol{D})T(\boldsymbol{A}) \tag{77}$$

and the result follows.

Similar results can be obtained by replacing ($<$) by ($\leq$) for the positive semidefinite case.

6.2 Generalised inverses

In this Appendix we will consider the basic definition of a generalised inverse and some important properties that are useful in our study.

B1: Let A be an $m \times n$ matrix. We will denote the generalised inverse of $\boldsymbol{A}$ as $\boldsymbol{A}^-$. This generalized or pseudo-inverse has the dimensions $n \times m$. It satisfies the following properties:

(i) $\boldsymbol{AA}^-$ is symmetric,
(ii) $\boldsymbol{A}^-\boldsymbol{A}$ is symmetric,
(iii) $\boldsymbol{AA}^-\boldsymbol{A} = \boldsymbol{A}$,
(iv) $\boldsymbol{A}^-\boldsymbol{AA}^- = \boldsymbol{A}^-$

For each matrix $\boldsymbol{A}$, there exists an $\boldsymbol{A}^-$ that is unique. The inverse of an $m \times n$ null matrix is an $n \times m$ null matrix. The generalized inverse of $(\boldsymbol{AB})$ is $(\boldsymbol{AB})^- = \boldsymbol{B}^-\boldsymbol{A}^-$.

In our work, the $\boldsymbol{H}$ matrix is an $n \times k$ matrix and it has a rank k. The generalised inverse of $\boldsymbol{H}$ is given by

$$\boldsymbol{H}^- = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T \tag{78}$$

B2: Let **H** be an $n \times k$ matrix with $n \geq k$, then

$$Tr(\boldsymbol{H}^T\boldsymbol{H}) = Tr(\boldsymbol{HH}^T)^- \tag{79}$$

This follows from $(\boldsymbol{HH}^T) = (\boldsymbol{H}^T)^-(\boldsymbol{H}^-)$. Using results from Section 6.1, we have

$$\begin{aligned} Tr\big[(\boldsymbol{H}^T)^-(\boldsymbol{H}^-)\big] &= Tr\big[\boldsymbol{H}^-(\boldsymbol{H}^T)^-\big] \\ &= Tr\big[(\boldsymbol{H}^T\boldsymbol{H})^{-1}\big] \end{aligned} \tag{80}$$

which proves the theorem.

6.3 Some interesting properties of H matrices

C1: The **H** matrix defined earlier has some interesting properties. Some of these are discussed below.

Let **H** be an $n \times 4$ matrix. Note that

$$\boldsymbol{H}_n \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{81}$$

which follows from the fact that the last column of $\boldsymbol{H}_n$ contains all ones. Therefore,

$$\boldsymbol{H}_n^- \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{82}$$

Furthermore,

$$\mathbf{1}_n^T(\boldsymbol{H}_n\boldsymbol{H}_n^T)^-\mathbf{1}_n = 1 \tag{83}$$

where $\mathbf{1}_4$ is a four-dimensional column vector of ones.

The above equation can be seen by noting that

$$(\boldsymbol{H}_n\boldsymbol{H}_n^T)^- = (\boldsymbol{H}_n^T)^-\boldsymbol{H}_n^{-1} \tag{84}$$

and

$$\mathbf{1}_n^T(\boldsymbol{H}_n^T)^-\boldsymbol{H}_n^-\mathbf{1}_n = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1 \tag{85}$$

C2: (regarding the sign pattern of $\boldsymbol{HH}^T$)

The ith row of $\boldsymbol{H}$ is given by

$$\boldsymbol{h}_i^T = \begin{pmatrix} \sin\theta_i\cos\phi_i & \sin\theta_i\sin\phi_i & \cos\theta_i & 1 \end{pmatrix} \tag{86}$$

and

$$\begin{aligned} \boldsymbol{h}_i^T\boldsymbol{h}_j^T &= \sin\theta_i\cos\phi_i\sin\theta_j\cos\phi_j \\ &\quad + \sin\theta_i\sin\phi_i\sin\theta_j\sin\phi_j + \cos\theta_i\cos\theta_j + 1 \qquad (87) \\ &= \sin\theta_i\sin\theta_j\big[\cos(\phi_i + \phi_j)\big] + \cos\theta_i\cos\theta_j + 1 \end{aligned}$$

Since θ_i and θ_j are elevation angles and therefore $0 \leq |\theta_i| \leq \pi/2$ for all i which implies that

$$\boldsymbol{h}_i^T\boldsymbol{h}_j \geq 1 \tag{88}$$

Note that the first term in the last line of eqn. 87 is always less than or equal to unity.