A Closed-Form Performance Upper-Bound on the Finite-Length MMSE-DFE with Arbitrary Decision Delay

Naofal Al-Dhahir *, Senior Member IEEE

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Abstract

A simple analytical upper–bound on the decision–point SNR of the unbiased finite–length MMSE–DFE with arbitrary decision delay is derived. The bound can be easily computed given the channel impulse response, input SNR, length of feedforward filter, and desired decision delay. The bound is shown to be tighter than the matched–filter bound. A by–product is a computationally–efficient method for setting the decision delay.

I. Introduction

The minimum–mean–square–error decision feedback equalizer (MMSE–DFE) is receiving considerable attention as an effective receiver structure for broadband packet–based transmission in a multipath wireless environment (see e.g. [6, 5]). For such high–sampling–rate scenarios, significant reductions in DFE implementation complexity can be achieved by using a short feedforward filter (FFF), a sufficiently long feedback filter (FBF) with QPSK modulation (since it can be implemented multiplier–free in this case), and an optimized decision delay [6, 5, 2].

Evaluating the DFE performance for a variety of channel impulse response profiles, number of equalizer taps, and decision delay settings using Monte Carlo simulations can be a very time–consuming task. Therefore, it is of great interest to develop analytical upper–bounds on the DFE performance that can be easily evaluated for different scenarios. A widely–used upper–bound is the matched–filter bound (MFB) for one–shot communication over a multipath noisy channel [4]. However, this bound becomes very loose when the FFF is short and the channel has severe multipath. In this paper, we derive a simple upper–bound on the decision–point SNR of the unbiased MMSE–DFE that has the following attractive features : 1) It is tighter than the MFB 2) It is very easy to evaluate numerically 3) It exhibits clearly the variation of the MMSE–DFE decision–point SNR as a function of the number of feedforward taps and decision delay setting. The rest of this paper is organized as follows. In Section II, we describe our model and review briefly the procedure for evaluating the performance of the finite–length MMSE–DFE. This is followed by deriving the new analytical performance upper–bound. Numerical results are presented in Section III and the paper is concluded in Section IV.

II. Analysis

II-A. Input-Output Model

We adopt the following standard discrete–time representation of an additive–noise multipath channel

$$\mathbf{y}_k = \sum_{m=0}^{\nu} \mathbf{h}_m x_{k-m} + \mathbf{n}_k \ ,$$

*Naofal Al-Dhahir is with AT&T Shannon Laboratory, 180 Park Ave., Florham Park, NJ 07932. Phone : (973)360−8407, FAX :(973)360−8178, Email : naofal@research.att.com

where $\mathbf{h}_m \stackrel{def}{=} \begin{bmatrix} h_{l-1,m} & \cdots & h_{0,m} \end{bmatrix}^t$ (the notation $(.)^t$ denotes the transpose) is the m^{th} (vector) channel impulse response (CIR) coefficient, assuming an oversampling factor of l, and ν is called the channel memory. The input symbols, $\{x_k\}$, and the noise symbols $\{\mathbf{n}_k\}$ are assumed to be complex, zero–mean, and uncorrelated random variables with average energies per complex dimension of S_x and lN_0, respectively. Furthermore, *the input symbols are assumed uniformly-distributed continuous random variables.*[1]

Over a block of N output symbols, we have the vector representation

$$\mathbf{y}_{k:k-N+1} = \mathbf{H}\mathbf{x}_{k:k-N-\nu+1} + \mathbf{n}_{k:k-N+1} \ , \tag{1}$$

where $\mathbf{y}_{k:k-N+1}$ is a block of N output symbols and $\mathbf{x}_{k:k-N-\nu+1}$ and $\mathbf{n}_{k:k-N+1}$ are the corresponding input and noise blocks. $\mathbf{H}$ is an $lN \times (N+\nu)$ block Toeplitz channel matrix whose first block row is $\begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 & \cdots & \mathbf{h}_\nu & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$ and ν is the channel memory. The N–dimensional noise probability density function (pdf) has the form

$$p_{\mathbf{n}}(\mathbf{u}) = \frac{1}{(2\pi N_0)^{\frac{N}{2}}} e^{-\frac{1}{2N_0}|\mathbf{u}|^2} \tag{2}$$

II-B. The FIR MMSE-DFE

The MMSE–DFE consists of two filters : a fractionally–spaced feedforward filter that has (lN) taps and a symbol–spaced feedback filter that has N_b strictly causal taps. It is a symbol–by–symbol estimator which processes N noisy and distorted observations $\mathbf{y}_{k:k-N+1}$ and N_b previous decisions $\hat{\mathbf{x}}_{k-\Delta-1:k-\Delta-N_b}$ to estimate the *unknown and random* symbol $x_{k-\Delta}$. The parameter Δ is called the decision delay. Its value can significantly affect performance for short length DFE's [2, 5]. Define the following Cholesky factorization

$$\mathbf{R} \stackrel{def}{=} \frac{1}{S_x}\mathbf{I}_{N_f+\nu} + \frac{1}{N_0}\mathbf{H}^*\mathbf{H} \stackrel{def}{=} \mathbf{LDL}^* \ , \tag{3}$$

where $\mathbf{I}_{N_f+\nu}$ denotes the identity matrix of size $(N_f+\nu)$, $(.)^*$ denotes complex–conjugate transpose, $\mathbf{L}$ is a lower–triangular matrix, and $\mathbf{D}$ is a diagonal matrix, $\mathbf{D} \stackrel{def}{=} diag(d_0, d_1, \cdots, d_{N_f+\nu-1})$.

The optimum decision–point SNR of the *unbiased* finite–length MMSE-DFE is given by [1]

$$SNR^{FIR}_{MMSE-DFE,U} = S_x d_{\Delta_{opt.}} - 1 \ , \tag{4}$$

where the optimum delay setting is given by

$$\Delta_{opt.} = argmax_{0 \leq i \leq (N_f+\nu-1)}\{d_i\} \ . \tag{5}$$

This result assumes the ideal DFE operating conditions of no error propagation and perfect channel knowledge. This is consistent with our objective of deriving an upper–bound on performance. However, the use of (4) as a performance upper–bound still requires performing intensive computations.

II-C. New Upper-Bound

Given the vector of N noisy observations $\mathbf{y}$, our objective is to estimate the *single* unknown and random symbol $x_{k-\Delta}$ *assuming perfect knowledge of the channel matrix* $\mathbf{H}$ *and all other elements in* $\mathbf{x}_{k:k-N-\nu+1}$. [2] A lower–bound on the error

[1] This assumption becomes more accurate for large signal constellations and when Tomlinson-Harashima precoding is used to eliminate error propagation in the MMSE-DFE (see e.g. [3] and page 463 in [4]).

[2] A tighter upper–bound can be derived by assuming only that the N_b post–cursor symbols $\mathbf{x}_{k-\Delta-1:k-\Delta-N_b}$ are known (as in DFE). However, the resulting bound is more computationally intense.

variance of any *unbiased* estimator of $x_{k-\Delta}$ is given by the *Stochastic Cramer–Rao bound* which states that (see p.72 in [7])

$$\sigma^2_{e,\Delta} \geq -(\mathrm{E}[\frac{\partial^2 L}{\partial^2 x_{k-\Delta}}])^{-1} \; : \; 0 \leq \Delta \leq N+\nu-1 \; , \tag{6}$$

where the expectation is over the random symbols $\mathbf{y}_{k:k-N+1}$ and $x_{k-\Delta}$ and L is defined as follows

$$\begin{aligned}
L &\stackrel{def}{=} \ln p(\mathbf{y}_{k:k-N+1}; x_{k-\Delta}) \; : \; \text{logarithm of joint probability density function} \\
&= \ln p(\mathbf{y}_{k:k-N+1}; \mathbf{x}_{k:k-N-\nu+1}) \; : \; \text{since other elements in } \mathbf{x}_{k:k-N-\nu+1} \text{ are assumed known} \\
&= \ln p(\mathbf{y}_{k:k-N+1}/\mathbf{x}_{k:k-N-\nu+1}) \; : \; \text{since the symbols } \mathbf{x}_{k:k-N-\nu+1} \text{ are assumed uniformly-distributed} \\
&= \ln p_{\mathbf{n}}(\mathbf{y}_{k:k-N+1} - \mathbf{H}\mathbf{x}_{k:k-N-\nu+1}) \\
&= -\frac{N}{2}\ln(2\pi N_0) - \frac{1}{2N_0}(\mathbf{y}_{k:k-N+1} - \mathbf{H}\mathbf{x}_{k:k-N-\nu+1})^*(\mathbf{y}_{k:k-N+1} - \mathbf{H}\mathbf{x}_{k:k-N-\nu+1}) \\
&= K - \frac{1}{2N_0} trace((\mathbf{y}_{k:k-N+1} - \mathbf{H}\mathbf{x}_{k:k-N-\nu+1})(\mathbf{y}_{k:k-N+1} - \mathbf{H}\mathbf{x}_{k:k-N-\nu+1})^*) \; .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 L}{\partial^2 x_{k-\Delta}} &= -\frac{1}{2N_0} trace(\frac{\partial^2}{\partial^2 x_{k-\Delta}} \mathbf{H}\mathbf{x}_{k:k-N-\nu+1}\mathbf{x}^*_{k:k-N-\nu+1}\mathbf{H}^*) \\
&= -\frac{1}{N_0} trace(H\mathbf{e}_\Delta \mathbf{e}^*_\Delta \mathbf{H}^*) \\
&= -\frac{1}{N_0}\mathbf{e}^*_\Delta \mathbf{H}^*\mathbf{H}\mathbf{e}_\Delta \; ,
\end{aligned} \tag{7}$$

where $\mathbf{e}^*_\Delta = \left[\begin{array}{ccc} \mathbf{0}_{1\times\Delta} & 1 & \mathbf{0}_{1\times(N+\nu-\Delta-1)} \end{array} \right]$.

Combining (6) and (7), we get [3]

$$\sigma^2_{e,\Delta} \geq N_0(\mathbf{e}^*_\Delta \mathbf{H}^*\mathbf{H}\mathbf{e}_\Delta)^{-1} \; .$$

If we define the SNR in estimating $x_{k-\Delta}$ as $SNR_{e,\Delta} \stackrel{def}{=} \frac{S_x}{\sigma^2_{e,\Delta}}$, then

$$SNR_{e,\Delta} \;\; \leq \;\; (\frac{S_x}{N_0})\mathbf{e}^*_\Delta \mathbf{H}^*\mathbf{H}\mathbf{e}_\Delta \stackrel{def}{=} SNR_{UB} \; .$$

Exploiting the fully–windowed block–Toeplitz structure of $\mathbf{H}$, this upper–bound can be further simplified as follows

$$\begin{aligned}
SNR_{UB} &= SNR \sum_{l=0}^{N-1} |\mathbf{h}_{\Delta-l}|^2 \\
&= SNR \sum_{max(0,\Delta-N+1)}^{min(\Delta,\nu)} |\mathbf{h}_i|^2 \; : \; 0 \leq \Delta \leq N+\nu-1 \\
&\leq SNR|\mathbf{h}|^2 = SNR_{MFB} \; .
\end{aligned} \tag{8}$$

Equation (8) is the desired upper–bound and we have shown that it is tighter than the MFB unless $\nu \leq \Delta \leq N-1$ in which case both bounds coincide. We conclude this section by showing that $SNR^{FIR}_{MMSE-DFE,U} \leq SNR_{UB}$. Equating the (Δ, Δ) element of both sides of (3), we have

$$\begin{aligned}
\frac{1}{S_x} + \frac{1}{N_0}\mathbf{e}^*_\Delta \mathbf{H}^*\mathbf{H}\mathbf{e}_\Delta &= \mathbf{e}^*_\Delta \mathbf{L}\mathbf{D}\mathbf{L}^*\mathbf{e}_\Delta \\
&= d_\Delta + \sum_{i=0}^{\Delta-1} |\mathbf{L}(\Delta, i)|^2 \\
&\geq d_\Delta \\
\Rightarrow SNR^{FIR}_{MMSE-DFE,U} &= S_x d_\Delta - 1 \leq SNR_{UB} \; ,
\end{aligned} \tag{9}$$

[3] When noise is correlated, the term $\frac{1}{N_0}\mathbf{e}^*_\Delta \mathbf{H}^*\mathbf{H}\mathbf{e}_\Delta$ is replaced by $\mathbf{e}^*_\Delta \mathbf{H}^*\mathbf{R}^{-1}_{nn}\mathbf{H}\mathbf{e}_\Delta$, where $\mathbf{R}_{nn}$ is the noise auto–correlation matrix.

with equality achieved if and only if $\mathbf{H}^*\mathbf{H}$ is diagonal, i.e., when the channel is *memoryless*.

III. Numerical Results

In our simulations, the $\frac{T}{2}$–spaced CIR coefficients are generated as zero–mean uncorrelated Gaussian random variables and the total CIR energy in each phase is normalized to unity. The input SNR level is fixed at 10 dB; therefore, $SNR_{MFB} =$ 13 dB. Figure 1 shows that when $N < (\nu + 1)$, the new upper–bound is much tighter than SNR_{MFB}. Moreover, unlike SNR_{MFB}, the new upper–bound varies with N and Δ in a manner similar to $SNR^{FIR}_{MMSE-DFE,U}$, as depicted in Figures 2 and 3. This suggests the use of SNR_{UB} as a computationally–efficient tool for setting Δ.

IV. Conclusions

We derived a simple upper–bound on the decision–point SNR of the finite–length unbiased MMSE–DFE. The bound is simple to compute, tighter than the popular matched–filter bound, and tracks closely performance variation with the number of feedforward taps and the decision delay setting. The bound can be used for a fast determination of decision delay setting for DFE's with short feedforward filters on severe multipath channels.

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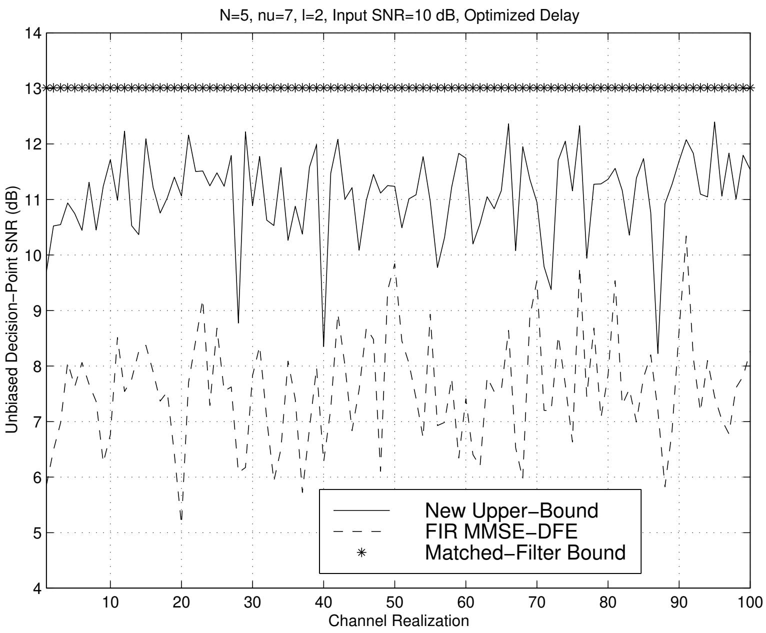

Fig. 1. Relation of $SNR^{FIR}_{MMSE-DFE,U}$ to the New Upper-Bound and the Matched-Filter Bound for Different Channel Realizations

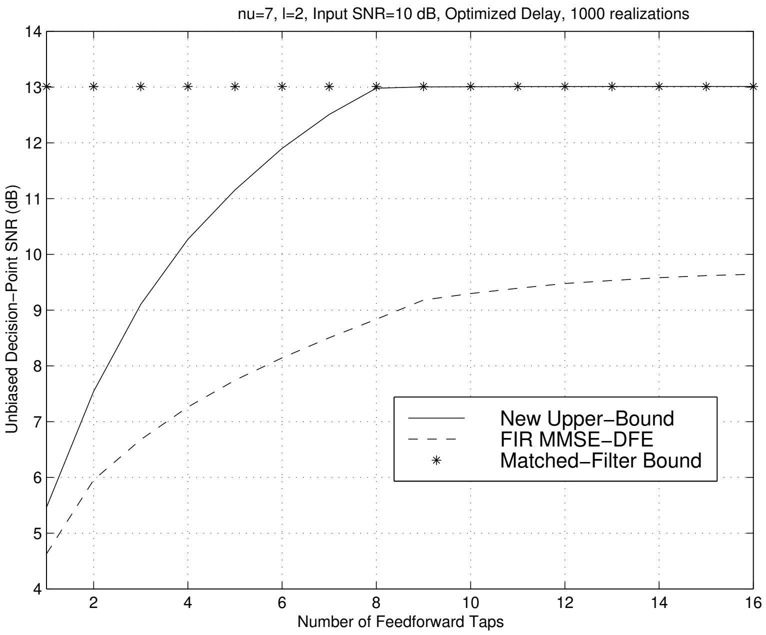

Fig. 2. Relation of $SNR^{FIR}_{MMSE-DFE,U}$ to the New Upper-Bound and the Matched-Filter Bound for Different Feedforward Filter Lengths

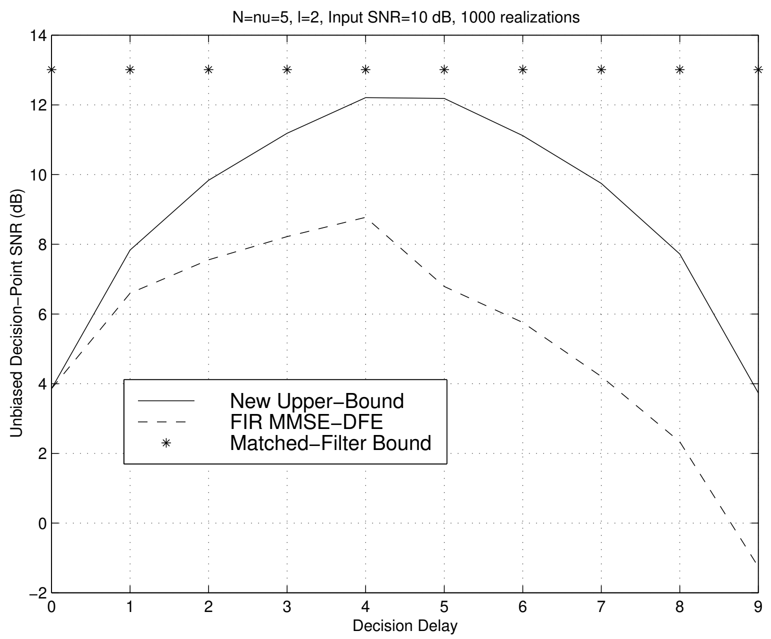

Fig. 3. Relation of $SNR^{FIR}_{MMSE-DFE,U}$ to the New Upper-Bound and the Matched-Filter Bound for Different Decision Delay Settings