Efficient Computation of the Delay-Optimized Finite-Length MMSE-DFE

Naofal Al-Dhahir *, Member, IEEE, and John M. Cioffi, Senior Member, IEEE

Abstract

We present new fast algorithms for computing the optimum settings of a finite–length minimum–mean–square–error decision feedback equalizer (MMSE–DFE) from channel and noise estimates.

These algorithms are based on *displacement structure theory* [5, 8] and generalize the algorithms of [2, 1] by including delay optimization. Both symbol–spaced and fractionally–spaced feedforward filters are considered.

I. Introduction

The minimum–mean–square–error decision feedback equalizer (MMSE–DFE) is a popular receiver structure on severe–ISI and noise channels [3]. Recently, there has been increased interest in channel–estimate–based DFEs; as opposed to the more traditional adaptive DFEs; since the former exhibit undeniable performance advantages over the latter (see [9] and the references therein). This performance improvement arises mainly because of the avoidance of the large eigenvalue spread problem that slows the convergence of LMS–based adaptive DFEs. Channel–estimate–based equalizers were also demonstrated in [4] to have better tracking performance than least–squares methods. Nevertheless, they have not yet found wide spread application in practice mainly due to the formidable computational complexity associated with computing the optimum settings of the DFE filters directly from channel and noise estimates.

A new theory for analyzing the finite–length MMSE–DFE based on the powerful analytical tools of *Cholesky factorization* and *displacement structure* was recently introduced by the present authors in [3]. This theory enabled us to derive new fast algorithms for computing the optimum settings of the *finite–length* MMSE–DFE by exploiting the *special structure* of the matrices involved in the computations [2, 1].

*Naofal Al-Dhahir is now with GE Corporate Research & Development, Schenectady, NY 12301.

This work was supported by NSF under Contract NCR-9203131, JSEP under Contract DAAL 03-91-C-0010, and NASA under Contract NAG2-842.

For simplicity, we assumed in deriving the algorithms of [2, 1] a particular choice for the decision delay of the finite-length MMSE–DFE. The optimality of this choice was verified through computer simulations for a wide range of practical channel and noise scenarios, especially when the feedforward filter is of sufficient length. However, in some situations where the channel is *maximum–phase* (or close to maximum–phase) and the short length of the feedforward filter is dictated by implementation–cost considerations, this fixed choice of the delay becomes suboptimum and could result in a significant performance degradation. For completeness of the theory and to accommodate these non–common, nevertheless practical, situations, we shall generalize in this paper the algorithms in [2, 1] to include delay optimization.

The rest of this paper is organized as follows. In Section II, we present closed–form expressions for the optimum feedback and feedforward filters of the finite–length delay–optimized MMSE–DFE that were derived in detail in [3]. Section III presents *new* fast algorithms for computing the taps of these filters for symbol–spaced and fractionally–spaced feedforward filters and illustrates them by a numerical example. We conclude the paper in Section IV.

II. The Finite-Length MMSE-DFE

We adopt the standard discrete–time representation an additive–noise dispersive channel given by

$$\mathbf{y}_k = \sum_{m=0}^{\nu} \mathbf{h}_m x_{k-m} + \mathbf{n}_k \ ,$$

where $\mathbf{h}_m \stackrel{def}{=} \begin{bmatrix} h_{l-1,m} & \cdots & h_{0,m} \end{bmatrix}^t$ ($(.)^t$ denotes the transpose) is the m^{th} (vector) channel coefficient (assuming an oversampling factor of l), and ν is called the channel memory. Both the input sequence, $\{x_k\}$, and the noise sequence, $\{\mathbf{n}_k\}$, are assumed to be complex, zero–mean, and uncorrelated with average energies of S_x and (lN_0) per complex dimension, respectively. An ideal (brick–wall) receive filter is assumed.

As shown in Figure 1, the MMSE–DFE consists of two filters : a fractionally–spaced anti–causal feedforward filter that has (lN_f) taps and is denoted by the vector $\mathbf{w}^* \stackrel{def}{=} \begin{bmatrix} \mathbf{w}^*_{-(N_f-1)} & \cdots & \mathbf{w}^*_0 \end{bmatrix}$, and a symbol–spaced feedback filter that is *assumed to have* $N_b (\leq \nu)$ *strictly causal taps* denoted by $\{-b_1, -b_2, \cdots, -b_{N_b}\}$.

For analytical convenience, we define the augmented vector $\tilde{\mathbf{b}}^* \stackrel{def}{=} \begin{bmatrix} \mathbf{0}_{1\times\Delta} & 1 & b_1^* & \cdots & b_{N_b}^* \end{bmatrix}$, where Δ is the delay of the feedforward filter, and $(.)^*$ denotes the conjugate transpose.

We showed in detail in [3] that the optimum feedback and feedforward filter settings that minimize the mean square of the error sequence, $e_k \stackrel{def}{=} z_k - \hat{x}_{k+N_f-1-\Delta}$, are given by

$$\tilde{\mathbf{b}}_{opt.} \quad = \quad \mathbf{L}\mathbf{e}_{(\Delta_{opt.}+1)} \qquad (1)$$

$$\mathbf{w}^*_{opt.} \;\; = \;\; d^{-1}_{\Delta_{opt.}} \mathbf{e}^*_{(\Delta_{opt.}+1)} \mathbf{L}^{-1} \mathbf{H}^* \; , \tag{2}$$

where $\mathbf{e}_i$ denotes the i^{th} unit vector and $\mathbf{H}$ is an $N_f \times (N_f + \nu)$ *fully–windowed Toeplitz* channel matrix whose first row is $\left[\begin{array}{cccccc} \mathbf{h}_0 & \mathbf{h}_1 & \cdots & \mathbf{h}_\nu & \mathbf{0} & \cdots & \mathbf{0} \end{array} \right]$. The optimum delay, $\Delta_{opt.}$, is given by

$$\Delta_{opt.} = argmax_{(N_f-1) \leq i \leq (N_f+\nu-1)} \{d_i\} \; . \tag{3}$$

The matrix $\mathbf{L}$ that appears in (1) and (2) and the scalars $\{d_i\}$ in (2) and (3) are defined by the following *Cholesky factorization* [3]

$$\mathbf{R} \stackrel{def}{=} \frac{l N_0}{S_x} \mathbf{I}_{N_f+\nu} + \mathbf{H}^* \mathbf{H} \stackrel{def}{=} \mathbf{L} \mathbf{D} \mathbf{L}^* \; , \tag{4}$$

where $\mathbf{I}_{N_f+\nu}$ denotes the identity matrix of size $(N_f + \nu)$, $\mathbf{L}$ is a lower–triangular monic (has ones along the main diagonal) matrix, and $\mathbf{D}$ is a diagonal matrix, $\mathbf{D} \stackrel{def}{=} diag(d_0, d_1, \cdots, d_{N_f+\nu-1})$.

Furthermore, the *unbiased* decision–point SNR of the MMSE–DFE is given by [3]

$$SNR_{MMSE-DFE,U} = \frac{l S_x}{N_0} d_{\Delta_{opt.}} - 1 \stackrel{def}{=} l SNR d_{\Delta_{opt.}} - 1 \; .$$

It is now clear that the task of computing the optimum MMSE–DFE settings efficiently is equivalent to that of efficiently computing the Cholesky factorization in (4).

Remark :

The reason behind setting the lower limit on the index search in (3) to $(N_f - 1)$ instead of 0 is that for the case of white input and noise sequences, we always have (a proof is provided in the appendix)

$$d_{N_f-1} \geq d_{N_f-2} \geq \cdots \geq d_0 \; , \tag{5}$$

However, a similar relationship between $\{d_{N_f-1}, d_{N_f}, \cdots, d_{N_f+\nu-1}\}$ does not exist in general.

III. Fast Computation of Optimum MMSE-DFE Settings

In this section, we shall present fast algorithms for computing the optimum MMSE–DFE settings from channel and noise estimates. These algorithms are based on the recent theory of *structured matrices* [5, 7]. In particular, they exploit the Toeplitz–like structure of the factored matrix in (4) to reduce the computational load. Due to space limitations, we refer the reader to [5, 7, 8] (and the references therein) for a detailed discussion on the theory of structured matrices and only present the algorithms in their final form here.

III-A. Symbol-Spaced Case

For a symbol–spaced feedforward filter, we showed in detail in [2, 1] that the *displacement rank* [5, 7, 8] of $\mathbf{R}$ is $\rho = 3$. However, in [2, 1], we assumed that $\Delta_{opt.} = N_f - 1$ which causes the displacement rank to drop to $\rho = 2$. Here, we handle the general case where the delay of the feedforward filter is optimized. In this case, it can be shown that a *generator* for $\mathbf{R}$ [7, 8] is given by [2, 1]

$$\mathbf{G}(D) = \left[\begin{array}{ccc} \frac{1}{\sqrt{SNR}} & h^*(D^*) & D^{N_f} h^*(D^*) \end{array} \right] ,$$

where $h(D) \stackrel{def}{=} \sum_{i=0}^{\nu} h_i D^i$ is the D–transform of the channel impulse response.

The following algorithm (whose derivation we omit for brevity but follows very closely the development in [2]) computes the optimum feedback filter settings in $O(3(N_f + \nu)^2)$ flops :

Algorithm 1 *(symbol–spaced case)*

Initial Condition :

$$\begin{array}{rcl} \mathbf{G}_0(D) & = & \mathbf{G}(D) = \left[\begin{array}{ccc} \frac{1}{\sqrt{SNR}} & h^*(D^*) & D^{N_f} h^*(D^*) \end{array} \right] \\ \mathbf{J} & = & diag(1, 1, -1) \; . \end{array}$$

Recursions :

For i=$0, 1, \cdots, N_f + \nu - 1$,

$$d_i = \mathbf{G}_i(0) \mathbf{J} \mathbf{G}_i^*(0) \; : \; \textit{diagonal elements of } \mathbf{D} \tag{6}$$

$$l_i(D) = D^i \mathbf{G}_i(D) \mathbf{J} \mathbf{G}_i^*(0) d_i^{-1} \; : \; \textit{columns of } \mathbf{L} \tag{7}$$

$$\left[\begin{array}{ccc} \alpha_i & \beta_i & \gamma_i \end{array} \right] = d_i^{-\frac{1}{2}} \mathbf{G}_i(0) \tag{8}$$

$$X_i \stackrel{def}{=} \frac{\beta_i}{\alpha_i} \; ; \; Y_i \stackrel{def}{=} \frac{-\gamma_i}{\alpha_i} \; ; \; \zeta_i^2 \stackrel{def}{=} 1 + |X_i|^2 - |Y_i|^2 = \frac{1}{|\alpha_i|^2} \tag{9}$$

$$D\mathbf{G}_{i+1}(D) = \mathbf{G}_i(D) \zeta_i^{-1} \left[\begin{array}{ccc} 1 & -X_i & Y_i \\ X_i^* & (1 - (1+\zeta_i)^{-1} |Y_i|^2) & X_i^* Y_i (1+\zeta_i)^{-1} \\ Y_i^* & -X_i Y_i^* (1+\zeta_i)^{-1} & (1 + (1+\zeta_i)^{-1} |X_i|^2) \end{array} \right] . \tag{10}$$

Output :

$$\begin{array}{rcl} b_{opt.}(D) & = & l_{\Delta_{opt.}}(D) \; : \; \textit{optimum feedback filter} \\ \Delta_{opt.} & = & argmax_{(N_f - 1) \le i \le N_f + \nu - 1} \{d_i\} \; : \; \textit{optimum delay.} \end{array}$$

Remark :

In (7), we assume the following vector–polynomial mapping

$$l_i(D) = \left[\begin{array}{cccc} 1 & D & D^2 & \cdots \end{array} \right] \mathbf{L} \mathbf{e}_i \; : \; i = 0, 1, \cdots$$

III-B. Fractionally-Spaced Case

We showed in [2, 1] that for any integer oversampling factor, l, the displacement rank of $\mathbf{R}$ becomes $\rho = (2l+1)$, and a generator for $\mathbf{R}$ is given by

$$\mathbf{G}(D) = \left[\begin{array}{cccccccc} \sqrt{\frac{l}{SNR}} & h^*_{l-1}(D^*) & \cdots & h^*_0(D^*) & D^{N_f} h^*_{l-1}(D^*) & \cdots & D^{N_f} h^*_0(D^*) \end{array}\right] ,$$

where $h^*_j(D^*) = \sum_{i=0}^{\nu} h^*_{j,i} D^i$ is the D–transform of the j^{th} phase of the oversampled channel impulse response. In this case, the following algorithm [6, 2] calculates the optimum feedback filter settings in $O((2l+1)(N_f+\nu)^2)$ flops :

Algorithm 2 *($\frac{T}{l}$–spaced case)*

Initial Condition:

$$\begin{aligned} \mathbf{G}_0(D) &= \mathbf{G}(D) = \left[\begin{array}{ccccccc} \sqrt{\frac{l}{SNR}} & h^*_{l-1}(D^*) & \cdots & h^*_0(D^*) & D^{N_f} h^*_{l-1}(D^*) & \cdots & D^{N_f} h^*_0(D^*) \end{array}\right] \\ \mathbf{J} &= diag(1, \mathbf{I}_l, -\mathbf{I}_l) \; . \end{aligned}$$

Recursions:

For $i = 0, 1, \cdots, N_f + \nu - 1$,

$$d_i = \mathbf{G}_i(0)\mathbf{J}\mathbf{G}_i^*(0) \tag{11}$$

$$l_i(D) = D^i \mathbf{G}_i(D)\mathbf{J}\mathbf{G}_i^*(0) d_i^{-1} \tag{12}$$

$$D\mathbf{G}_{i+1}(D) = \mathbf{G}_i(D)\mathbf{U}_i diag(D, \mathbf{I}_{2l}) \; . \tag{13}$$

If we define

$$d_i^{-\frac{1}{2}} \mathbf{G}_i(0)\mathbf{J} \stackrel{def}{=} \left[\begin{array}{cccc} \alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{i(2l+1)} \end{array}\right] , \tag{14}$$

then $\mathbf{U}_i$ *is found as follows (see [6] for the general form of* $\mathbf{U}_i$*)*

$$\begin{aligned} \mathbf{k}_i &\stackrel{def}{=} \left[\begin{array}{cccc} \frac{\alpha_{i2}}{\alpha_{i1}} & \frac{\alpha_{i3}}{\alpha_{i1}} & \cdots & \frac{\alpha_{i(2l+1)}}{\alpha_{i1}} \end{array}\right] \stackrel{def}{=} \left[\begin{array}{cccc} k_{i1} & k_{i2} & \cdots & k_{i(2l)} \end{array}\right] \\ \eta_i &= (1 - \mathbf{k}_i diag(-\mathbf{I}_l, \mathbf{I}_l)\mathbf{k}_i^*)^{\frac{1}{2}} \\ \mathbf{U}_i &= \eta_i \left[\begin{array}{cc} 1 & \mathbf{k}_i diag(\mathbf{I}_l, -\mathbf{I}_l) \\ -\mathbf{k}_i^* & \eta_i \mathbf{I}_{2l} - (1+\eta_i)^{-1}\mathbf{k}_i^*\mathbf{k}_i diag(\mathbf{I}_l, -\mathbf{I}_l) \end{array}\right]^{-1} . \end{aligned} \tag{15}$$

Output :

$$\begin{aligned} b_{opt.}(D) &= l_{\Delta_{opt.}}(D) \; : \; \textit{optimum feedback filter} \\ \Delta_{opt.} &= argmax_{(N_f-1)\leq i \leq N_f+\nu-1}\{d_i\} \; : \; \textit{optimum delay.} \end{aligned}$$

It is shown in [2, 1] that when $\Delta_{opt.} = (N_f - 1)$, ρ drops to $(l+1)$.

Remark :

For a given value of l, a closed–form expression for $\mathbf{U}_i$ can be found by evaluating the inverse of the $(2l+1) \times (2l+1)$ matrix in (15) using a symbolic manipulator software. As an example, for the popular case of a $\frac{T}{2}$–spaced feedforward filter, $\rho = 5$, $\mathbf{J} = diag(1,1,1,-1,-1)$, $\eta_i = \sqrt{1 + |k_{i1}|^2 + |k_{i2}|^2 - |k_{i3}|^2 - |k_{i4}|^2}$, and $\mathbf{U}_i$ is given by

$$\mathbf{U}_i = \frac{1}{(\eta_i + \eta_i^2)} \begin{bmatrix} 1+\eta_i & -(1+\eta_i)k_{i1} & -(1+\eta_i)k_{i2} & (1+\eta_i)k_{i3} & (1+\eta_i)k_{i4} \\ (1+\eta_i)k_{i1}^* & \eta_i + \eta_i^2 - |k_{i1}|^2 & -k_{i1}^* k_{i2} & k_{i1}^* k_{i3} & k_{i1}^* k_{i4} \\ (1+\eta_i)k_{i2}^* & -k_{i1}k_{i2}^* & \eta_i + \eta_i^2 - |k_{i2}|^2 & k_{i2}^* k_{i3} & k_{i2}^* k_{i4} \\ (1+\eta_i)k_{i3}^* & -k_{i1}k_{i3}^* & -k_{i2}k_{i3}^* & \eta_i + \eta_i^2 + |k_{i3}|^2 & k_{i3}^* k_{i4} \\ (1+\eta_i)k_{i4}^* & -k_{i1}k_{i4}^* & -k_{i2}k_{i4}^* & k_{i3}k_{i4}^* & \eta_i + \eta_i^2 + |k_{i4}|^2 \end{bmatrix} .$$

III-C. Computing the Feedforward Filter

It is clear from (2) that we only need to compute the $(\Delta_{opt.}+1)^{st}$ row of $\mathbf{L}^{-1}$ to determine the optimum feedforward filter. Because of the lower triangular nature of $\mathbf{L}$, we have

$$\mathbf{w}_{opt.}^* = d_{\Delta_{opt.}}^{-1} \left[\begin{array}{cc} \mathbf{v}_{\Delta_{opt.}}^* & \mathbf{0}_{N_f+\nu-\Delta_{opt.}-1} \end{array} \right] \mathbf{H}^* ,$$

where $\mathbf{v}_{\Delta_{opt.}}^*$ is the $(\Delta_{opt.}+1)^{st}$ row of the inverse of the $(\Delta_{opt.}+1) \times (\Delta_{opt.}+1)$ leading submatrix of $\mathbf{L}$. If we define $\mathbf{v}_{\Delta_{opt.}}^* \stackrel{def}{=} \left[\begin{array}{cccc} v^*(0) & v^*(1) & \cdots & v^*(\Delta_{opt.}) \end{array} \right]$, then the elements of $\mathbf{v}_{\Delta_{opt.}}^*$ can be calculated using the *back substitution method* (B.S.M.) as follows [2, 1]

$$v^*(k) = -\sum_{j=k+1}^{\Delta_{opt.}} \mathbf{L}(j,k) v^*(j) \quad : k = \Delta_{opt.}-1, \Delta_{opt.}-2, \cdots, 0.$$

Note that $v^*(\Delta_{opt.}) = 1$ since $\mathbf{L}$ is a monic lower–triangular matrix.

Subsequently, the feedforward filter taps can be efficiently computed using the relation [2, 1]

$$\mathbf{w}_{i-N_f+1}^* = d_{\Delta_{opt.}}^{-1} \sum_{k=0}^{\min(\nu, \Delta_{opt.}-i)} v^*(k+i)\mathbf{h}_k^* \quad : i = 0, 1, \cdots, N_f - 1.$$

IV. Computational Complexity Evaluation

In Tables I and II, we give a worst–case estimate of the number of complex multiplications and additions required to compute the feedback and (symbol–spaced) feedforward filter, respectively.

Computation	Complex multiplies	Complex adds
Equation (6)	$3(N_f+\nu)$	$2(N_f+\nu)$
Equation (7)	$3(N_f+\nu)(\nu+2)$	$2(N_f+\nu)(\nu+1)$
Equation (8)	$3(N_f+\nu)$	0
Equation (9)	$3(N_f+\nu)$	0
Equation (10)	$9(N_f+\nu)(\nu+2)$	$3(N_f+\nu)(2\nu+3)$
Total	$3(N_f+\nu)(4\nu+11)$	$(N_f+\nu)(8\nu+13)$

Table I. Complexity of Algorithm 1

Range	Complex multiplies	Complex adds
for $\Delta_{opt.} \geq \nu$	$\frac{\Delta_{opt.}(\Delta_{opt.}-1)}{2}+(\Delta_{opt.}-\nu+1)(\nu+1)$ $+N_f+(\frac{N_f-1-\Delta_{opt.}+\nu}{2})(\Delta_{opt.}+\nu-N_f+2)$	$\frac{\Delta_{opt.}(\Delta_{opt.}-1)}{2}+\nu(\Delta_{opt.}-\nu+1)$ $+(\frac{N_f-1-\Delta_{opt.}+\nu}{2})(\Delta_{opt.}+\nu-N_f)$
for $\Delta_{opt.} < \nu$	$\frac{\Delta_{opt.}(\Delta_{opt.}-1)}{2}+N_f+\Delta_{opt.}N_f-\frac{N_f(N_f-1)}{2}$	$\frac{\Delta_{opt.}(\Delta_{opt.}-1)}{2}-\frac{N_f(N_f-1)}{2}+N_f(\Delta_{opt.}-1)$

Table II. Computational Complexity of Feedforward Filter for Symbol-Spaced Case

IV-A. A Numerical Example

Consider the ISI channel $h(D) = 0.2 - 0.5D + D^2 - 0.2D^3$ with an input SNR of 20 dB and a 3–tap *symbol–spaced* feedforward filter. Therefore, $N_f = 3$ and $\nu = 3$.

Iterations of Algorithm 1 :

- $\underline{i=0:}$

$$\begin{aligned} \mathbf{G}_0(D) &= \left[\begin{array}{ccc} 0.1 & 0.2-0.5D+D^2-0.2D^3 & D^3(0.2-0.5D+D^2-0.2D^3) \end{array} \right] \; ; \; d_0 = 0.05 \\ l_0(D) &= 1-2D+4D^2-0.8D^3 \; ; \; X_0 = 2 \; ; \; Y_0 = 0 \; ; \; \zeta_0 = 2.236 \; . \end{aligned}$$

- $\underline{i=1:}$

$$\begin{aligned} \mathbf{G}_1(D) &= \frac{1}{\sqrt{5}} \left[\begin{array}{ccc} 0.5-D+2D^2-0.4D^3 & -0.5+D-0.2D^2 & D^2(0.4472-1.118D+2.2361D^2-0.4472D^3) \end{array} \right] \; ; \; d_1 = 0.1 \\ l_1(D) &= D-2D^2+2.2D^3-0.4D^4 \; ; \; X_1 = -1 \; ; \; Y_1 = 0 \; ; \; \zeta_1 = \frac{1}{\sqrt{0.5}} \; . \end{aligned}$$

- $\underline{i=2:}$

$$\begin{aligned} \mathbf{G}_2(D) &= \sqrt{0.1} \left[\begin{array}{ccc} 1-2D+2.2D^2-0.4D^3 & 1.8D-0.4D^2 & D(0.6325-1.5812D+3.1623D^2-0.6325D^3) \end{array} \right] \; ; \; d_2 = 0.1 \\ l_2(D) &= D^2-2D^3+2.2D^4-0.4D^5 \; ; \; X_2 = 0 \; ; \; Y_2 = 0 \; ; \; \zeta_2 = 1 \; . \end{aligned}$$

- $\underline{i=3:}$

$$\begin{aligned}\mathbf{G}_3(D) &= \sqrt{0.1}\begin{bmatrix} 1-2D+2.2D^2-0.4D^3 & 1.8-0.4D & 0.6325-1.5812D+3.1623D^2-0.6325D^3 \end{bmatrix} ; \ d_3 = 0.384 \\ l_3(D) &= D^3-0.4479D^4+0.0521D^5 ; \ X_3 = 1.8 ; \ Y_3 = -0.6325 ; \ \zeta_3 = 1.9596 .\end{aligned}$$

- $\underline{i = 4 :}$

$$\begin{aligned}\mathbf{G}_4(D) &= \begin{bmatrix} 0.6197-0.2776D+0.0323D^2 & 0.427-0.4427D+0.0769D^2 & -0.3055+0.8444D-0.173D^2 \end{bmatrix} ; \ d_4 = 0.4729 \\ l_4(D) &= D^4-0.2179D^5 ; \ X_4 = 0.689 ; \ Y_4 = 0.493 ; \ \zeta_4 = 1.1097 .\end{aligned}$$

- $\underline{i = 5 :}$

$$\mathbf{G}_5(D) = \begin{bmatrix} 0.6878-0.1498D & -0.3032+0.0664D & 0.7445-0.1654D \end{bmatrix} ; \ d_5 = 0.0105 ; \ l_5(D) = D^5 .$$

It is clear that $d_4 = \max_{0\le i\le 5}\{d_i\}$, therefore, $\Delta_{opt.} = 4$. The optimum feedback filter settings are given by the coefficients of $l_4(D)$, i.e., $\mathbf{b}^*_{opt.} = \begin{bmatrix} 1 & -0.2179 \end{bmatrix}$.

The optimum feedforward filter is calculated using B.S.M. as follows

$$v^*(4) = 1 ; \ v^*(3) = -\mathbf{L}(4,3)v^*(4) = 0.4479 ; \ v^*(2) = -\sum_{j=3}^{4}\mathbf{L}(j,2)v^*(j) = -1.3042;$$

$$v^*(1) = -\sum_{j=2}^{4}\mathbf{L}(j,1)v^*(j) = -3.1938 ; \ v^*(0) = -\sum_{j=1}^{4}\mathbf{L}(j,0)v^*(j) = -0.8124 .$$

$$w^*_{-2} = d_4^{-1}\sum_{k=0}^{3} v^*(k)h^*_k = 0.0859 ; \ w^*_{-1} = d_4^{-1}\sum_{k=0}^{3} v^*(k+1)h^*_k = 0.5524 ; \ w^*_0 = d_4^{-1}\sum_{k=0}^{2} v^*(k+2)h^*_k = 1.0893 .$$

$$\Rightarrow \mathbf{w}^*_{opt.} = \begin{bmatrix} 0.0859 & 0.5524 & 1.0893 \end{bmatrix} .$$

The unbiased decision–point SNR of the MMSE–DFE is given by

$$SNR_{MMSE-DFE,U} = 10\log_{10}(d_4 SNR - 1) = 16.6554 \ dB .$$

Note that if we had assumed that $\Delta_{opt.} = N_f - 1 = 2$, then

$$SNR_{MMSE-DFE,U} = 10\log_{10}(d_2 SNR - 1) = 9.5424 \ dB ,$$

which represents a loss of 7.113 dB from the maximum achievable decision–point SNR.

V. Conclusions

In this paper, we extended our algorithms in [2, 1] for efficiently computing the optimum settings of the finite–length MMSE–DFE by taking delay optimization into account. Delay optimization was shown to increase the complexity involved in computing the MMSE–DFE filters, however, it could result in an appreciable improvement in the decision–point SNR. Finally, we generalized the presented fast algorithms to accommodate the case of fractionally–spaced feedforward filters.

Appendix

A. A proof of Equation (5)

In this appendix, we prove Equation (5) by considering the general case of a fractionally–spaced feedforward filter. Over the first $(N_f - 1)$ recursions of Algorithm 2, we have $\mathbf{J} = \mathbf{I}_{l+1}$ [2, 1]. Therefore, using (13) and (14) and the property $\mathbf{G}_i(0)\mathbf{U}_i = d_i^{\frac{1}{2}} \mathbf{e}_1^*$ [6, 7], we get

$$\begin{aligned} D\mathbf{G}_{i+1}(D) &= \mathbf{G}_i(D)\mathbf{U}_i diag(D, \mathbf{I}_l) \\ \Rightarrow \mathbf{G}_{i+1}(0)\mathbf{e}_1 &= \mathbf{G}_i(0)\mathbf{U}_i\mathbf{e}_1 \\ \Rightarrow d_{i+1}^{\frac{1}{2}} \alpha_{(i+1)1} &= d_i^{\frac{1}{2}} \mathbf{e}_1^* \mathbf{e}_1 \\ \Rightarrow \frac{d_{i+1}}{d_i} &= \frac{1}{|\alpha_{(i+1)1}|^2} \ : \ 0 \le i \le (N_f - 2) \ . \end{aligned}$$

Combining (11) and (14), we get

$$\begin{aligned} d_{i+1} &= |\mathbf{G}_{i+1}(0)|^2 \\ &= d_{i+1}[|\alpha_{(i+1)1}|^2 + \cdots + |\alpha_{(i+1)(l+1)}|^2] \\ \Rightarrow |\alpha_{(i+1)1}|^2 &\le 1 \\ \Rightarrow d_{i+1} &\ge d_i \ : \ 0 \le i \le (N_f - 2) \ . \end{aligned}$$

References

[1] N. Al-Dhahir and J. Cioffi. "Fast Algorithms for the Computation of the FIR MMSE-DFE". In *IEEE International Conference on Acoustics, Speech, and Signal Processing*, pages IV533–IV536, San Francisco, March 1992.

[2] N. Al-Dhahir and J. Cioffi. "Fast Computation of Channel-Estimate Based Equalizers in Packet Data Transmission". *IEEE Transactions on Signal Processing*, November 1995.

[3] N. Al-Dhahir and J. Cioffi. "MMSE Decision Feedback Equalizers : Finite-Length Results". *IEEE Transactions on Information Theory*, 41(4):961–976, July 1995.

[4] S. Fechtel and H. Meyer. "An Investigation of Channel Estimation and Equalization Techniques for Moderately Rapid Fading HF Channels". In *ICC Proceedings, Denver, CO.*, pages 25.2.1–25.2.5, 1991.

[5] T. Kailath, S.Y. Kung, and M. Morf. "Displacement Ranks of a Matrix". *Bulletin of the American Mathematical Society*, 1(5):769–773, September 1979.

[6] H. Lev-Ari and T. Kailath. "Lattice Filter Parametrization and Modeling of Nonstationary Processes". *IEEE Transactions on Information Theory*, IT-30(1), January 1984.

[7] H. Lev-Ari and T. Kailath. "Triangular Factorization of Structured Hermitian Matrices". *I.Schur Methods in Operator Theory and Signal Processing*, 18:301–324, 1986.

[8] A. H. Sayed. *"Displacement Structure in Signal Processing and Mathematics"*. PhD thesis, Stanford University, August 1992.

[9] R. Ziegler, N. Al-Dhahir, and J. Cioffi. "Nonrecursive Adaptive Decision-Feedback Equalization from Channel Estimates". In *Proceedings of IEEE Vehicular Technology Society* 42^{nd} *VTS Conference*, volume 2, pages 600–603, Denver, May 1992.

Figures

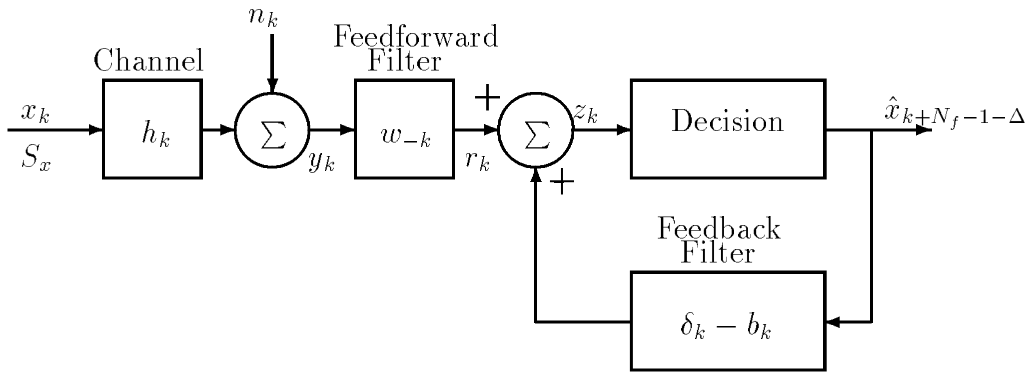

Fig. 1. Block diagram of the MMSE–DFE