Maximum Throughput Loss of Noisy ISI Channels Due to Narrowband Interference

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Abstract

Narrowband interference could occur in transmission media such as twisted pair or coaxial cable. In this letter, we analyze the effect of such interference on the data throughput for finite–blocklength transmission over noisy inter-symbol interference channels. It is shown that the worst narrowband interference spreads its power over the sweet spots of the signal. More precisely, the auto–correlation matrix of worst–case narrowband (rank-deficient) interference is shown to have the same eigendirections as the signal. We derive the exact spectral shape of the worst narrowband interference and its associated minimum channel block throughput in closed form.

1 Introduction

Narrowband interference (NBI) is a major impairment for broadband transmission over wired and wireless media. Examples include the upstream channel in hybrid fiber coaxial (HFC) networks [10], spread spectrum transmissions in the ISM band [8], and very high speed digital subscriber line (VDSL) transmissions [3].

In many situations, radio frequency interference (RFI) from short–wave radio, Citizen's Band (CB) radio, and amateur (HAM) radio is the most serious source of NBI. While the spectrum allocations for these services are fixed, the exact spectral characteristics (spectral shape and frequency support) of RFI at any particular time are generally unknown and hence need to be adaptively estimated prior to cancellation [2]. Ineffective cable shielding in HFC networks also causes the *ingress* of external signals from electrical devices *inside* the subscriber premises such as TV sets, computers, microwave ovens, etc. Again, the exact spectral characteristics of this *ingress noise* are generally unknown and vary with time.

In addition to the above–described *un-intentional* NBI, military communication systems often suffer from *intentional* NBI, commonly known as jamming. Whatever the cause of NBI is, the following main features often hold : its exact spectral characteristics are usually unknown, it could result in significant performance degradation if not suppressed, and it is often present in conjunction with other impairments such as intersymbol interference (ISI) and additive noise.

The objective of this letter is to determine the spectral characteristics of worst–case NBI that minimize the block throughput of noisy ISI channels. This characterization enables us to predict maximum performance degradation due to NBI and evaluate various NBI suppression techniques under this worst–case NBI scenario. We assume that the transmitter does not adjust its signal to the interference and therefore we do not have a game–theoretic problem as in [4, 7]. Another distinction of this work is that we assume the interference to have a limited spectrum, which causes its auto–correlation matrix to be rank deficient.

2 Analysis

2.1 Model and Assumptions

We adopt the following discrete–time baseband representation of an additive noise linear ISI channel corrupted by additive interference

$$\mathbf{y}_k = \sum_{m=0}^{\nu} h_m x_{k-m} + \mathbf{z}_k \ , \qquad (1)$$

where h_m is the m^{th} channel impulse response (CIR) coefficient and ν is called the channel memory. Both the input sequence, $\{x_k\}$, and the noise sequence, $\{\mathbf{z}_k\}$, are assumed to be complex, zero–mean, and have positive–definite auto–correlation matrices denoted by $\mathbf{R}_{xx}$ and $\mathbf{R}_{zz}$, respectively. In addition, the noise is *independent* of the input and consists of two components :

- A white Gaussian component due to background noise with variance equal to N_0 per complex dimension.
- A colored component due to interference with auto–correlation matrix $\mathbf{R}_{II}$ and total energy E_I, i.e., $trace(\mathbf{R}_{II}) = E_I$.

Over any block of N output symbols, (1) becomes (our notation here follows that in [1])

$$\mathbf{y}_{k+N-1:k} = \mathbf{H}\mathbf{x}_{k+N-1:k-\nu} + \mathbf{z}_{k+N-1:k} \ , \qquad (2)$$

where $\mathbf{H}$ is a fully–windowed Toeplitz channel matrix with first row equal to the CIR coefficients appended by zeros.

2.2 Worst-Case Narrow-band Interference

Assuming interference to be Gaussian (see Remark 1 below), then the channel block throughput, or equivalently the mutual information (in bits/symbol) between the input and output blocks, is given by [1]

$$I(\mathbf{X};\mathbf{Y}) = \log|\mathbf{I}_N + \mathbf{H}\mathbf{R}_{xx}\mathbf{H}^*\mathbf{R}_{zz}^{-1}| = \log|\mathbf{I}_N + \tilde{\mathbf{R}}_{xx}(N_0\mathbf{I}_N + \mathbf{R}_{II})^{-1}| \ , \qquad (3)$$

where $|.|$, $(.)^*$, and $\mathbf{I}_N$ denote the determinant, the complex–conjugate transpose, and the size–N identity matrix, respectively. Furthermore, we defined $\tilde{\mathbf{R}}_{xx} \overset{def}{=} \mathbf{H}\mathbf{R}_{xx}\mathbf{H}^*$ and $\mathbf{R}_{zz} \overset{def}{=} N_0\mathbf{I}_N + \mathbf{R}_{II}$.

Our objective is to find the worst–case interference auto–correlation matrix $\mathbf{R}_{II}$ that minimizes $I(\mathbf{X};\mathbf{Y})$ subject to the following fixed total interference energy and dimensionality constraints

$$trace(\mathbf{R}_{II}) = E_I \qquad \text{and} \qquad rank(\mathbf{R}_{II}) = \bar{N} \ , \qquad (4)$$

where $\bar{N} \leq N$ is an integer denoting the number of dimensions (or subchannels) occupied by NBI, and E_I is the total NBI energy.

Main Result

The worst–case rank–$\bar{N}$ NBI auto–correlation matrix $\mathbf{R}_{II}$ has the same $\bar{N}$ eigenvectors as those corresponding to the $\bar{N}$ *largest* eigenvalues of $\tilde{\mathbf{R}}_{xx}$. Moreover, the $\bar{N}$ non–zero eigenvalues of $\mathbf{R}_{II}$ are given by (14) below.

Proof

Define the following eigen–decomposition [1]

$$\tilde{\mathbf{R}}_{xx} = \mathbf{U}_x\mathbf{\Delta}_x\mathbf{U}_x^* = \mathbf{U}_x diag(\delta_{x,1},\cdots,\delta_{x,N})\mathbf{U}_x^* \ . \qquad (5)$$

[1] Assume without loss of generality that $\delta_{x,1} \geq \delta_{x,2} \geq \cdots \geq \delta_{x,N}$.

Furthermore, consider the following eigen–decomposition of the rank–$\bar{N}$ matrix $\mathbf{R}_{II}$

$$\mathbf{R}_{II} = \mathbf{U}_I \begin{bmatrix} \mathbf{\Delta}_I & \\ & \mathbf{0}_{N-\bar{N}} \end{bmatrix} \mathbf{U}_I^* = \mathbf{U}_I diag(\delta_{I,1}, \cdots, \delta_{I,\bar{N}}, 0, \cdots, 0)\mathbf{U}_I^* \; . \tag{6}$$

After substituting (5) and (6) in (3), we can express the channel block throughput as follows

$$\begin{aligned} I(\mathbf{X}\,;\mathbf{Y}) &= \log|\mathbf{I}_N + \mathbf{U}_x\mathbf{\Delta}_x\mathbf{U}_x^*(N_0\mathbf{I}_N + \mathbf{U}_I \begin{bmatrix} \mathbf{\Delta}_I & \\ & \mathbf{0}_{N-\bar{N}} \end{bmatrix} \mathbf{U}_I^*)^{-1}| & (7) \\ &= \log|\mathbf{I}_N + \mathbf{\Delta}_x^{\frac{*}{2}}\mathbf{U}_x^*\mathbf{U}_I \begin{bmatrix} (N_0\mathbf{I}_{\bar{N}} + \mathbf{\Delta}_I)^{-1} & \\ & N_0^{-1}\mathbf{I}_{N-\bar{N}} \end{bmatrix} \mathbf{U}_I^*\mathbf{U}_x\mathbf{\Delta}_x^{\frac{1}{2}}| & (8) \\ &\stackrel{def}{=} \log|\mathbf{I}_N + \mathbf{K}^{-1}| & \\ &= \log\prod_{i=1}^{N}(1+\lambda_i(\mathbf{K}^{-1})) \quad : \text{ where } \lambda_i(.) \text{ denotes the } i^{th} \text{ eigenvalue} & \\ &= \sum_{i=1}^{N}\log(1+\frac{1}{\lambda_i(\mathbf{K})}) \; , & (9) \end{aligned}$$

where Equation (8) above used the orthogonality of $\mathbf{U}_I$ and the matrix identity $|\mathbf{I}+\mathbf{AB}| = |\mathbf{I}+\mathbf{BA}|$. Since the function $g(x) = \log(1+\frac{1}{x})$ is *convex* for $x > 0$, then the function $\sum_{i=1}^{N} g(x_i)$ is *Schur convex* (see page 64 of [9]). Moreover, since $\mathbf{K}$ is a Hermitian positive–definite matrix, we have the following inequality (see page 223 of [9])

$$\sum_{i=1}^{N}\log(1+\frac{1}{\lambda_i(\mathbf{K})}) \geq \sum_{i=1}^{N}\log(1+\mathbf{K}^{-1}(i,i)) \; , \tag{10}$$

where $\mathbf{K}^{-1}(i,i)$ denotes the (i,i) diagonal element of $\mathbf{K}^{-1}$. Combining (9) and (10), we get

$$I(\mathbf{X}\,;\mathbf{Y}) \geq \sum_{i=1}^{N}\log(1+\mathbf{K}^{-1}(i,i)) \; ,$$

where equality, and hence minimization of $I(\mathbf{X}\,;\mathbf{Y})$, is achieved if and only if $\mathbf{K}^{-1}$ is a diagonal matrix. It can be readily verified through direct substitution in (8) that diagonalization of $\mathbf{K}^{-1}$ is achieved by choosing $\mathbf{U}_I$ to have the form

$$\mathbf{U}_I = \mathbf{U}_x \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_{\bar{N}} & ? & \cdots & ? \end{bmatrix} , \tag{11}$$

where $\mathbf{e}_i$ is the i^{th} unit vector and ? denotes a set of $N-\bar{N}$ orthonormal vectors spanning a subspace orthogonal to $\{\mathbf{e}_1, \cdots, \mathbf{e}_{\bar{N}}\}$. Equation (11) implies that the $\bar{N}$ dominant eigenvectors of $\tilde{\mathbf{R}}_{xx}$ and $\mathbf{R}_{ii}$ are identical.

To determine the eigenvalues $\delta_{I,k}$, we substitute (11) in (7) to get

$$\begin{aligned} I(\mathbf{X};\mathbf{Y}) &= \log|\mathbf{I}_N + \mathbf{\Delta}_x \begin{bmatrix} \mathbf{\Delta}_I + N_0\mathbf{I}_{\bar{N}} & \\ & N_0\mathbf{I}_{N-\bar{N}} \end{bmatrix}^{-1}| & \\ &= \sum_{k=1}^{\bar{N}}\log(1+\frac{\delta_{x,k}}{N_0+\delta_{I,k}}) + \sum_{k=\bar{N}+1}^{N}\log(1+\frac{\delta_{x,k}}{N_0}) \; . & (12) \end{aligned}$$

Differentiating (12) with respect to $\delta_{I,k}$ subject to (4), it can be easily shown that the eigenvalues of the worst–case NBI auto–correlation matrix satisfy the *quadratic equation*

$$\delta_{I,k}^2 + (2N_0+\delta_{x,k})\delta_{I,k} + (N_0^2 + N_0\delta_{x,k} - \frac{\delta_{x,k}}{\alpha}) = 0 \; . \tag{13}$$

The only meaningful solution [2] to (13) is given by

$$\delta_{I,k} = max(0, -(N_0 + \frac{\delta_{x,k}}{2}) + \sqrt{\frac{\delta_{x,k}^2}{4} + \frac{\delta_{x,k}}{\alpha}}) \ : \ 1 \leq k \leq \bar{N} \ , \tag{14}$$

where the constant α is computed by applying the trace constraint in (4), from which we get

$$\sum_{k=1}^{\bar{N}} \sqrt{\frac{\delta_{x,k}^2}{4} + \frac{\delta_{x,k}}{\alpha}} = E_I + \sum_{k=1}^{\bar{N}} (\frac{\delta_{x,k}}{2} + N_0) \ . \tag{15}$$

□

In summary, the following offline procedure [3] can be used to compute the worst–case NBI auto–correlation matrix :

1. Perform an eigen–decomposition of $\tilde{\mathbf{R}}_{xx}$ such that the eigenvalues are arranged in descending order, i.e, $\delta_{x,1} \geq \delta_{x,2} \geq \cdots \geq \delta_{x,N}$.
2. Form the matrix $\bar{\mathbf{U}}_x$ from the first $\bar{N}$ eigenvectors of $\tilde{\mathbf{R}}_{xx}$ corresponding to its $\bar{N}$ largest eigenvalues.
3. The $\bar{N}$ non–zero eigenvalues of the worst–case NBI auto–correlation matrix are given by (14) where the constant α is computed from (15).
4. The corresponding worst–case NBI auto–correlation matrix and minimum mutual information are given by

$$\begin{aligned} \mathbf{R}_{II}^{worst} &= \bar{\mathbf{U}}_x diag(\delta_{I,1}, \cdots, \delta_{I,\bar{N}}) \bar{\mathbf{U}}_x^* \\ I_{min}(\mathbf{X};\mathbf{Y}) &= \sum_{k=\bar{N}+1}^{N} \log(1 + \frac{\delta_{x,k}}{N_0}) + \sum_{k=1}^{\bar{N}} (1 + \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{\alpha\delta_{x,k}}} - \frac{1}{2}}) \ . \end{aligned}$$

<u>Remarks</u>

1. It was shown in [7, 5] that assuming NBI to be Gaussian minimizes the block mutual information over all other probability density functions.
2. For a stationary input sequence, i.e., when $\mathbf{R}_{xx}$ is Toeplitz, the matrix $\tilde{\mathbf{R}}_{xx}$ is also Toeplitz (due to the fully–windowed Toeplitz structure of $\mathbf{H}$). Hence, it is *asymptotically* (as $N \to \infty$) equivalent to a *circulant* matrix [6]. Therefore, we can view its eigenvalues as uniformly–spaced samples of a power spectral density (psd) and the subspace occupied by NBI as its frequency support. However, for the finite blocklength scenario, the rank deficient representation of the narrowband property is an abstraction.
3. The case of wide–band interference follows as a special case by setting $\bar{N} = N$.
4. The complexity of the offline procedure outlined above is dominated by the complexity of the eigen–decomposition of $\tilde{\mathbf{R}}_{xx}$ in Step 1.
5. It can be readily verified that the auto–correlation matrix of worst–case NBI is *non-Toeplitz* in general. Hence, for the interferer to cause the highest channel block throughput loss, it has to employ a *non-stationary* signaling scheme.

[2] The other solution to the quadratic equation results in *negative* NBI energy and is therefore rejected.

[3] This procedure is meant to be a tool for analysis and the focus of this letter is *not* on algorithms for interference suppression, which has a vast literature.

6. In our numerical experiments, we observed that the worst–case NBI psd is approximately flat over its optimized frequency support. This is the *dual* of an observation made in [1] that the best–case input psd (that *maximizes* the channel block throughput), under a fixed total input energy constraint, is approximately flat, as long as its frequency support is optimized.

7. The case of multiple frequency–disjoint narrowband interferers can be easily accommodated by taking the sum in (15) over the disjoint NBI frequency ranges (or equivalently subspaces).

3 Numerical Example

We consider the ISI channel $h(D) = 1 + 0.9D$ and normalize its energy to unity. The input sequence is assumed white with unity variance, i.e., $\mathbf{R}_{xx} = \mathbf{I}_{N+\nu}$. The input SNR level is fixed at 10 dB, i.e., $N_0 = 0.1$. In Figure 1, we show the energy distribution of worst–case NBI when the total interference energy is fixed at $E_I = 10$. NBI is assumed to occupy 6 subchannels; i.e., $\bar{N} = 6$ out of a total of $N = 32$ subchannels. It is intuitively appealing that the energy of worst–case NBI is concentrated in the 6 subchannels of the channel spectrum that have the highest gain. It can also be seen from Figure 1 that the shape of worst–case NBI is almost flat. While the shape of worst–case NBI power spectral density is not critical, its frequency support is. In other words, the NBI frequency occupancy within the transmission bandwidth can significantly affect the achievable channel block throughput.

4 Conclusions

An analytical framework for upper–bounding the throughput loss due to NBI on linear noisy ISI channels was presented. We derived the spectral characteristics of NBI that minimize the channel block throughput in closed form. It was shown that the dimension-limited interference can inflict the maximal harm by focusing its attention on the sweet spots of signal transmission. This allows the designer to easily predict the maximum throughput loss due to NBI and serves as a useful tool for modeling NBI in a conservative design.

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List of Figures

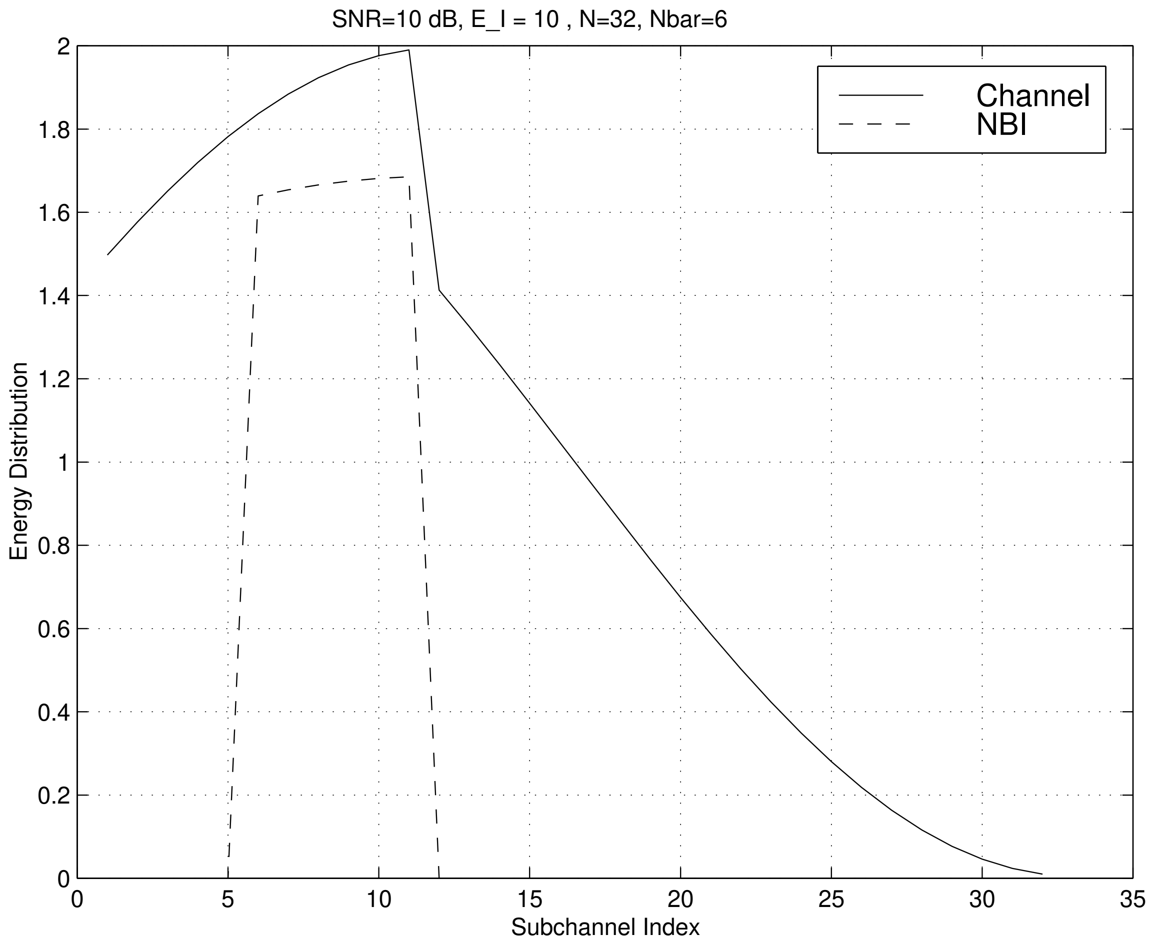

Figure 1: Worst–case NBI Energy Distribution Example