Throughput-Maximizing FIR Transmit Filters for Linear Dispersive Channels

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Abstract

Optimum FIR transmit filters for symbol–by–symbol transmission on linear dispersive additive–Gaussian–noise channels are derived by maximizing the channel throughput, subject to a fixed average input energy constraint. This maximized throughput is compared with that achievable with water–pour and flat transmit filters. The effect of transmit filter optimization on the receiver performance is investigated by considering the popular MMSE–DFE receiver structure.

I. Introduction

Maximizing the achievable throughput of noisy intersymbol interference (ISI) channels requires optimization of both the transmitter and receiver ends of the communication system. In multicarrier transceivers, an orthogonal transformation (such as FFT) is used to convert the wideband ISI channel into a large number of parallel ISI–free narrow–band subchannels that can be individually decoded. On the other hand, single carrier transceivers commonly employ finite–impulse–response (FIR) filters at both the transmitter and receiver ends to mitigate the ISI and noise.

The FIR minimum–mean–square–error decision feedback equalizer (MMSE–DFE) is a widely–used receiver structure on severe–ISI channels. Optimizing the MMSE–DFE receiver filter settings for the infinite–length and finite–length cases was treated in [8, 5] and [2], respectively. While there have been several studies on the transmitter optimization problem, they either assume an infinite–length transmit filter as in [8, 5] or block–by–block transmission as in [3]. For block–based transmission, the throughput–maximizing input covariance process was shown in [3] to be *nonstationary* and the eigenvalues of its auto–correlation matrix obey a water–pour distribution. Although this input covariance process was shown to have a special structure that results in a generalized constant–parameter lattice filter implementation of the transmit filter, the synthesis procedure is somewhat involved. Moreover, FIR synthesis filters for this nonstationary process are *time–varying* and hence costly to implement.

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Our objective in this paper is to maximize the throughput of noisy ISI linear channels by passing the white input sequence through an FIR linear–time–invariant (LTI) transmit filter that introduces correlation between the samples of its input sequence while preserving its stationarity and average input energy.

Notation

Throughout this paper, vectors are denoted in lower–case bold and matrices in upper–case bold. The transpose is denoted by $(.)^t$ and the complex–conjugate transpose by $(.)^*$. Furthermore, $\mathbf{0}_{i\times j}$ is the all–zeros matrix with i rows and j columns, and $\mathbf{e}_j$ denotes a column vector with an j^{th} element of unity and zeros everywhere else. When the component of a vector are to be emphasized, the first and last components, separated by a colon, are given as a subscript to the vector as in $\mathbf{x}_{k+N-1:k-\nu}$.

II. Transmit Filter Optimization

We start in this section by describing the input–output model. Then, we formulate the channel throughput maximization problem in terms of the FIR transmit filter coefficients.

A. Input-Output Model

We adopt the following discrete–time representation of an additive–noise dispersive channel

$$\mathbf{y}_k = \sum_{m=0}^{\nu} \mathbf{h}_m x_{k-m} + \mathbf{n}_k \ , \tag{1}$$

where $\mathbf{h}_m \stackrel{def}{=} \begin{bmatrix} h_{l-1,m} & \cdots & h_{0,m} \end{bmatrix}^t$ is the m^{th} (vector) channel impulse response (CIR) coefficient, assuming an oversampling factor of l, and ν is called the channel memory. We assume a continuous transmission bandwidth and perfect knowledge of the channel and noise characteristics at the transmitter and receiver ends. Both the input sequence, $\{x_k\}$, and the noise sequence, $\{\mathbf{n}_k\}$, are assumed to be stationary, zero–mean, and have (non–singular) Toeplitz auto–correlation matrices denoted by $\mathbf{R}_{xx}$ and $\mathbf{R}_{nn}$, respectively. The input sequence is generated by an FIR transmit filter according to

$$x_k = \sum_{n=0}^{\nu_t} p_n \epsilon_{k-n} \ , \tag{2}$$

where $\{p_i\}_{i=0}^{\nu_t}$ are the transmit filter coefficients and $\{\epsilon_k\}$ is a white unit–energy sequence.

Over any block of N output symbols, (1) becomes

$$\mathbf{y}_{k+N-1:k} = \mathbf{H}\mathbf{x}_{k+N-1:k-\nu} + \mathbf{n}_{k+N-1:k} \ ,$$

where $\mathbf{H}$ is a fully–windowed Toeplitz channel matrix whose first block–row is equal to the $(\nu+1)$ CIR coefficients appended by zeros. Similarly, the vector representation of (2) is

$$\mathbf{x}_{k+N-1:k-\nu} = \mathbf{P}\epsilon_{k+N-1:k-\nu-\nu_t} \ . \tag{3}$$

B. Maximizing the Channel Throughput

It was shown in [3] that for Gaussian input and noise sequences, the (normalized per input symbol) channel throughput is given by

$$\bar{I} = \frac{1}{(N+\nu)} \log_2(|\mathbf{I}_{N+\nu} + \mathbf{H}^*\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{R}_{xx}|) \ , \tag{4}$$

where $\mathbf{I}_{N+\nu}$ is the identity matrix of size $(N+\nu)$.

We want to find the optimum $\mathbf{R}_{xx}$ that maximizes (4) **subject** to the constraint of a unit–energy transmit filter, i.e., $\sum_{i=0}^{\nu_t} |p_i|^2 = 1$. This constraint holds the average input energy to the channel at its level with the white input sequence $\{\epsilon_k\}$.

Remarks:

- From Equation (3), $\mathbf{R}_{xx} = \mathbf{P}\mathbf{P}^*$ which guarantees that the input correlation matrix will be a Hermitian positive semi–definite matrix whose size, $(N+\nu)$, is *independent* of the transmit filter length. Perhaps less obvious is the fact that since $\mathbf{P}$ is a *fully-windowed* Toeplitz matrix, $\mathbf{R}_{xx}$ will also be Toeplitz with an (i,j) element of $\mathbf{R}_{xx}(i,j) = \sum_{k=0}^{\nu_t} p_k p^*_{k+|j-i|}$. In addition, the constraint $\sum_{i=0}^{\nu_t} |p_i|^2 = 1$ is equivalent to $\frac{trace(\mathbf{R}_{xx})}{(N+\nu)} = 1$. Hence, the FIR transmit filter preserves the stationarity and average input energy level of its input sequence $\{\epsilon_k\}$, as desired.

- Without the Toeplitzness and invertibility constraints, the optimum Hermitian positive semi–definite $\mathbf{R}_{xx}$ that maximizes (4) was shown in [3] to have the same *eigenvectors* as the rank–N matrix $\mathbf{H}^*\mathbf{R}_{nn}^{-1}\mathbf{H}$. Moreover, its *eigenvalues*, denoted here by δ_i, are computed according to a *water-pour* distribution. The achievable throughput with this optimum nonstationary $\mathbf{R}_{xx}$ is given by [3]

$$\bar{I}_{water-pour} = \frac{1}{(N+\nu)} \sum_{i=1}^{\bar{N}} \log_2(1+\sigma_i\delta_i) \ , \tag{5}$$

  where σ_i are the eigenvalues of $\mathbf{H}^*\mathbf{R}_{nn}^{-1}\mathbf{H}$ and $\bar{N}$ $(\leq N)$ is the number of usable input dimensions as determined by the water–pour procedure. This procedure optimizes the transmission bandwidth and is employed in the generalized decision feedback equalizer (GDFE) structure of [6] to achieve optimum performance even when $\mathbf{R}_{xx}$ is singular. The GDFE is a block–by–block processing time–varying structure, hence, a guard sequence is needed to eliminate inter–block interference resulting in a throughput penalty.

C. Computational Algorithm

The optimum FIR transmit filter coefficients $\mathbf{p} = \begin{bmatrix} p_0 & p_1 & \cdots & p_{\nu_t} \end{bmatrix}^t$ are computed by solving the following constrained non–linear optimization problem

$$\min_{\mathbf{p}} f(\mathbf{p}) \stackrel{def}{=} -\bar{I} \quad \text{subject to} \quad g(\mathbf{p}) \stackrel{def}{=} \mathbf{p}^*\mathbf{p} - 1 = 0 \ .$$

We used the sequential quadratic programming (SQP) algorithm, as implemented by the *constr* function in the *MATLAB Optimization Toolbox* [1], to compute optimum $\mathbf{p}$. The SQP algorithm solves the Kuhn–Tucker equations for the Lagrangian cost function $L(\mathbf{p},\lambda) = f(\mathbf{p}) + \lambda g(\mathbf{p})$ by executing the following two steps

1. For each iteration k, solve the following quadratic programming subproblem for d_k

$$\min_{d_k} \frac{1}{2} d_k^* \mathbf{H}_k d_k + (\nabla f(\mathbf{p}_k))^* d_k \quad \text{subject to} \quad (\nabla g(\mathbf{p}_k))^* d_k + g(\mathbf{p}_k) = 0$$

   where ∇ denotes the gradient operator and $\mathbf{H}_k$ is a numerical approximation of the Hessian matrix of $L(\mathbf{p},\lambda)$.

2. Use d_k computed above to update the transmit filter coefficients according to

$$\mathbf{p}_{k+1} = \mathbf{p}_k + \alpha_k d_k \ ,$$

   where α is the step size.

III. Application to MMSE-DFE

The FIR MMSE–DFE consists of a fractionally–spaced feedforward filter that has (lN) taps and is denoted by the vector $\mathbf{w}^*$, and a symbol–spaced feedback filter that is *assumed to have N_b strictly causal taps* denoted by $\{-b_1, -b_2, \cdots, -b_{N_b}\}$. For analytical convenience, we define the augmented vector $\tilde{\mathbf{b}}^* \stackrel{def}{=} \begin{bmatrix} \mathbf{0}_{1\times\Delta} & 1 & b_1^* & \cdots & b_{N_b}^* \end{bmatrix}$, where Δ is the delay of feedforward filter.

We showed in detail in [2] that the optimum feedback and feedforward filter settings are given by

$$\tilde{\mathbf{b}}_{opt.} = \mathbf{L}\mathbf{e}_{(\Delta_{opt.}+1)} \tag{6}$$

$$\mathbf{w}^*_{opt.} = d^{-1}_{\Delta_{opt.}} \mathbf{e}^*_{(\Delta_{opt.}+1)} \mathbf{L}^{-1}\mathbf{H}^*\mathbf{R}^{-1}_{nn} \ . \tag{7}$$

The optimum delay, $\Delta_{opt.}$, is given by

$$\Delta_{opt.} = argmax_{0\le i\le(N+\nu-1)}\{d_i\} \ . \tag{8}$$

The matrix $\mathbf{L}$ that appears in (6) and (7) and the scalars $\{d_i\}$ in (7) and (8) are defined by the following *Cholesky factorization* [2]

$$\mathbf{R} \stackrel{def}{=} \mathbf{R}_{xx}^{-1} + \mathbf{H}^*\mathbf{R}_{nn}^{-1}\mathbf{H} \stackrel{def}{=} \mathbf{L}\mathbf{D}\mathbf{L}^* \ , \tag{9}$$

where $\mathbf{L}$ is a lower–triangular monic (has ones along the main diagonal) matrix and $\mathbf{D}$ is a diagonal matrix with entries $d_0, d_1, \cdots, d_{N+\nu-1}$. Furthermore, the *unbiased* decision–point SNR of the FIR MMSE–DFE is given by [2]

$$SNR_{MMSE-DFE,U} = S_x d_{\Delta_{opt.}} - 1 \ , \tag{10}$$

where $S_x \stackrel{def}{=} \frac{trace(\mathbf{R}_{xx})}{(N+\nu)}$ is the average energy of the $(N+\nu)$ input symbols.

IV. Simulation Results

In our computer simulations we consider the CIR $h(D) = 0.407 + 0.815D + 0.407D^2$, which was also used in [7], and will be denoted here by Channel 1. Unless otherwise stated, the transmit filter memory is assumed equal to the channel memory, i.e., $\nu_t = \nu = 2$. The SQP algorithm was initialized with the a unit–impulse transmit filter, i.e., $\mathbf{p} = \mathbf{e}_1$. Figures 1 and 2 show that the optimized 3–tap FIR filter achieves a throughput improvement of about 0.2 bits/symbol over the flat transmit filter case. For comparison purposes, the upper–bound on throughput achieved by a water–pour transmit filter and computed according to (5) is also plotted.

Next, we investigated the effect of transmit filter optimization on the unbiased decision–point SNR of the MMSE–DFE defined in (10). We assume a symbol–spaced feedforward filter, i.e., $l = 1$. Figure 3 shows that appreciable performance improvements are achieved, especially at low input SNR. At very high input SNR, transmit filter shaping becomes less important and even the water–pour energy distribution becomes essentially flat [5, 3]. On the other hand, the effect of transmit filter optimization becomes more significant for highly–dispersive channels, as shown in Figure 4 for the EPR6 channel $h(D) = \frac{(1+D)^4(1-D)}{\sqrt{28}}$, which we call here Channel 2.

As expected, further improvements in performance are achievable by increasing the length of the FIR transmit filter. Instead of implementing the resulting long FIR transmit filter, we can synthesize the corresponding Toeplitz input correlation matrix using all–pole or lattice filters with less than $(N+\nu)$ coefficients, irrespective of the transmit filter length, as illustrated by the following example.

Numerical Example

Consider Channel 1 with $N = 5$ feedforward taps and an input SNR of 20 dB. The optimization procedure results in the 3–tap unit–energy transmit filter $p(D) = 0.7798 + 0.5994D - 0.1806D^2$. With a flat input, the optimum settings and decision–point SNR of the MMSE–DFE were computed to be

$$
\begin{aligned}
\Delta_{flat} &= 5 \\
\mathbf{b}_{flat} &= \begin{bmatrix} 1 & 1.3291 & 0.4999 \end{bmatrix}^t \\
\mathbf{w}_{flat} &= \begin{bmatrix} -0.0648 & 0.2128 & -0.4585 & 0.8045 & 1.2254 \end{bmatrix}^t
\end{aligned}
$$

$$SNR^{flat}_{MMSE-DFE,U} = 15.09 \text{ dB} ,$$

while with the optimum 3–tap FIR transmit filter, we get

$$
\begin{aligned}
\Delta_{opt} &= 4 \\
\mathbf{b}_{opt} &= \begin{bmatrix} 1 & 0.9699 & 0.3905 & -0.0271 \end{bmatrix}^t \\
\mathbf{w}_{opt} &= \begin{bmatrix} 0.093 & -0.3629 & 0.868 & 0.8389 & 0 \end{bmatrix}^t
\end{aligned}
$$

$$SNR^{optimized}_{MMSE-DFE,U} = 16.78 \text{ dB} .$$

By increasing ν_t to 15, we achieve $SNR^{optimized}_{MMSE-DFE,U} = 19.43$ dB. The non–zero correlations lags of the optimum Toeplitz $\mathbf{R}_{xx}$ in this case were computed to be $\{1, 0.3626, -0.2203, 0.0857, 0.0493, -0.1956, 0.3156\}$. Instead of implementing the 16–tap FIR transmit filter, we can efficiently compute reduced–parameter all–pole and lattice implementations using the Levinson algorithm [4]. The corresponding pole polynomial, lattice filter reflection coefficients, and prediction error variances of orders 0 through $N+\nu-1 = 6$ are given in Table 1.

V. Conclusions

We showed that appreciable gains in the decision–point SNR of the MMSE–DFE can be achieved by optimizing the transmit filter. The optimum transmit filters derived in this paper are FIR filters that maximize the channel throughput while preserving the stationarity and average input energy level of the input sequence. Because of the Toeplitz structure of the optimized input correlation matrix, alternative all–pole or lattice implementations of the transmit filter can be computed efficiently using the Levinson algorithm [4].

References

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Order i	Reflection Coefficient	Prediction Error Variance	i^{th} order Pole Polynomial
0	0	1	1
1	-0.3626	0.8685	$1 - 0.3626D$
2	0.405	0.7260	$1 - 0.5095D + 0.405D^2$
3	-0.4749	0.5623	$1 - 0.7018D + 0.647D^2 - 0.4749D^3$
4	0.579	0.3738	$1 - 0.9768D + 1.0216D^2 - 0.8813D^3 + 0.579D^4$
5	-0.6632	0.2094	$1 - 1.3608D + 1.606D^2 - 1.5588D^3 + 1.2268D^4 - 0.6632D^5$
6	-0.0794	0.2081	$1 - 1.3081D + 1.5087D^2 - 1.435D^3 + 1.0993D^4 - 0.5552D^5 - 0.0794D^6$

Table 1: Iterations of the Levinson algorithm on optimized $\mathbf{R}_{xx}$ with 16–tap FIR filter

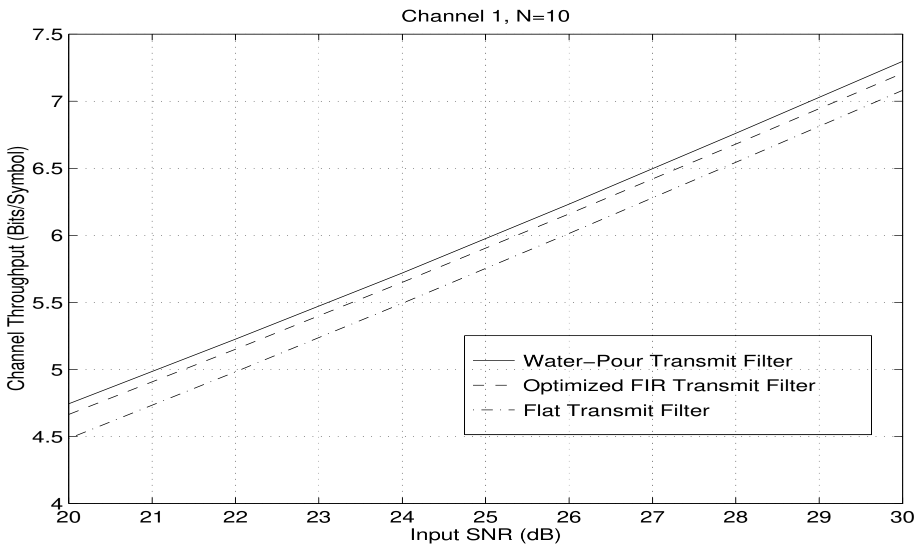

Figure 1: Channel throughput versus input SNR with water-pour, flat, and optimized FIR transmit filters

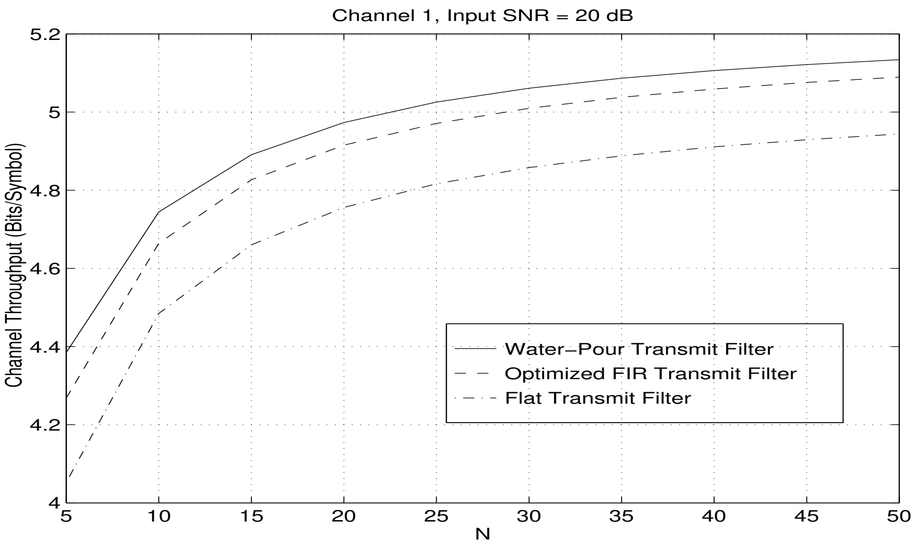

Figure 2: Channel throughput versus N with water-pour, flat, and optimized FIR transmit filters

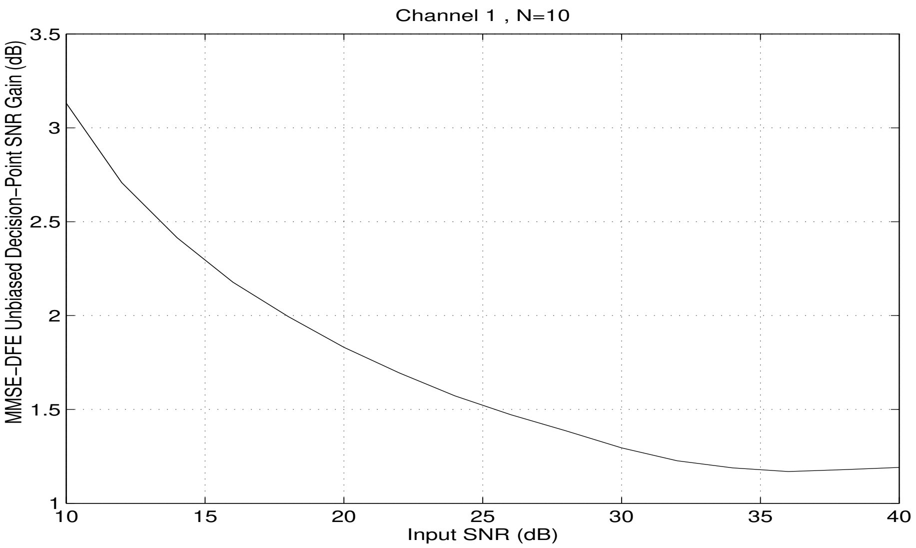

Figure 3: $SNR_{MMSE-DFE,U}$ improvement with transmit filter optimization

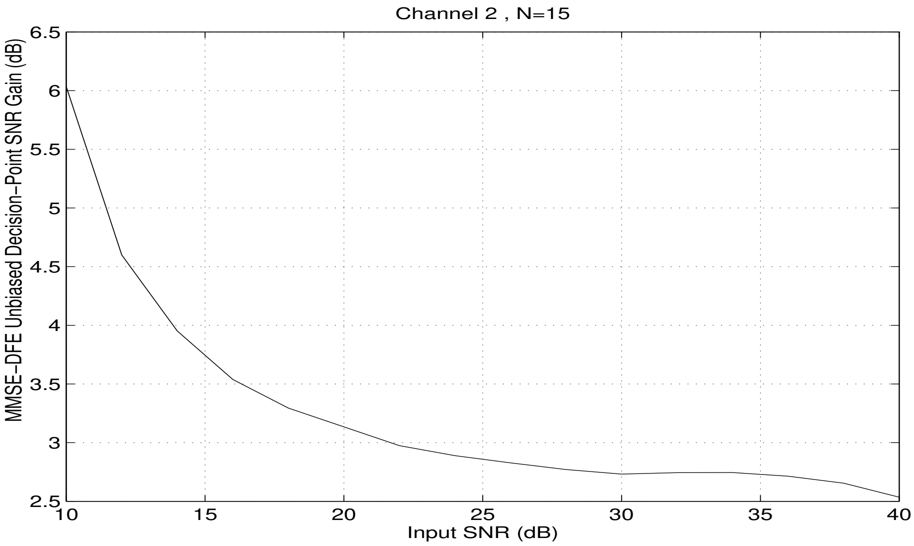

Figure 4: $SNR_{MMSE-DFE,U}$ improvement with transmit filter optimization