Relative Performance Auctions

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Abstract

We study a multi-unit auction environment in which the agents who carry out the bidding are rewarded according to the ranking of their surpluses. We show that in a symmetric equilibrium of our auction game – if one exists – an efficient allocation may not be attainable. Our auction mechanism may be viewed as combining the features of a discriminatory auction and of a tournament. The inefficient outcome of our auction is remarkable given the efficiency of each of its components.

1 Introduction

A common implication of the principal-agent literature is that the agent's payoff should be, at least in part, contingent on performance. While the early principal-agent literature focused on absolute measures of the agent's performance, theorists have long recognized that pay based on relative measures of performance may implement efficient levels of effort. One of the common examples of relative payment schemes is a tournament (Lazear and Rosen, 1981). A growing body of empirical evidence supports the hypothesis that the outcome of many payment schemes is consistent with tournament theory (see e.g. Main, O'Reilly and Wade, 1993; Eriksson, 1999).

In his paper we study auctions in which bidders care not only about winning, but also about the ranking of their ex-post profits. The theoretical principal-agent literature suggests that tying an agent's compensation to performance ensures that the agent's actions are consistent with the principal's objectives. It is well known (Lazear and Rosen, 1981; Nalebuff and Stiglitz, 1982) that tournaments with risk neutral agents implement the first best allocation. In addition, since common shocks to performance may be "filtered out" in a tournament, risk averse agents may prefer relative performance compensation to compensation schemes based on absolute performance.

Bidding in most procurement settings is carried out by agents who are rewarded according to their performance. In this paper we investigate the efficiency properties of an auction in which the bidders' rewards are structured as in a tournament. The main purpose of this paper is to show that a particular auction environment in which agents seek to increase their profits relative to the profits of their competitors may not be allocatively efficient. This is remarkable since, when considered separately, the two ingredients of our model – the auction and the rank-order tournament – yield efficient outcomes.

Before fleshing out our model, we turn to a short review of the related literature. Nalebuff and Stiglitz (1983) discuss the use of absolute or relative performance standards and the design of contests. They show that agents may prefer tournaments to individualistic evaluation methods and argue that while tournaments can implement perfect-information first-best levels of effort, the principal's problem involves an essential trade-off between efficiency and the amount of risk borne by tournament participants. McAfee and McMillan (1986) derive the optimal linear bidding contract in the presence of moral hazard. They show that the optimal contract involves some risk sharing, even in situations when the bidders are risk neutral. Fullerton and McAfee (1999) discuss the use of auctions as means for selecting tournament participants. They show that both discriminatory and uniform auctions may fail to result in an efficient selection of participants. While the model considered by Fullerton and McAfee discusses situations in which auctions are used only as a selection device *prior* to the tournament, our models consider auction models in which payoffs are affected by the *ex-post* ranking of profits. Our analysis is related to the work of Morgan, Stieglitz and Reis (2003) who show that aggressive bidding may arise when bidders have preferences that depend on the payoff of a rival bidder. While the outcomes of the auctions considered by Morgan *et al.* are efficient, they show that revenue equivalence does not hold when bidders are motivated by relative profit considerations.

2 The model

Our focus is on a simple discriminatory auction environment with three risk-neutral bidders and two objects in which the winner who realizes the highest *ex-post* surplus (valuation net of the bid) receives a positive payoff, while the other winner receives a zero payoff. While our model focuses on an auction in which bidders place buying bids, application to a procurement (selling) setting is straightforward. Our model may be viewed as the last stage in a two-stage game. While we do not model the first stage, one could think of the bidders' valuations as the outcome of a process whereby bidders exert effort to learn their valuations (or to reduce their costs in a procurement setting). We assume that bidders draw their valuations independently according to a differentiable and strictly increasing c.d.f. $F(\cdot)$. We denote the density of valuations by $f(\cdot)$ and assume that bidders demand only one unit. The auction assigns the objects to the highest two bidders, who pay their bids. Denote the symmetric equilibrium bid function (if one exists) by $b(v)$ and the associated surplus function $v - b(v)$ by $s(v)$. We assume that the winning bidder who has the highest surplus receives payoff $\phi(s(v))$, with $\phi(s(v)) > (=)0$ as $v > (=)0$. We assume that $\phi(\cdot)$ is differentiable and weakly increasing and that any tied bids are resolved with equal probability. We also assume that if the surplus of the winners is equal, each winner receives $\phi(s(v))/2$. While relatively simple, our setting is general enough to capture the essential features of a tournament ($\phi(\cdot)$ is constant) and permits a characterization of the allocative efficiency of the auction. Our model also accommodates situations in which the winning bidder who has the highest surplus is paid a proportional bonus ($\phi(\cdot)$ is linear). We are not concerned with the existence or computation of an equilibrium; however, we show that if a symmetric equilibrium exists in this game, the auction may not allocate the goods to the bidders who have the highest values.

The following Lemma shows that the bid and surplus function cannot be both increasing in equilibrium.

Lemma 1 *In a symmetric equilibrium of the bidding game, the bid and surplus functions cannot be both strictly increasing.*

Proof. Suppose that $b(\cdot)$ and $s(\cdot)$ are strictly increasing; thus, both functions are invertible and differentiable a.e. Bidder i solves

$$v_i \in \underset{x}{\operatorname{argmax}}\, \phi(v - b(x)) \Pr\left[b(x) > \min\{b(v_{-i})\}; v - b(x) > \max\{s(v_{-i})\}\right].$$

Let $\Pr\left[\min\{v_{-i}\} < a; \max\{v_{-i}\} < b\right] = G(a,b)$, where $G(a,b) = F(a)(2F(b) - F(a))$ if $a \leq b$ and $G(a,b) = F^2(b)$ if $b < a$. In an equilibrium characterized by strictly increasing bid and surplus functions, the following must hold:

$$v_i \in \underset{x}{\operatorname{argmax}}\, \phi\left(v_i - b(x)\right) G(x, s^{-1}(v_i - b(x))). \tag{1}$$

Since $G(a,b)$ is differentiable, the first order condition in a symmetric equilibrium can be written as

$$-\phi'(s(v))b'(v)G(v,v) + \phi(s(v))\left[G_1(v,v) - G_2(v,v)\frac{b'(v)}{s'(v)}\right] = 0.$$

Since $G_1(v,v) = 0$, the first order condition requires that for all v

$$s'(v) = -2\frac{f(v)}{F(v)}\frac{\phi(s(v))}{\phi'(s(v))} < 0,$$

contradicting the assumption that $s(\cdot)$ is strictly increasing. ■

Lemma 2 *In any symmetric equilibrium of the bidding game, a bidder's expected payoff is monotone in his type.*

Proof. Player i chooses his bid β to maximize his expected payoff

$$\Pi(v_i, \beta) = \phi(v - \beta) \Pr[\beta \geq \min b(v_{-i}), v_i - \beta \geq v_J - b(v_J)], \tag{2}$$

where J is the index of the bidder who submits the highest bid among player i's opponents. Note that, for any $v' > v$, $\Pi(v, b(v)) < \Pi(v', b(v))$. Since a higher type can obtain a greater ex-ante expected payoff than a lower type by bidding the same value as the lower type, his equilibrium expected payoff must be greater than the equilibrium expected payoff of a lower type. ■

We outline next some of the properties of the equilibrium bid and surplus functions.

Lemma 3 *If a symmetric monotone equilibrium exists, the bid function must be continuous. In addition, there does not exist $\bar{v} \in [0, 1)$ so that $s(v) = s(\bar{v})$ for all $v \geq \bar{v}$.*

Proof. The proof for the continuity of the bid function is an application of Theorem 3 and Corollary 4.4 in Jackson and Swinkels (2005). Note that if there exists $\bar{v} \in [0, 1)$ so that $s(v) = s(\bar{v})$ for all $v \geq \bar{v}$, then a bidder with type $v \geq \bar{v}$ who lowers his bid (and the probability of winning) by a small amount receives a discrete jump in expected payoff by becoming the high surplus bidder. ■

We are now ready to present our main result.

Theorem 1 *This game does not have a strictly increasing symmetric equilibrium.*

Proof. We established in Lemma 1 that there does not exist a symmetric equilibrium in which both the bidding function $b(\cdot)$ and the surplus $s(v) = v - b(v)$ are strictly increasing. We will now establish that this game has no symmetric equilibrium characterized by a strictly increasing bid function. Suppose that the symmetric bid function is strictly increasing. Lemma 3 guarantees that the surplus function is continuous and that no symmetric equilibrium exists in which the surplus function $s(v)$ takes the same value for all types $v \in [0, 1]$. Note that in any equilibrium $s(0) = 0$ and $s(v) > 0$ for all $v > 0$, since any positive type can guarantee himself a positive expected surplus by bidding 0. For all $\varepsilon \in (0, 1)$, define $v_\varepsilon \in [\varepsilon, 1]$ as

$$v_\varepsilon = \inf(\arg\min_{x \in [\varepsilon, 1]} s(x)).$$

Similarly, define

$$\underline{v}_\varepsilon = \inf(x | x \in [0, v_\varepsilon], s(x) = s(v_\varepsilon))$$

and

$$\hat{v}_\varepsilon \in \arg\max_{x \in [\varepsilon, v_\varepsilon]} s(x).$$

Note that the three objects above are well-defined, since in any symmetric equilibrium $s(\cdot)$ must be continuous. Furthermore, since we assumed that $s(\cdot)$ is not strictly increasing and since $s(\cdot)$ cannot be constant over a range of values in the neighborhood of 1, there exists $\varepsilon^* \in (0, 1)$ such that $v_{\varepsilon^*} > \varepsilon^*$. This implies that $\underline{v}_{\varepsilon^*} < \hat{v}_{\varepsilon^*} < v_{\varepsilon^*}$ (recall that $s(\cdot)$ cannot be constant in equilibrium). Denote the set of types against which type v has positive payoff by

$$A(v) = \{(\varpi_1, \varpi_2) | \varpi_1 \leq \varpi_2, b(v) \geq b(\varpi_1), s(v) \geq s(\varpi_2)\}; \tag{3}$$

since $b(\cdot)$ is assumed to be monotone, we have

$$A(v) = \{(\varpi_1, \varpi_2) | \varpi_1 \leq \varpi_2, v \geq \varpi_1, s(v) \geq s(\varpi_2)\}.$$

Note that since $s(\underline{v}_{\varepsilon^*}) = s(v_{\varepsilon^*})$, we can write

$$A(v_{\varepsilon^*}) = A(\underline{v}_{\varepsilon^*}) \cup \{A(v_{\varepsilon^*}) \cap \{\varpi | \varpi_1 \leq \varpi_2, v_{\varepsilon^*} < \varpi_2 < \underline{v}_{\varepsilon^*}\}\}.$$

Choose now $\varpi = (\varpi_1, \varpi_2)$ such that $v_{\varepsilon^*} < \varpi_2 < \underline{v}_{\varepsilon^*}$ and observe that $\varpi \notin A(v_{\varepsilon^*})$. Thus, $A(v_{\varepsilon^*}) = A(\underline{v}_{\varepsilon^*})$ and, similarly, $A(v_{\varepsilon^*}) \subset A(\hat{v})$. Finally, note that the expected equilibrium profit of a player with type v is $\Pi(v, b(v)) = \phi(s(v)) \Pr[A(v)]$. It follows that $\Pi(v_2, b(v_2)) \geq \Pi(v_0, b(v_0))$, and since by construction $\hat{v}_{\varepsilon^*} < v_{\varepsilon^*}$, we have established a contradiction with the result of Lemma 2 and the proof of the Theorem is concluded. ■

3 Conclusions

The key insight into our inefficiency result is that the allocation of goods in an auction in which bidders have *ex-post* profit motives may not be efficient. The intuition is that decreasing one's bid, while decreasing the probability of winning an object, increases the surplus in the event that an object is won. Since there are two objects, but only the highest *ex-post* profit maker has positive payoff, an increase in surplus has a higher effect on expected payoff than a decrease in the probability of winning. It is noteworthy that this intuition fails in case of a uniform pricing rule: in a uniform auction a bidder can not affect the surplus ranking by manipulating his bid.

The revenue ranking of the uniform and discriminatory auctions is known to be ambiguous. Multi-unit auction design, however, seems to favor the discriminatory auction format primarily because outcomes resembling collusion are likely to obtain in the uniform auction (Back and Zender, 1993). Our model shows that the presence of relative performance incentives in the preferred discriminatory auction format may generate *ex-post* allocative inefficiencies.

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