A. Neishtadt, „On adiabatic perturbation theory for systems with impacts".

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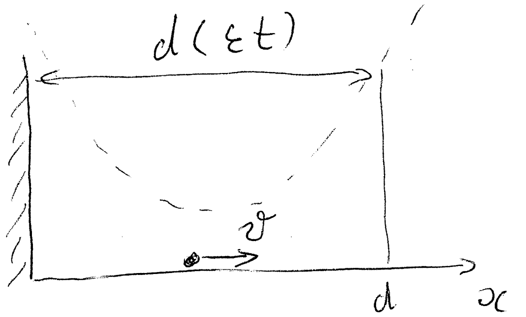
Topics:

1. Adiabatic description of motions with impacts.
2. Jump of adiabatic invariant at transition between modes of motion.

1. Model: generalised Fermi-Ulam problem: one-dimensional motion between slowly moving walls in presence of slowly varying in time potential field.



$0 < \varepsilon \ll 1$

Distance between walls: $d = d(\varepsilon t)$

Potential: $U = U(x, \varepsilon t)$

Between collisions:

$$\ddot{x} = -\frac{\partial U(x, \varepsilon t)}{\partial x}$$

Collisions are elastic.

$\tau = \varepsilon t$ - slow time

$d = d(\tau)$, $U = U(x, \tau)$

$d \in C^\infty$, $U \in C^\infty$

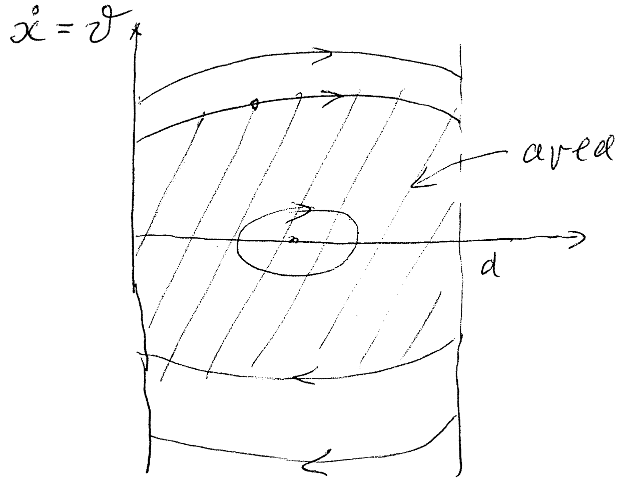
$C^{-1} < d < C$, derivatives of d, U are bounded

For simplicity let only the right wall move.

First consider the problem for frozen τ:

$\tau = const$

This is a Hamiltonian system with 1 degree of freedom. We can draw its phase portrait



Trajectories are level lines of the Hamiltonian

$$E = \frac{v^2}{2} + U(x, \tau)$$

Introduce action-angle variables I, φ:

$(v, x) \mapsto (I, \varphi)$ (canonical transformation)

$$I = \frac{\text{"area"}}{2\pi}$$

$$\varphi = 2\pi \frac{\text{"time"}}{T} \mod 2\pi$$

(T - period)

$W(x, I, \tau)$ - generating function

$$W = \int_0^x v(I, q, \tau)\, dq$$

$$v = \frac{\partial W}{\partial x}, \quad \varphi = \frac{\partial W}{\partial I}$$

$$E = E(I, \tau)$$

Equations of motion:

$$\dot{I} = 0$$

$$\dot{\varphi} = \frac{\partial E}{\partial I}$$

Now switch on slow changing of τ:

$\mathring{\tau} = \varepsilon \neq 0$

Make the canonical transformation of variables $(\vartheta, x) \mapsto (I, \varphi)$ with ~~the same~~ generating function W as before.

Change of I, φ between collisions is described by the Hamiltonian system with the Hamiltonian

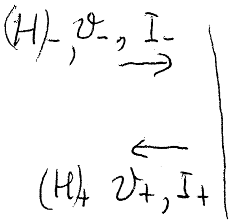
$$H(I,\varphi,\tau) = E(I,\tau) + \frac{\partial W}{\partial t} = E(I,\tau) + \varepsilon H_1(I,\varphi,\tau)$$

$$H_1 = \frac{\partial W}{\partial \tau}.$$

Between collisions:

$$\begin{cases} \mathring{I} = -\varepsilon \dfrac{\partial H_1}{\partial \varphi} \\ \mathring{\varphi} = \dfrac{\partial E}{\partial I} + \varepsilon \dfrac{\partial H_1}{\partial I} \end{cases}$$

At collision with the moving wall



$$\vartheta_+ = -\vartheta_- + 2\varepsilon d'$$

<u>Lemma</u> At collision with the moving wall the value of H is preserved: $(H)_- = (H)_+$, i.e.

$$E(I_-,\tau) + \varepsilon H_1(I_-, \pi - 0, \tau) =$$

$$= E(I_+,\tau) + \varepsilon H_1(I_+, \pi + 0, \tau)$$

Proof. We use variational principle:

$$\delta \int_{t_0}^{t_1} (\vartheta\, dx - E\, dt) = 0 \Rightarrow \delta \int_{t_0}^{t_1} (I\, d\varphi - H\, dt) = 0$$

$$\delta \left(\int_{t_0}^{t_*} I\, d\varphi - H\, dt + \int_{t_*}^{t_1} I\, d\varphi - H\, dt \right) = 0$$

(t_* is the time moment of collision).

$$(I\,\delta\varphi - H\,\delta t)_{t_* - 0} = (I\,\delta\varphi - H\,\delta t)_{t_* + 0}$$

$\delta\varphi = 0$ (collision is always at $\varphi = \pi$) $\Rightarrow$

$$(H)_{t_* - 0} = (H)_{t_* + 0} \qquad \blacksquare$$

Now we will perform a standard procedure of perturbation theory. We eliminate dependence of the Hamiltonian on the fast angle up to very high order in ε. Then the conjugate to this angle action variable will be an approximate first integral of the motion. This is a standard approach developed for smooth systems. We will show that it works also for systems with impacts.

Transformation of variables:

$$(I,\varphi)\mapsto(\hat{I},\hat{\varphi})$$

Generating function

$$\hat{I}\varphi+\varepsilon S(\hat{I},\varphi,\tau,\varepsilon)$$

Old and new variables are related as follows:

$$I=\hat{I}+\varepsilon\frac{\partial S}{\partial\varphi},\quad \hat{\varphi}=\varphi+\varepsilon\frac{\partial S}{\partial I}$$

We are looking for S in the form

$$S=S_1+\varepsilon S_2+\dots+\varepsilon^{r-2}S_{r-1}$$

and the new Hamiltonian in the form

$$\mathcal{H}(\hat{I},\hat{\varphi},\tau,\varepsilon)=\mathcal{H}_{\Sigma,r}(\hat{I},\tau,\varepsilon)+\varepsilon^{r}H_r(\hat{I},\varphi(\hat{I},\hat{\varphi},\tau,\varepsilon),\tau,\varepsilon$$

$$\mathcal{H}_{\Sigma,r}=E+\varepsilon\mathcal{H}_1+\dots+\varepsilon^{r-1}H_{r-1}$$

We know that

$$\mathcal{H}=H+\varepsilon^2\frac{\partial S}{\partial\tau}\Rightarrow$$

$$\mathcal{H}(\hat{I},\varphi,\tau,\varepsilon)=H(\hat{I}+\varepsilon\frac{\partial S}{\partial\varphi},\varphi,\tau,\varepsilon)+\varepsilon^2\frac{\partial S}{\partial\tau}$$

We will equate terms of the same order in ε in the left and right hand sides.

E.g. for terms of order ε:

$$\mathcal{H}_1(\hat{I},\tau)=\frac{\partial E}{\partial I}\frac{\partial S_1}{\partial\varphi}+H_1(\hat{I},\varphi,\tau)$$

This implies:

$$\mathcal{H}_1(\hat{I},\tau)=\langle H_1\rangle^{\varphi}=0.\qquad (\langle\cdot\rangle^{\varphi}\text{ - average over }\varphi)$$

$$\frac{\partial S_1}{\partial\varphi}=-\frac{1}{\partial E/\partial\tau}H_1(\hat{I},\varphi,\tau)$$

$$S_1(\hat{I},\varphi,\tau)=-\frac{1}{\partial E/\partial\tau}\int_0^{\varphi}H_1(\hat{I},\vartheta,\tau)\,d\vartheta$$

We will be able to construct all S_i, H_i:

$$H_i \in C^\infty, \quad S_i \in C^{\infty;\,0}_{I,\tau;\,\varphi}$$

$S_i \in C^\infty$ for $\varphi \neq \pi$.

Between collisions

$$\dot{\hat{I}} = -\varepsilon^r \frac{\partial H_r}{\partial \varphi}$$

$$\dot{\hat{\varphi}} = \frac{\partial \mathcal{H}_{\Sigma,r}}{\partial \hat{I}} + \varepsilon^r \frac{\partial H_r}{\partial \hat{I}}$$

<u>Theorem</u>. For $\hat{I}, \hat{\varphi}$ introduced above

$$\hat{I}(t) = \hat{I}(0) + O(\varepsilon^r), \quad 0 \le t \le 1/\varepsilon^k$$

$$\hat{\varphi}(t) = \hat{\varphi}(0) + \frac{1}{\varepsilon} \int_0^\tau \frac{\partial \mathcal{H}_{\Sigma,r}(\hat{I}(0), v)}{\partial \hat{I}} \, dv + O(\varepsilon^{r-k})$$

Here k is any prefixed number.

(So, the accuracy is the same as in smooth systems.

<u>Proof</u>

Transformation of variables

$$(I, \varphi) \mapsto (\tilde{I}, \tilde{\varphi})$$

Generating function:

$$\tilde{I}\varphi + \varepsilon S(\tilde{I}, \varphi, \tau, \varepsilon)$$

$$S = S_1 + \dots \varepsilon^{r-2} S_{r-1} + \dots + \varepsilon^{r+k-2} S_{r+k-1}$$

New Hamiltonian $\tilde{\mathcal{H}} = H + \varepsilon^2 \frac{\partial \tilde{S}}{\partial \tau}$.

$$\tilde{\mathcal{H}} = \mathcal{H}_{\Sigma, r+k}(\tilde{I}, \tau, \varepsilon) + \varepsilon^{r+k} H_{r+k}(\tilde{I}, \varphi(\tilde{I}, \tilde{\varphi}, \tau, \varepsilon), \tau, \varepsilon)$$

$$\hat{I} = \tilde{I} + O(\varepsilon^r), \quad \hat{\varphi} = \tilde{\varphi} + O(\varepsilon^r).$$

So, it is enough to get estimates for $\tilde{I}, \tilde{\varphi}$.

Between collisions

$$\dot{\tilde{I}} = -\varepsilon^{r+k}\frac{\partial H_{r+k}}{\partial \hat{\varphi}} = O(\varepsilon^{r+k})$$

Thus change of $\hat{I}$ between two collisions in $O(\varepsilon^{r+k})$.
During time $\sim 1/\varepsilon^k$ total change of $\hat{I}$ between all ($\sim 1/\varepsilon^k$) collisions in $O(\varepsilon^r)$.

At collision:

$$H_- = H_+$$

$$\left(\mathcal{H} - \varepsilon^2\frac{\partial\tilde{S}}{\partial\tau}\right)_- = \left(\mathcal{H} - \varepsilon^2\frac{\partial\tilde{S}}{\partial\tau}\right)_+ \Rightarrow$$

$$\mathcal{H}_{\Sigma,r+k}(\hat{I}_-,\tau,\varepsilon) - \varepsilon^2\frac{\partial\tilde{S}(\tilde{I}_-,\pi,\tau,\varepsilon)}{\partial\tau} =$$

$$= \mathcal{H}_{\Sigma,r+k}(\tilde{I}_+,\tau,\varepsilon) - \varepsilon^2\frac{\partial\tilde{S}(\tilde{I}_+,\pi,\tau,\varepsilon)}{\partial\tau} + O(\varepsilon^{r+k})$$

We use here that S is a continuos function of φ at $\varphi = \pi$ (i.e. at collision).

$$\mathcal{H}_{\Sigma,r+k}(I,\tau,\varepsilon) = E(I,\tau) + O(\varepsilon)$$

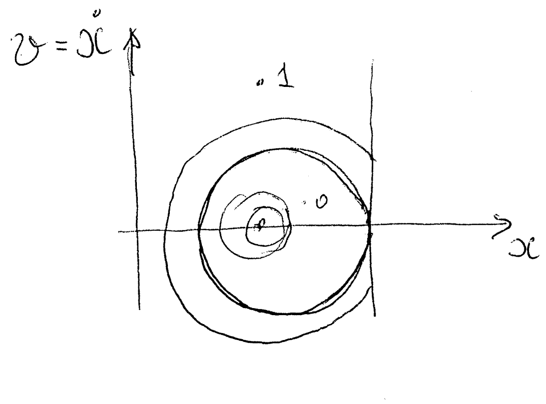
This implies that change of $\hat{I}$ at a collision is $O(\varepsilon^{r+k})$: $\hat{I}_+ - \hat{I}_- = O(\varepsilon^{r+k})$

Total change of $\hat{I}$ for all $\sim 1/\varepsilon^k$ collisions is $O(\varepsilon^r)$.
Therefore the total change of $\hat{I}$ during time $1/\varepsilon^k$ is $O(\varepsilon^r)$.

We skip the proof of the estimate for $\tilde{\varphi}$ here. ∎

2.

For frozen τ there are two modes of motion: without collisions with the walls and with such collisions. When τ changes the system may pass from one mode of motion to anothe. On the phase portrait this means passage through a "separatrix" - the curve $E = const$ tangent to a wall.



For each mode of motion we can introduce "improved adiabatic invariant" - a function of phase variables whose value remains a constant along trajectory with an accuracy $O(\varepsilon^2)$ for motions far from the separatrix. For motions with collisions we just have introduced such function:

$$J = I - \varepsilon \frac{\partial S_1}{\partial \varphi} .$$

We would like to calculate the change in J for the case of separatrix crossings.

Denote

$S(\tau)$ - area surrounded by the separatrix

$$\Theta(\tau) = -\frac{d S(\tau)}{d\tau}$$

$$A(\tau) = \frac{2\sqrt{2}}{3\pi \, \partial U/\partial x|_{x=d(\tau)}}$$

Let I_0 be an initial value of action (at $\tau = \tau_0$)
Then the moment τ_* of separatrix crossing calculated in adiabatic approximation is a root of the equation

$$S(\tau_*) = 2\pi I_0$$

We assume that

$$\Theta_* = \Theta(\tau_*) \neq 0$$

$$A_* = A(\tau_*) \neq 0$$

Denote J_0 - an initial value of the improved adiabatic invariant J (far from the separatrix, in the mode without collisions), J_1 - a final value of J (far from the separatrix, in the mode with collisions)

Then

$$J_1 - J_0 = A_* (\varepsilon \Theta_*)^{3/2} f(\xi) + O(\varepsilon^{5/3})$$

(hypothetical estimate of accuracy here is $O(\varepsilon^2)$)

Here

$$f(\xi) = \frac{3}{4\sqrt{\pi}} \int_0^\infty t^{-3/2} \left(\frac{1}{2} - \xi + \frac{1}{t} - \frac{e^{-\xi t}}{1 - e^{-t}} \right) dt$$

$$\xi \in [0, 1]$$

$$\xi = \operatorname{Frac}\left\{ \frac{1}{2} - \frac{1}{2\pi}\left(\varphi_0 + \frac{1}{\varepsilon} \int_{\tau_0}^{\tau_*} \omega_0(I, \tau)\, d\tau + O(\varepsilon) \right\}\right.$$

$$\omega_0(I, \tau) = \frac{\partial E(I, \tau)}{\partial I}.$$

For small ε the value ξ is very sensitive to small changes in initial conditions. It can be treated as a random value uniformly distributed on $[0, 1]$.

Numerical check of formula for jump of adiabatic invariant. $U=(x-x_0)^2/2$, $0<x_0<d(\xi)$. 10

Dashed line - $f(\xi)$. Solid line - $(J_1-J_0)/[A_*(\varepsilon\Theta)^{3/2}]$

δ - the mean square deviation of these curves.