MATH 2413 Fall 2012, Lectures 12-16

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Outline of Sections 3.1–3.3

1. Derivatives of polynomials and exponential functions: \( y = c \) (\( c \) is a constant), \( y = x \), \( y = x^r \), \( r \in \mathbb{R} \) is a constant, \( y = e^x \);

\[
\frac{d}{dx} (c) = 0, \quad \frac{d}{dx} (x) = 1, \quad \frac{d}{dx} (x^r) = rx^{r-1}
\]

2. Basic differentiation rules: \( (f + g)' \), \( (f - g)' \), \( (cf)' \) where \( c \) is a constant, \( (fg)' \), \( (f/g)' \); Assume that \( f \) and \( g \) are differentiable. Then we have

\[
(f(x) + g(x))' = f'(x) + g'(x),
\]

\[
(f(x) - g(x))' = f'(x) - g'(x),
\]

\[
(cf(x))' = cf'(x),
\]

\[
(f(x)g(x))' = f'(x)g(x) + f(x)g'(x),
\]

\[
\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.
\]
Exercises:

1. If $f$, $g$ and $h$ are differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, can you develop a differentiation rule for $(fgh)(x)$?

2. Is it true that if $f$ is a differentiable function from $\mathbb{R}$ to $\mathbb{R}$, then for every $n \in \mathbb{N}$, $[f(x)]^n$ is also differentiable?

3. Can you develop a differentiation rule for the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = |x|$? (Note $x \neq 0$.)

4. If one guesses possible functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$\frac{d}{dx}f(x) = f(x).$$

What are the possible candidates for $f$?
Appendix: Proof of the Chain Rule

Let $y = f(u)$ and $u = g(x)$. If $g$ is differentiable at $x$ and $f$ is differentiable at $u = g(x)$, then the composite function $f \circ g$, defined by $(f \circ g)(x) = f(g(x))$, is differentiable at $x$ and

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

or

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$
Proof of the Chain Rule: Define the function $E(k)$ by

$$E(k) = \begin{cases} \frac{f(u+k) - f(u)}{k} - f'(u), & \text{if } k \neq 0, \\ 0, & \text{if } k = 0. \end{cases}$$

Note $f(x)$ is differentiable, then for any $k \in \mathbb{R}$, $E(k)$ is continuous and $f(u+k) - f(u) = (f'(u) + E(k))k$. Let $u = g(x)$, $k = g(x + h) - g(x)$, then $u + k = g(x + h)$ and

$$f(g(x + h)) - f(g(x)) = (f'(g(x)) + E(k))(g(x + h) - g(x)).$$

Note $g(x)$ is differentiable and $\lim_{h \to 0} E(k) = E(\lim_{h \to 0} k) = E(0) = 0$, one has

$$\frac{d}{dx} f(g(x)) = \lim_{h \to 0} \frac{f(g(x + h)) - f(g(x))}{h} = \lim_{h \to 0} \frac{(f'(g(x)) + E(k))(g(x + h) - g(x))}{h} = f'(g(x))g'(x).$$

That is $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. 