MATH 2413 Fall 2012, Lectures 4-5

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Outline of Section 2.1–2.2

1. Limiting process,
   a) limit of $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}$;
   b) limit of an inscribed $n$-gon in a disc;
   c) the area problem with $y = x^2$, $x \in [0, 1]$;
   d) the tangent line problem;

2. Limits of functions: explain $f(x) \rightarrow L$ as $x \rightarrow a$;

3. One-sided limits; the Heaviside function;
Appendix: Basic logic

a. Important results in mathematics are called **Theorems**, which are often stated in the form "If $P$ then $Q$". In abbreviation, $P \Rightarrow Q$.

b. Converse of $P \Rightarrow Q$: $Q \Rightarrow P$.

c. Negation of the statement $P$ is "not $P$": $\sim P$.

d. Contrapositive of $P \Rightarrow Q$: $\sim Q \Rightarrow \sim P$.

(i) If the statement $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are both true, then we call $P$ and $Q$ are equivalent. i.e.

$$P \iff Q$$

(ii) A statement is equivalent to its contrapositive. i.e.

If $P$ then $Q \iff$ If $\sim Q$ then $\sim P$. 
Examples for logic statement

Example 1. Let $P =$"$n^2$ is even", $Q =$"$n$ is even". Then the statement "If $n^2$ is even, then $n$ is even" can be written as $P \Rightarrow Q$. Then we have

(i) The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$, i.e. "If $n$ is even, then $n^2$ is even".
(ii) The negation of $P$ is $\sim P$, i.e. "$n^2$ is not even."
(iii) The contrapositive of $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$, i.e. "If $n$ is not even, then $n^2$ is not even."

Example 2. Note that the converse of a true statement may NOT be true. For instance, the statement: If $0 < x < y$, then $x^2 < y^2$" is true, but its converse " If $x^2 < y^2$, then $0 < x < y$" is false.
For more details of mathematical logic, please refer to some references, e.g., the book "A Mathematical Introduction to Logic", Enderton, H. B. (Section 1.2 is available as preview at Amazon).
Exercises

1. Let $L \in \mathbb{R}$ and $a \in \mathbb{R}$ be constants. Suppose that the function $f$ is defined in a neighborhood of $a$. Use plain language to explain the following limits.

   a) $\lim_{x \to a} f(x) = L$,
   b) $\lim_{x \to a^+} f(x) = L$,
   c) $\lim_{x \to a^-} f(x) = L$,
   d) $\lim_{x \to a} f(x) = +\infty$,
   e) $\lim_{x \to a^+} f(x) = +\infty$,
   f) $\lim_{x \to a^-} f(x) = +\infty$.

2. Find the following limits

   a) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$,
   b) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$,
   c) $\lim_{x \to 1^-} \frac{|x^3 - 1|}{x - 1}$.

3. Sketch the graph of the function and use it to determine the values of $a$ for which $\lim_{x \to a} f(x)$ exists.

   
   \[
   f(x) = \begin{cases} 
   1 + \sin x & \text{if } x < 0, \\
   \cos x & \text{if } 0 \leq x \leq \pi, \\
   \sin x & \text{if } x > \pi.
   \end{cases}
   \]
4. Find an interval $I \subseteq \mathbb{R}$ such that $y = f(x) = 1/x^3$ is larger than $10^{3000}$ for every $x \in I$.

5. Recall that for every $a \in \mathbb{R}$ and $b \in \mathbb{R}$, we have

$$||a| - |b|| \leq |a \pm b| \leq |a| + |b|.$$ 

If we want $|x^2 - 4| \leq 10^{-30000}$, how close should $x$ be to 2? Hint: We can firstly require that $x$ is close to 2 with $|x - 2| < 1$. With this condition, we will discover that we also need to require $|x - 2| < a$ for some $a$ so that $|x^2 - 4| < 10^{-30000}$. Then $|x - 2| < \text{min}\{1, a\}$ is sufficient to imply $|x^2 - 4| \leq 10^{-30000}$, where $\text{min}\{1, a\}$ stands for the minimum between 1 and $a$.

6. Let $f$ and $g$ be two functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $f(x)g(x) = 1$ for all $x \in \mathbb{R}$ and $\lim_{x \to 1} f(x) = 0$. Does the limit $\lim_{x \to 1} g(x)$ exist? Justify your answer.