2.3 SAMPLING BANDPASS SIGNALS

Although satisfying the majority of sampling requirements, the sampling of low-pass signals, as in Figure 2–6, is not the only sampling scheme used in practice. We can use a technique known as bandpass sampling to sample a continuous bandpass signal that is centered about some frequency other than zero Hz. When a continuous input signal’s bandwidth and center frequency permit us to do so, bandpass sampling not only reduces the speed requirement of A/D converters below that necessary with traditional low-pass sampling; it also reduces the amount of digital memory necessary to capture a given time interval of a continuous signal.

By way of example, consider sampling the band-limited signal shown in Figure 2–7(a) centered at \( f_c = 20 \text{ MHz} \), with a bandwidth \( B = 5 \text{ MHz} \). We use the term bandpass sampling for the process of sampling continuous signals whose center frequencies have been translated up from zero Hz. What we’re calling bandpass sampling goes by various other names in the literature, such as IF sampling, harmonic sampling[2], sub-Nyquist sampling, and undersampling[3]. In bandpass sampling, we’re more concerned with a signal’s bandwidth than its highest frequency component. Note that the negative frequency portion of the signal, centered at \(-f_c\), is the mirror image of the positive frequency portion—as it must be for real signals. Our bandpass signal’s highest frequency component is 22.5 MHz. Conforming to the Nyquist criterion (sampling at twice the highest frequency content of the signal) implies that the sampling frequency must be a minimum of 45 MHz. Consider the effect if the sample rate is 17.5 MHz shown in Figure 2–7(b). Note that the original spectral components remain located at \( \pm f_c \) and spectral replications are located exactly

![Figure 2-7](image-url)
at baseband, i.e., butting up against each other at zero Hz. Figure 2–7(b) shows that sampling at 45 MHz was unnecessary to avoid aliasing—instead we've used the spectral replicating effects of Eq. (2–5) to our advantage.

Bandpass sampling performs digitization and frequency translation in a single process, often called sampling translation. The processes of sampling and frequency translation are intimately bound together in the world of digital signal processing, and every sampling operation inherently results in spectral replications. The inquisitive reader may ask, “Can we sample at some still lower rate and avoid aliasing?” The answer is yes, but, to find out how, we have to grind through the derivation of an important bandpass sampling relationship. Our reward, however, will be worth the trouble because here's where bandpass sampling really gets interesting.

Let's assume we have a continuous input bandpass signal of bandwidth $B$. Its carrier frequency is $f_c$ Hz, i.e., the bandpass signal is centered at $f_c$ Hz, and its sampled value spectrum is that shown in Figure 2–8(a). We can sample that continuous signal at a rate, say $f_s$, Hz, so the spectral replications of the positive and negative bands, P and Q, just butt up against each other exactly at zero Hz. This situation, depicted in Figure 2–8(a), is reminiscent of Figure 2–7(b). With an arbitrary number of replications, say $m$, in the range of $2f_c - B$, we see that

$$mf_s = 2f_c - B \quad \text{or} \quad f_s = \frac{2f_c - B}{m}. \quad (2–6)$$

In Figure 2–8(a), $m = 6$ for illustrative purposes only. Of course $m$ can be any positive integer so long as $f_s$ is never less than $2B$. If the sample rate $f_s$ is in-

![Figure 2–8](image)

**Figure 2–8** Bandpass sampling frequency limits: (a) sample rate $f_s = (2f_c - B)/6$; (b) sample rate is less than $f_s$; (c) minimum sample rate $f_s' < f_s$. 
creased, the original spectra (bold) do not shift, but all the replications will shift. At zero Hz, the P band will shift to the right, and the Q band will shift to the left. These replications will overlap and aliasing occurs. Thus, from Eq. (2–6), for an arbitrary \( m \), there is a frequency that the sample rate must not exceed, or

\[
f_s \leq \frac{2f_c - B}{m} \quad \text{or} \quad \frac{2f_c - B}{m} \geq f_s^*.
\]  

(2–7)

If we reduce the sample rate below the \( f_s^* \) value shown in Figure 2–8(a), the spacing between replications will decrease in the direction of the arrows in Figure 2–8(b). Again, the original spectra do not shift when the sample rate is changed. At some new sample rate \( f_{s''} \), where \( f_{s''} < f_s^* \), the replication \( P' \) will just butt up against the positive original spectrum centered at \( f_c \) as shown in Figure 2–8(c). In this condition, we know that

\[
(m + 1)f_{s''} = 2f_c + B \quad \text{or} \quad f_{s''} = \frac{2f_c + B}{m + 1}.
\]  

(2–8)

Should \( f_{s''} \) be decreased in value, \( P' \) will shift further down in frequency and start to overlap with the positive original spectrum at \( f_c \) and aliasing occurs. Therefore, from Eq. (2–8) and for \( m + 1 \), there is a frequency that the sample rate must always exceed, or

\[
f_{s''} \geq \frac{2f_c + B}{m + 1}.
\]  

(2–9)

We can now combine Eqs. (2–7) and (2–9) to say that \( f_s \) may be chosen anywhere in the range between \( f_{s''} \) and \( f_s^* \) to avoid aliasing, or

\[
\frac{2f_c - B}{m} \geq f_s \geq \frac{2f_c + B}{m + 1}.
\]  

(2–10)

where \( m \) is an arbitrary, positive integer ensuring that \( f_s \geq 2B \). (For this type of periodic sampling of real signals, known as real or first-order sampling, the Nyquist criterion \( f_s \geq 2B \) must still be satisfied.)

To appreciate the important relationships in Eq. (2–10), let's return to our bandpass signal example, where Eq. (2–10) enables the generation of Table 2–1. This table tells us that our sample rate can be anywhere in the range of 22.5 to 35 MHz, anywhere in the range of 15 to 17.5 MHz, or anywhere in the range of 11.25 to 11.66 MHz. Any sample rate below 11.25 MHz is unacceptable because it will not satisfy Eq. (2–10) as well as \( f_s \geq 2B \). The spectra resulting from several of the sampling rates from Table 2–1 are shown in Figure 2–9 for our bandpass signal example. Notice in Figure 2–9(f) that when \( f_s \) equals 7.5 MHz \( (m = 5) \), we have aliasing problems because neither
### Table 2-1: Equation (2-10) Applied to the Bandpass Signal Example

<table>
<thead>
<tr>
<th>$m$</th>
<th>$(2f_c - B)/m$</th>
<th>$(2f_c + B)/(m + 1)$</th>
<th>Optimum sampling rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35.0 MHz</td>
<td>22.5 MHz</td>
<td>22.5 MHz</td>
</tr>
<tr>
<td>2</td>
<td>17.5 MHz</td>
<td>15.0 MHz</td>
<td>17.5 MHz</td>
</tr>
<tr>
<td>3</td>
<td>11.66 MHz</td>
<td>11.25 MHz</td>
<td>11.25 MHz</td>
</tr>
<tr>
<td>4</td>
<td>8.75 MHz</td>
<td>9.0 MHz</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>7.0 MHz</td>
<td>7.5 MHz</td>
<td>—</td>
</tr>
</tbody>
</table>

**Figure 2-9** Various spectral replications from Table 2-1: (a) $f_s = 35$ MHz; (b) $f_s = 22.5$ MHz; (c) $f_s = 17.5$ MHz; (d) $f_s = 15$ MHz; (e) $f_s = 11.25$ MHz; (f) $f_s = 7.5$ MHz.
the greater than relationships in Eq. (2–10) nor \( f_s \geq 2B \) have been satisfied. The 
\( m = 4 \) condition is also unacceptable because \( f_s \geq 2B \) is not satisfied. The last 
column in Table 2–1 gives the optimum sampling frequency for each acceptable \( m \) value. Optimum sampling frequency is defined here as that frequency 
where spectral replications do not butt up against each other except at zero 
Hz. For example, in the \( m = 1 \) range of permissible sampling frequencies, it is 
much easier to perform subsequent digital filtering or other processing on the 
signal samples whose spectrum is that of Figure 2–9(b), as opposed to the 
spectrum in Figure 2–9(a).

The reader may wonder, “Is the optimum sample rate always equal to 
the minimum permissible value for \( f_s \) using Eq. (2–10)?” The answer depends 
on the specific application—perhaps there are certain system constraints that 
must be considered. For example, in digital telephony, to simplify the follow-
on processing, sample frequencies are chosen to be integer multiples of 
8 kHz [4]. Another application-specific factor in choosing the optimum \( f_s \) is the 
shape of analog anti-aliasing filters [5]. Often, in practice, high-performance 
A/D converters have their hardware components fine-tuned during manufac-
ture to ensure maximum linearity at high frequencies (>5 MHz). Their use at 
lower frequencies is not recommended.

An interesting way of illustrating the nature of Eq. (2–10) is to plot the 
minimum sampling rate, \( (2f_c+B)/(m+1) \), for various values of \( m \), as a function 
of \( R \) defined as

\[
R = \frac{\text{highest signal frequency component}}{\text{bandwidth}} = \frac{f_c + B}{B}/2.
\]

(2–11)

If we normalize the minimum sample rate from Eq. (2–10) by dividing it by 
the bandwidth \( B \), we get a curve whose axes are normalized to the bandwidth 
shown as the solid curve in Figure 2–10. This figure shows us the minimum 
normalized sample rate as a function of the normalized highest frequency 
component in the bandpass signal. Notice that, regardless of the value of \( R \), 
the minimum sampling rate need never exceed \( 4B \) and approaches \( 2B \) as the 
carrier frequency increases. Surprisingly, the minimum acceptable sampling 
frequency actually decreases as the bandpass signal’s carrier frequency in-
creases. We can interpret Figure 2–10 by reconsidering our bandpass signal 
example from Figure 2–7 where \( R = 22.5/5 = 4.5 \). This \( R \) value is indicated by 
the dashed line in Figure 2–10 showing that \( m = 3 \) and \( f_s/B \) is 2.25. With \( B = 5 \) 
MHz, then, the minimum \( f_s \) is 11.25 MHz in agreement with Table 2–1. The 
leftmost line in Figure 2–10 shows the low-pass sampling case, where the 
sample rate \( f_s \) must be twice the signal’s highest frequency component. So the 
normalized sample rate \( f_s/B \) is twice the highest frequency component over \( B \) 
or \( 2R \).
Figure 2-10 Minimum bandpass sampling rate from Eq. (2-10).

Figure 2-10 has been prominent in the literature, but its normal presentation enables the reader to jump to the false conclusion that any sample rate above the minimum shown in the figure will be an acceptable sample rate[6–12]. There's a clever way to avoid any misunderstanding[13]. If we plot the acceptable ranges of bandpass sample frequencies from Eq. (2–10) as a function of R we get the depiction shown in Figure 2-11. As we saw from Eq. (2–10), Table 2–1, and Figure 2–9, acceptable bandpass sample rates are a series of frequency ranges separated by unacceptable ranges of sample rate frequencies, that is, an acceptable bandpass sample frequency must be above the minimum shown in Figure 2–10, but cannot be just any frequency above that minimum. The shaded region in Figure 2–11 shows those normalized bandpass sample rates that will lead to spectral aliasing. Sample rates within the white regions of Figure 2–11 are acceptable. So, for bandpass sampling, we want our sample rate to be in the white wedged areas associated with some value of m from Eq. (2–10). Let's understand the significance of Figure 2–11 by again using our previous bandpass signal example from Figure 2–7.

Figure 2–12 shows our bandpass signal example R value (highest frequency component/bandwidth) of 4.5 as the dashed vertical line. Because that line intersects just three white wedged areas, we see that there are only three frequency regions of acceptable sample rates, and this agrees with our results from Table 2–1. The intersection of the R = 4.5 line and the borders of the white wedged areas are those sample rate frequencies listed in Table 2–1. So Figure 2–11 gives a depiction of bandpass sampling restrictions much more realistic than that given in Figure 2–10.

Although Figures 2–11 and 2–12 indicate that we can use a sample rate that lies on the boundary between a white and shaded area, these sample
Figure 2-11  Regions of acceptable bandpass sampling rates from Eq. (2-10), normalized to the sample rate over the signal bandwidth ($f_s/B$).

Figure 2-12  Acceptable sample rates for the bandpass signal example ($B = 5$ MHz) with a value of $R = 4.5$.  

36
rates should be avoided in practice. Nonideal analog bandpass filters, sample rate clock generator instabilities, and slight imperfections in available A/D converters make this ideal case impossible to achieve exactly. It’s prudent to keep $f_s$ somewhat separated from the boundaries. Consider the bandpass sampling scenario shown in Figure 2–13. With a typical (nonideal) analog bandpass filter, whose frequency response is indicated by the dashed line, it’s prudent to consider the filter’s bandwidth not as $B$, but as $B_{gb}$ in our equations. That is, we create a guard band on either side of our filter so that there can be a small amount of aliasing in the discrete spectrum without distorting our desired signal, as shown at the bottom of Figure 2–13.

We can relate this idea of using guard bands to Figure 2–11 by looking more closely at one of the white wedges. As shown in Figure 2–14, we’d like to set our sample rate as far down toward the vertex of the white area as we can—lower in the wedge means a lower sampling rate. However, the closer we operate to the boundary of a shaded area, the more narrow the guard band must be, requiring a sharper analog bandpass filter, as well as the tighter the tolerance we must impose on the stability and accuracy of our A/D clock generator. (Remember, operating on the boundary between a white and shaded area in Figure 2–11 causes spectral replications to butt up against each other.) So, to be safe, we operate at some intermediate point

![Diagram of bandpass sampling and aliasing](image)

**Figure 2-13** Bandpass sampling with aliasing occurring only in the filter guard bands.
Figure 2-14  Typical operating point for \( f_s \) to compensate for nonideal hardware.

away from any shaded boundaries as shown in Figure 2-14. Further analysis of how guard bandwidths and A/D clock parameters relate to the geometry of Figure 2-14 is available in reference \([13]\). For this discussion, we’ll just state that it’s a good idea to ensure that our selected sample rate does not lie too close to the boundary between a white and shaded area in Figure 2-11.

There are a couple of ways to make sure we’re not operating near a boundary. One way is to set the sample rate in the middle of a white wedge for a given value of \( R \). We do this by taking the average between the maximum and minimum sample rate terms in Eq. (2-10) for a particular value of \( m \), that is, to center the sample rate operating point within a wedge we use a sample rate of

\[
f_{s_{	ext{ave}}} = \frac{1}{2} \left[ \frac{2f_c - B}{m} + \frac{2f_c + B}{m+1} \right] = \frac{f_c - B/2}{m} + \frac{f_c + B/2}{m+1} \quad (2-12)
\]

Another way to avoid the boundaries of Figure 2-14 is to use the following expression to determine an intermediate \( f_{s_i} \) operating point:

\[
f_{s_i} = \frac{4f_c}{m_{\text{odd}}} \quad (2-13)
\]

where \( m_{\text{odd}} \) is an odd integer\([14]\). Using Eq. (2-13) yields the useful property that the sampled signal of interest will be centered at one fourth the sample rate \( (f_{s_i}/4) \). This situation is attractive because it greatly simplifies follow-on complex downconversion (frequency translation) used in many digital communications applications. Of course the choice of \( m_{\text{odd}} \) must ensure that the
Figure 2-15 Intermediate $f_s$ and $f_c$ operating points, from Eqs. (2-12) and (2-13), to avoid operating at the shaded boundaries for the bandpass signal example. $B = 5$ MHz and $R = 4.5$.

Nyquist restriction of $f_s > 2B$ be satisfied. We show the results of Eqs. (2-12) and (2-13) for our bandpass signal example in Figure 2-15.

2.4 SPECTRAL INVERSION IN BANDPASS SAMPLING

Some of the permissible $f_s$ values from Eq. (2-10) will, although avoiding aliasing problems, provide a sampled baseband spectrum (located near zero Hz) that is inverted from the original positive and negative spectral shapes, that is, the positive baseband will have the inverted shape of the negative half from the original spectrum. This spectral inversion happens whenever $m$, in Eq. (2-10), is an odd integer, as illustrated in Figures 2–9(b) and 2–9(e). When the original positive spectral bandpass components are symmetrical about the $f_c$ frequency, spectral inversion presents no problem and any nonaliasing value for $f_s$ from Eq. (2-10) may be chosen. However, if spectral inversion is something to be avoided, for example, when single sideband signals are being processed, the minimum applicable sample rate to avoid spectral inversion is defined by Eq. (2-10) with the restriction that $m$ is the largest even inte-
ger such that \( f_s \geq 2B \) is satisfied. Using our definition of optimum sampling rate, the expression that provides the optimum noninverting sampling rates and avoids spectral replications butting up against each other, except at zero Hz, is

\[
f_s = \frac{2f_c - B}{m_{\text{even}}},
\]

where \( m_{\text{even}} = 2, 4, 6, \) etc. For our bandpass signal example, Eq. (2-14) and \( m = 2 \) provide an optimum noninverting sample rate of \( f_{s_0} = 17.5 \) MHz, as shown in Figure 2-9(c). In this case, notice that the spectrum translated toward zero Hz has the same orientation as the original spectrum centered at 20 MHz.

Then again, if spectral inversion is unimportant for your application, we can determine the absolute minimum sampling rate without having to choose various values for \( m \) in Eq. (2-10) and creating a table like we did for Table 2-1. Considering Figure 2-16, the question is “How many replications of the positive and negative images of bandwidth \( B \) can we squeeze into the frequency range of \( 2f_c + B \) without overlap?” That number of replications is

\[
R = \frac{\text{frequency span}}{\text{twice the bandwidth}} = \frac{2f_c + B}{2B} = \frac{f_c + B/2}{B}.
\]

To avoid overlap, we have to make sure that the number of replications is an integer less than or equal to \( R \) in Eq. (2-15). So, we can define the integral number of replications to be \( R_{\text{int}} \) where

\[
R_{\text{int}} \leq R < R_{\text{int}} + 1,
\]

or

\[
R_{\text{int}} \leq \frac{f_c + B/2}{B} < R_{\text{int}} + 1.
\]

With \( R_{\text{int}} \) replications in the frequency span of \( 2f_c + B \), then, the spectral repetition period, or minimum sample rate \( f_{s_{\text{min}}} \), is

![Figure 2-16](image)

**Figure 2-16** Frequency span of a continuous bandpass signal.
\[ f_{\text{min}} = \frac{2f_c + B}{R_{\text{int}}} \quad (2-17) \]

In our bandpass signal example, finding \( f_{\text{min}} \) first requires the appropriate value for \( R_{\text{int}} \) in Eq. (2-16) as

\[ R_{\text{int}} \leq \frac{22.5}{5} < R_{\text{int}} + 1, \]

so \( R_{\text{int}} = 4 \). Then, from Eq. (2-17), \( f_{\text{min}} = (40+5)/4 = 11.25 \text{ MHz} \), which is the sample rate illustrated in Figures 2–9(e) and 2–12. So, we can use Eq. (2-17) and avoid using various values for \( m \) in Eq. (2-10) and having to create a table like Table 2–1. (Be careful though. Eq. (2–17) places our sampling rate at the boundary between a white and shaded area of Figure 2–12, and we have to consider the guard band strategy discussed above.) To recap the bandpass signal example, sampling at 11.25 MHz, from Eq. (2-17), avoids aliasing and inverts the spectrum, while sampling at 17.5 MHz, from Eq. (2-14), avoids aliasing with no spectral inversion.

Now here’s some good news. With a little additional digital processing, we can sample at 11.25 MHz, with its spectral inversion and easily reinvert the spectrum back to its original orientation. The discrete spectrum of any digital signal can be inverted by multiplying the signal’s discrete-time samples by a sequence of alternating plus ones and minus ones (1, −1, 1, −1, etc.), indicated in the literature by the succinct expression \((-1)^n\). This scheme allows bandpass sampling at the lower rate of Eq. (2-17) while correcting for spectral inversion, thus avoiding the necessity of using the higher sample rates from Eq. (2-14). Although multiplying time samples by \((-1)^n\) is explored in detail in Section 13.1, all we need to remember at this point is the simple rule that multiplication of real signal samples by \((-1)^n\) is equivalent to multiplying by a cosine whose frequency is \(f_s/2\). In the frequency domain, this multiplication flips the positive frequency band of interest, from zero to \(+f_s/2\) Hz, about \(f_s/4\) Hz, and flips the negative frequency band of interest, from \(-f_s/2\) to zero Hz, about \(-f_s/4\) Hz as shown in Figure 2–17. The \((-1)^n\) sequence is not only used for inverting the spectra of bandpass sampled sequences; it can be used to invert the spectra of low-pass sampled signals. Be aware, however, that, in the low-pass sampling case, any DC (zero Hz) component in the original continuous signal will be translated to both \(+f_s/2\) and \(-f_s/2\) after multiplication by \((-1)^n\). In the literature of DSP, occasionally you’ll see the \((-1)^n\) sequence represented by the equivalent expressions \(\cos(n\pi)\) and \(e^{jn\omega}\).

We conclude this topic by consolidating in Table 2–2 what we need to know about bandpass sampling.
Figure 2-17 Spectral inversion through multiplication by \((-1)^n\): (a) original spectrum of a time-domain sequence; (b) new spectrum of the product of original time sequence and the \((-1)^n\) sequence.

Table 2-2 Bandpass Sampling Relationships

<table>
<thead>
<tr>
<th>Requirement</th>
<th>Sample rate expression</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acceptable ranges of (f_s) for bandpass sampling: Eq. (2–10)</td>
<td>(\frac{2f_c - B}{m} \geq f_s \geq \frac{2f_c + B}{m + 1})</td>
<td>(m = \text{any positive integer so that } f_s \geq 2B).</td>
</tr>
<tr>
<td>Sample rate in the middle of the acceptable sample rate bands: Eq. (2–12)</td>
<td>(f_{s_{\text{ctr}}} = \frac{f_c - B/2}{m} + \frac{f_c + B/2}{m + 1})</td>
<td>(m = \text{any positive integer so that } f_{s_{\text{ctr}}} \geq 2B).</td>
</tr>
<tr>
<td>Sample rate forcing signal to reside at one fourth the sample rate: Eq. (2–13)</td>
<td>(f_{s_i} = \frac{4f_c}{m_{\text{odd}}})</td>
<td>(m_{\text{odd}} = \text{any positive odd integer so that } f_{s_i} \geq 2B). (Spectral inversion occurs when (m_{\text{odd}} = 3, 7, 11, \text{etc.}))</td>
</tr>
<tr>
<td>Optimum sample rate to avoid spectral inversion: Eq. (2–14)</td>
<td>(f_{s_o} = \frac{2f_c - B}{m_{\text{even}}})</td>
<td>(m_{\text{even}} = \text{any even positive integer so that } f_{s_o} \geq 2B). where (R_{\text{int}} \leq \frac{f_c + B/2}{B} &lt; R_{\text{int}} + 1).</td>
</tr>
<tr>
<td>Absolute minimum (f_s) to avoid aliasing: Eq. (2–17)</td>
<td>(f_{s_{\text{min}}} = \frac{2f_c + B}{R_{\text{int}}})</td>
<td></td>
</tr>
</tbody>
</table>


REFERENCES


