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NONLINEAR CONTROL

Exercises

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Problem 1.

Consider the state equations of the controlled Van der Pol oscillator:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2 + u,
\end{align*}
\]

where \( \epsilon > 0 \). Assume \( u \equiv 0 \) for objectives 1-2.

1. Find all equilibrium points of this system.
2. Linearize the system about all equilibrium points to determine local stability properties. Confirm your conclusion using Matlab. Do you observe any nonlinear phenomenon? How does behavior change for small or large \( \epsilon \)?
   \textit{Hint:} Use \texttt{odesolve.m} to plot solution trajectories. Use \texttt{vectline.m} to plot vector field arrows and see the direction of flow at discrete points.
3. Design a linear feedback controller for input \( u \) to stabilize any unstable equilibrium points. If a periodic orbit exists, what happens to it in the presence of linear feedback?

Problem 2.

Consider the state equations of the time-reversed Van der Pol oscillator:

\[
\begin{align*}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1 - (1 - x_1^2)x_2.
\end{align*}
\]

1. Linearize the system about the equilibrium point at the origin.
2. Using Matlab, solve the Lyapunov equation with some choice of \( Q \) to verify asymptotic stability of the linearized system.
   \textit{Hint:} Type \texttt{help lyap} in Matlab.
3. Use this quadratic Lyapunov function to fit contours (i.e., level sets) inside the limit cycle from problem 1. How does your choice for \( Q \) effect the fit?
   \textit{Hint:} Visualize contours of your Lyapunov function by:
   
   \[
   \begin{align*}
   [x1,x2]=\text{meshgrid}(-2:0.1:2); \ V=1/2*(x1.^ 2+x2.^ 2); \\
   [c, h] = \text{contour}(x1,x2,V); \ \text{clabel}(c, h);
   \end{align*}
   \]
Problem 3.

Repeat steps 1-3 of Problem 2 for the following system:

\[
\begin{align*}
\dot{z}_1 &= \frac{1}{4} z_1^2 - z_1 + \frac{1}{2} z_1 z_2 + \frac{1}{4} z_2^2 \\
\dot{z}_2 &= \frac{1}{4} (z_1 + z_2)^2 - z_2.
\end{align*}
\]

Problem 4.

Conceptual Exercise. Let \( \bar{x}(t) = \phi(t, \bar{x}_0) \) be a solution trajectory along a limit cycle of some time-invariant system \( \dot{x} = f(x) \). Let error \( \delta x(t) = x(t) - \bar{x}(t) \) and define asymptotic stability in the tracking sense: \( \delta x(t) = 0 \) is stable and there exists \( \gamma > 0 \) such that if \( |\delta x(t_0)| < \gamma \), then \( \delta x(t) \to 0 \) as \( t \to \infty \). Let orbit \( \mathcal{O} = \{ x | x = \bar{x}(t) \text{ for some } t \} \) and distance \( d(x, \mathcal{O}) := \inf_{y \in \mathcal{O}} ||x - y|| \), and define asymptotic stability in the orbital sense: \( d(x, \mathcal{O}) = 0 \) is stable and there exists \( \gamma > 0 \) such that if \( d(x(t_0), \mathcal{O}) < \gamma \), then \( d(x(t), \mathcal{O}) \to 0 \) as \( t \to \infty \). Are both forms of stability possible for this system? How about in the case of a time-varying system?

Hint: \( \phi_t(\phi_r(x)) = \phi_{t+r}(x) \) for time-invariant systems.
Problem 1.

**Lemma 1 (Poincaré Bendixson Criterion).** Consider planar system \( \dot{x} = f(x) \), where \( f(x) \) is continuously differentiable. Let \( M \) be a closed bounded subset of the plane \( \mathbb{R}^2 \) such that

1. \( M \) contains no equilibrium points, or contains only one equilibrium point such that the Jacobian matrix \( Df \) at this point has eigenvalues with positive real parts (i.e., the point is unstable).
2. Every trajectory starting in \( M \) stays in \( M \) for all future time (i.e., \( M \) is a positively invariant set).

Then, \( M \) contains a periodic orbit of the system.

Consider again the state equations of the controlled Van der Pol oscillator:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \epsilon (1 - x_1^2)x_2 + u,
\end{align*}
\]

where \( \epsilon > 0 \). Use the Poincaré-Bendixson Criterion to prove the existence of a periodic orbit for some choice of \( u \).

**Hint:** Find a function \( V(x) \) such that \( \dot{V} = L_f V \leq 0 \) for all \( x \) outside some bounded set \( M \subset \mathbb{R}^2 \) around the origin. Choosing a non-zero \( u \) might make this task easier, as long as it does not stabilize the origin!

Problem 2.

Now for the same choice of \( u \) and \( V \), consider the time-reversed dynamics

\[
\begin{align*}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1 - \epsilon (1 - x_1^2)x_2 - u,
\end{align*}
\]

What bound on state does \( \dot{V} = L_f V \leq 0 \) now imply? Use this result (possibly with LaSalle’s Invariance Principle) to prove the origin is asymptotically stable and to compute a positively invariant subset of the region of attraction.
Problem 3.

**Input-to-State Stability.** Find two 0-GAS systems such that their cascade interconnection is not GAS.

Problem 4.

**Conceptual Exercise.** Consider the class of systems

\[ \dot{x} = Ax + g(x)u \]  \hspace{1cm} (5)

where \( A \) is Hurwitz (all eigenvalues are in the left-half plane) and \( g(x) \) is a globally Lipschitz matrix function. This class includes bilinear systems. Show the following:

1. There is a system in this class that is not ISS.
2. Every system in this class is integral-ISS.
Problem 1.

**Driftless Systems.** Consider the car with trailer

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\psi}
\end{pmatrix} =
\begin{pmatrix}
\cos(\theta) \\
\sin(\theta) \\
0 \\
\sin(\psi)
\end{pmatrix} u_1 +
\begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix} u_2.
\]  

(1)

Determine whether the

1. linear approximation about the origin is controllable
2. nonlinear system is nonholonomic
3. nonlinear system is small-time locally controllable

Problem 2.

**Input-Output Linearization.** Consider the controlled *Van der Pol* oscillator

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \epsilon (1 - x_1^2) x_2 + u,
\end{align*}
\]  

(2) \hspace{1cm} (3)

where \(\epsilon > 0\). Let \(y = x_1\).

1. Determine the relative degree of this input-output system. Does this hold globally?
2. Derive an I-O linearizing controller that stabilizes the output dynamics.
3. Derive the zero dynamics to determine whether the I-O system is minimum phase.

Repeat steps 1-3 for output \(y = x_2\) and then \(y = x_1 + x_2^2\).

Problem 3.
Momentum Control of a Mechanical System. Consider the class of mechanical systems described by the dynamics

\[ M\ddot{q} + N(q) = u, \]

where \( q \in \mathbb{R}^n \) is the joint configuration vector, inertia/mass matrix \( M \in \mathbb{R}^{n \times n} \) is constant and positive definite, and input \( u \in \mathbb{R}^n \). Define the generalized momentum as \( p = M\dot{q} \in \mathbb{R}^n \). We wish to drive \( q \) to constant vector \( \bar{q} \in \mathbb{R}^n \) by zeroing the output \( y = p - (-K(q - \bar{q})) \), for constant positive-definite matrix \( K \in \mathbb{R}^{n \times n} \).

1. The state vector can be expressed as either \((q, \dot{q})\) or \((q, p)\) in \( \mathbb{R}^{2n} \). Pick one representation and write the state equations in control-affine form.

2. What is the relative degree with respect to this output? Derive an output linearizing controller that renders the output dynamics exponentially stable.

3. Express your closed-loop system in normal form. What are the zero dynamics? Is your system minimum phase?
Definition 1. Control system

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x),
\end{align*}
\]  \tag{1}

is (input/output) passive if there exists a differentiable nonnegative scalar function \(S : \mathbb{R}^{2n} \to \mathbb{R}\), called the storage function, such that

\[
\dot{S} \leq y^T u.
\]

Lemma 1. Suppose that system (1) is passive with positive-definite storage function \(S\). Given output feedback control \(u = \gamma(y)\), where \(\gamma\) is any continuous function satisfying \(y^T \gamma(y) \leq 0\), the origin is stable in the sense of Lyapunov, i.e., for every \(\epsilon > 0\) there exists a \(\delta > 0\) such that if \(|x(0)| < \delta\) then \(|x(t)| < \epsilon\) for all \(t > 0\). Moreover, the largest invariant set in \(\{x | \gamma(h(x)) = 0\}\) is attractive.

Problem 1.

**Passivity-Based Control of Robots.** Given a robotic system with configuration vector \(q\) and joint velocity vector \(\dot{q}\), the dynamical equations

\[
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla_q \mathcal{V}(q) = \tau, \tag{2}
\]

\(M(q)\) invertible, can be massaged into state-space form (1) with state \(x = (q^T, \dot{q}^T)^T\). Dynamics (2) can be used to prove the robot passivity property

\[
\dot{E} = \dot{q}^T \tau, \tag{3}
\]

where the storage function is mechanical energy \(E = \mathcal{K} + \mathcal{V}\) for kinetic energy \(\mathcal{K} = \frac{1}{2}\dot{q}^T M(q)\dot{q}\) and potential energy \(\mathcal{V}\). The map from torque input \(\tau\) to joint velocity output \(\dot{q}\) is passive, implying that passive feedback \(\tau = -p\dot{q}, p > 0\), renders (2) stable in the sense of Lyapunov and \(\dot{q} \to 0\) as \(t \to \infty\) by Lemma 1.

1. Use the control-affine state equations from (2) to show that \(\dot{q}\) is the passive output corresponding to storage function \(E\), i.e., \(L_q E = \dot{q}\).

2. This is by no means the only passive mapping for this system. Consider the storage function

\[
S(q, \dot{q}) = \frac{1}{2}(E(q, \dot{q}) - E_{\text{ref}})^2, \tag{4}
\]
for some constant $E_{\text{ref}}$. Your goal is to design a passive feedback law to force the total energy $E$ of system (2) to this constant reference. Use the passivity property (3) to find a new passive input-output mapping that yields a passive feedback law implying $E \to E_{\text{ref}}$ as $t \to \infty$ by Lemma 1.

*Hint:* Multiple passive mappings can exist for the same input. What is $L_yS$?

3. Can we still invoke Lemma 1 in the presence of actuator saturation $\tau = \text{sat}(\gamma(y))$, i.e., when passive feedback $\gamma(y)$ is upper and lower bounded by some constant vector $K$ and $-K$, respectively?

### Problem 2.

**Passivity-Based Control of the Compass-Gait Biped.**

Go through the instructions in README.txt to familiarize yourself with the biped simulator. For the sake of clarity, “passive” walking refers to walking downhill with no control torques (i.e., unactuated), which should not be confused with the above notion of “passivity-based” control.

Type `opp` to clear the environment, then `pop` and enter 2 for a passive dynamic walker that has been pre-loaded with the passivity-based control law from the previous exercise. Note that $x_i$ is the initial condition corresponding to the post-impact state on the hybrid limit cycle of a passive walking gait. Set control gain $p = 0$ and run `eigenmap2(x_i)` to compute the eigenvalues of the hybrid limit cycle. Set $p = 10$ and re-compute the eigenvalues. Are they different? What does this mean in terms of transient response to perturbations?

Let’s now examine how passivity-based control effects the basin of attraction of the locally stable hybrid limit cycle (defined at the post-impact event). Perturb the initial conditions for the hybrid limit cycle and study the biped’s response: `pop, p=0, xi(1) = xi(1)+0.03, walk2(xi,4,1)`.

*Note:* You can watch the animation by running `walk2(xi,4,2)`.

Does the biped recover with the help of passivity-based control? `pop, p=10, xi(1) = xi(1)+0.03, walk2(xi,4,1)`

Try perturbing other states in the initial conditions. What does passivity-based control do to the basin of attraction?
Consider the second-order, single-input system
\[
\dot{x}_1 = -ax_1 + x_1^2 - x_2^2, \quad (1)
\]
\[
\dot{x}_2 = u, \quad (2)
\]
where \(a \in \mathbb{R}\) is a constant parameter. Your task is to construct a real-valued state feedback function \(u = k(x_1, x_2)\) with \(k(0, 0) = 0\) such that the origin \((x_1, x_2) = (0, 0)\) of the closed-loop system is globally asymptotically stable. This task is impossible if \(a < 0\), even for local asymptotic stability (why?), so we will assume \(a > 0\) for now and consider the case \(a = 0\) at the end of the assignment. Your feedback law should be continuous everywhere, and at least locally Lipschitz continuous (if not \(C^1\) or smoother) everywhere except possibly at the origin.

You will measure the performance of a particular feedback law using the the cost functional
\[
J = \int_0^\infty \left[ x_1^2(t) + x_2^2(t) + u^2(t) \right] dt, \quad (3)
\]
which is to be evaluated along trajectories from fixed initial states. The Jacobi linearization of the system at the origin is
\[
\dot{x}_1 = -ax_1, \quad (4)
\]
\[
\dot{x}_2 = u, \quad (5)
\]
which is stabilizable but not controllable. The linear feedback law \(u = -x_2\) is optimal for this system (4)–(5) with respect to the cost (3), and the resulting minimum cost from each initial state \((x_1(0), x_2(0))\) is
\[
\min J = \frac{1}{2a} x_1^2(0) + x_2^2(0). \quad (6)
\]
When applied to the original nonlinear system (1)–(2), this linear feedback \(u = -x_2\) renders the origin locally asymptotically stable.

1. For the system (1)–(2) with the linear feedback \(u = -x_2\), is the origin globally asymptotically stable? If not, estimate its region of attraction when \(a = 1\).

2. Show that the system (1)–(2) is not feedback linearizable at the origin.
This system admits a global control Lyapunov function of the form

$$V(x_1, x_2) = \frac{1}{2a}x_1^2 + [x_2 + \mu(x_1)]^2,$$

(7)

where $\mu : \mathbb{R} \to \mathbb{R}$ is an appropriately chosen function having the following properties:

(a) $\mu$ is differentiable everywhere, and its derivative $\mu'$ is locally Lipschitz continuous (this ensures that $L_fV$ is locally Lipschitz continuous),

(b) $\mu(0) = \mu'(0) = 0$ (this ensures that the control Lyapunov function (7) is approximately the same as the minimum cost value function (6) for points near the origin).

3. For $a = 1$, find a function $\mu$ which satisfies (a) and (b) and which makes (7) a control Lyapunov function for the system (1)–(2). You may find it convenient to piece together various functions on various intervals, but keep in mind the smoothness constraint in (a). There are many possible valid choices for this function $\mu$.

Using your choice for $\mu$, you can construct two globally stabilizing feedback laws, the first using Sontag’s formula

$$k(x_1, x_2) = \begin{cases} -\frac{L_fV(x_1, x_2) + \sqrt{[L_fV(x_1, x_2)]^2 + [L_gV(x_1, x_2)]^2}}{L_gV(x_1, x_2)} & \text{when } L_gV(x_1, x_2) \neq 0 \\ 0 & \text{when } L_gV(x_1, x_2) = 0, \end{cases}$$

(8)

and the second using the modification

$$k(x_1, x_2) = \begin{cases} -\frac{L_fV(x_1, x_2) + \sqrt{[L_fV(x_1, x_2)]^2 + \left(x_1^2 + x_2^2\right)[L_gV(x_1, x_2)]^2}}{L_gV(x_1, x_2)} & \text{when } L_gV(x_1, x_2) \neq 0 \\ 0, & \text{when } L_gV(x_1, x_2) = 0. \end{cases}$$

(9)

You can simulate the resulting closed-loop performance from a given initial state $(x_1(0), x_2(0))$ by using MATLAB to solve the initial value problem

$$\begin{align*}
\dot{x}_1 &= -ax_1 + x_1^2 - x_2^2 \\
\dot{x}_2 &= k(x_1, x_2) \\
\dot{x}_3 &= x_1^2 + x_2^2 + k^2(x_1, x_2)
\end{align*}$$

(10) \hspace{1cm} (11) \hspace{1cm} (12)

over a sufficiently long interval $[0, t_f]$. If $t_f$ is large enough, then $x_3(t_f)$ will be a good approximation of the cost (3) from the given initial state. You can tell if you chose a large enough $t_f$ by plotting $x_3(t)$ versus $t$ (if $x_3(t)$ has converged to its final value by time $t = t_f$, then your $t_f$ is large enough).
4. For $a = 1$, simulate the performance of the closed-loop system (1)–(2) with $u = k(x_1, x_2)$ using both the feedback law (8) and the feedback law (9). Calculate the costs for the two control laws for initial states chosen over a grid of points covering the square $|x_i(0)| \leq B$, where $B$ is a positive parameter describing the size of the region of interest. The grid should contain 50 equally-spaced points in each direction, for a total of 2500 initial states.

For small initial states, say $B = 0.01$, the optimal cost will be approximately equal to (6). Which of the two control laws is closer to the optimal over this region? Why?

For larger initial states, say $B = 5$, neither control law may be near optimal, but one may be better than the other. Which one would you chose for initial states over this larger region, and why?

5. Optional question: can you find a continuous, globally asymptotically stabilizing feedback law when $a = 0$?