

Teaching Mathematics Through Problem Solving: A Personal Perspective

Tom Butts

tbutts@utdallas.edu

Quickies

- 1. The hungry mathematician bought a dozen cookies and ate all but three of them. How many did she have left?**
- 2. In going over his books one day a bookkeeper was entering figures for a toy store and noticed the word "balloon" had two consecutive sets of double letters. I wonder, he thought, if there is a word with three consecutive sets of double letters. Is there such a word?**
- 3. Rearrange the letters of NEW DOOR to form one word.**
- 4. Which is larger: the number of seven-letter English words ending in ING, or the number of seven-letter words with I as the fifth letter?**
- 5. If it takes 12 seconds to cut a log into three pieces, how long does it take to cut a log into four pieces?**
- 6. There were 64 teams in the NCAA Basketball Tournament. How many games were played to determine the winner?**
- 7. Can you place 10 coins in 3 cups so each cup contains an odd number of coins?**

Research Themes in Problem Solving

[Stanic Kilpatrick 1988 summary; other findings to 2004]

• **Problem solving as context. - Teaching *for* problem solving**

has the longest tradition in the school mathematics curriculum.

“Applications” and “context” are key words

Five subcategories

- Problem solving has been used as ***justification*** for teaching mathematics. To persuade students of the value of mathematics, the content is related to **real-world problem-solving** experiences.
- Problem solving also has been used to ***motivate*** students, sparking their interest in a specific mathematical topic or algorithm by providing a contextual (real-world) example of its use.
- Problem solving has been used as ***recreation***, a fun activity often used as a reward or break from routine studies.
- Problem solving as ***practice***, probably the most widespread use, has been used to reinforce skills and concepts that have been taught directly.
- Problem solving as ***vehicle***. [later]

When problem solving is used as **context** for mathematics, the emphasis is on finding interesting and engaging tasks or problems that help illuminate a mathematical concept or procedure. To use problem solving as context, a teacher might present the concept of fractions, for example, assigning groups of students the problem of dividing two pieces of licorice so that each gets an equal share. By providing this problem-solving context, the teacher's goals are multiple: to create opportunities for students to make discoveries about fraction concepts using a familiar and desirable medium (*motivation*); to help make the concepts more concrete (*practice*); and to offer a rationale for learning about fractions (*justification*).

Research Says: Real-World Applications

“We know from years of unsuccessful experience that for most students, decontextualized learning does not last. Students retain what they learn from their own efforts to address challenging problems that arise from situations that resonate with their own interests.”

Research Says: Linking Subject Matter to Contexts

Students who have trouble solving mathematical problems in school can solve comparable problems in out-of-school situations that are more meaningful to them. ["Folk Math" as one author called this phenomenon.]. As such, teachers should make an effort to link subject matter to contexts that are realistic to students.

- **Problem solving as skill.**

- **Teaching *about* problem solving**

- Problem solving is viewed as a *skill* in which component parts of the process can be taught and learned separately. After students learn the component parts, the parts can be combined for students to solve real problems.

Problem-Solving Skills

George Polya (1943) articulated a problem-solving process as one involving four phases:

1. Understanding the Problem,
2. Devising a Plan,
3. Carrying Out the Plan
4. Looking Back.

Most textbook series have their own interpretation. One example **Read and Understand; Plan and Solve; Look Back and Check.**

Two typical list of those skills are:

Problem-Solving Skills

- Read and Understand
- Plan and Solve
- Look Back and Check
- Too Much Information
- Too Little Information
- Reading Comprehension
- Writing to Compare
- Writing to Explain
- Writing to Persuade
- Writing to Justify

Problem Solving Strategies

- Draw a Diagram
- Make a Graph
- Make a Model
- Make an Organized List
- Make a Table and Look for a Pattern
- Simulate a Problem
- Solve by Graphing
- Solve a Simpler Problem
- Try, Check, and Revise
- Use Logical Reasoning
- Use a Problem Solving Plan
- Work Backward
- Write an Equation
- Write a Proportion
- Account for All Possibilities

Research Says:

- There should be direct instruction on the teaching of problem-solving strategies, such as draw a picture, make a table, look for a pattern, write an equation, and so on.
- Problem-solving performance is enhanced by teaching students to use a variety of strategies.
- Problem-solving practice without direct instruction on strategies did not produce improvement. Developing skill with diagrams gave the most pronounced effects on problem solving performance (at all grades), followed by training in the translation of words into symbols.
- Students can learn how and when to use problem-solving strategies to successfully solve problems when provided with explicit instruction on the strategies.

Caution: When problem solving is viewed as a collection of skills, however, the skills are often placed in a hierarchy in which students are expected to first master the ability to solve routine problems before attempting nonroutine problems. Consequently, nonroutine problem solving is often taught only to advanced students rather than to *all* students. When defining the learning objectives of a problem-solving activity, teachers will want to be aware of the distinction between teaching problem solving as a separate skill and infusing problem solving throughout the curriculum to develop conceptual understanding as well as basic skills.

Nonroutine problems are difficult for students.

Nonroutine, open-ended problems are often, by their nature, difficult for many students. Watching their students struggle in frustration is often very difficult for teachers. Knowing when to give hints and how much help to give requires striking a delicate balance that comes with experience and knowing students' capabilities.

Teachers are concerned about content coverage.

The TIMSS research characterized the U.S. curriculum as “a mile wide and an inch deep” compared to the mathematics curriculum in other countries. Teachers in the U.S. are generally expected to cover large areas of content each year. Yet solving challenging, nonroutine problems takes time. Often a single problem can occupy a class for a whole period or more. Therefore, it's essential that content and **skills be integrated within the context of problem solving**. By selecting rich, engaging, and worthwhile tasks, teachers can ensure that time is well-spent.

Textbooks present few nonroutine problems .

Although they are improving, many textbooks do not provide an adequate number of nonroutine problems from which teachers can choose. Many teachers are not comfortable straying from the scope and sequence the textbook provides, but they must develop the confidence to search out and develop other materials to supplement their texts.

Problem solving as *art*.

In his classic book, *How To Solve It*, George Polya (1943) introduced the idea that problem solving could be taught as a practical *art*, like playing the piano or swimming. Polya saw problem solving as an act of discovery and introduced the term “modern heuristics” (the art of inquiry and discovery) to describe the abilities needed to successfully investigate new problems. He encouraged presenting mathematics not as a finished set of facts and rules, but as an experimental and inductive science consisting of facts and "know-how". The aim of teaching problem solving as art is to develop students' abilities to become skillful and enthusiastic problem solvers; to be independent thinkers who are capable of dealing with open-ended, ill-defined problems.

"A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime."

George Polya *How to Solve It*, 1943

Some distort Polya's artistic view and attempt to reduce these "rule-of-thumb" heuristics to procedural skills. A heuristic becomes a skill, even paradoxically an algorithm.

- **Problem solving as vehicle.** -
Teaching *through* problem solving.

Problem solving, then, is not a topic, not a content strand, not a standard, but a philosophy of teaching.

Allow Mathematics to Be Problematic

allowing mathematics to be problematic means

- posing problems that are just within the student's reach,
- allowing them to **struggle** to find solutions, and
- then examining the methods they have used.

allowing mathematics to be problematic requires

- a different mind set about what mathematics is,
- thinking about how students learn mathematics with understanding
- considering what role the teacher should play
- believing that all students **need to struggle** with [challenging] problems if they are to truly learn mathematics.

This notion is in direct conflict with what many mathematics teachers feel is their main goal: to step in and remove the struggle and the challenge.

allowing mathematics to be problematic does not require importing lots of special problems, rather it requires allowing the problems that are taught every day to be *problems*.

Information should be shared, but when?

present alternate methods of solution not suggested by students as choices to be examined, not "teacher's method" to be memorized.

Research Says "Most, if not all, important mathematics concepts and procedures can best be taught through problem solving." Develops understanding of the underlying mathematical ideas.

Research Says: Students are able to learn new skills and concepts while they are solving challenging problems. It is not necessary for teachers to focus first on skill development and then move on to problem solving. Both can be done together. In fact, students who develop conceptual understanding through problem solving early perform best on procedural knowledge later.

Research Says: Conceptual understanding is the goal of teaching mathematics through problem solving. Mathematical understanding "is supported best through a delicate balance among engaging students in solving challenging problems, examining increasingly better solution methods, and providing information for students at just the right times."

How can you construct problems?

Anatomy of a Problem

A problem can be thought of as consisting of three parts:

Data; information; conditions

(Question[s] [and Hidden questions for multi-step problems])

Answer[s] and Solution[s]

2 out of 3 Principle:

In a traditional problem, of course, the two given parts are the data/information, and the question. and the unknown third part is a single numerical answer. There are many worthwhile problems that can be constructed by giving two other parts [or portion thereof] and asking the solver to find alternatives for the third part.

Example:

Wraps Clothing Store is having a sale where everything is marked 25% off. The regular price of a coat is \$120 and the sales tax rate is 10%. Maria bought a coat on sale. How much did she pay for the coat?

[Consider ways to find the price before tax:

$$120 - .25(120); \quad 120 - \left(\frac{1}{4}\right)(120); \quad .75(120) \quad \frac{3}{4} (120)]$$

Missing Question

Wraps Clothing Store is having a sale where everything is marked 25% off and the sales tax rate is 10%. Maria bought a coat and paid a total \$99. _____ ?

Complete the statement of this problem so its answer is \$120.

Wraps Clothing Store is having a sale where everything is marked 25% off. The regular price of a coat is \$120 and the sales tax rate is 10%. _____ ?

Complete the statement of this problem so its answer is

A) \$1 B) \$198

Missing Data/Information/Condition

Wraps Clothing Store is having a sale _____ .
The regular price of a coat is \$120 and the sales tax rate is 10%.
Maria bought a coat. How much change did Maria receive from the \$100 check his grandfather had given him?

Complete the statement of this problem so its answer is \$1.

Another example:

One angle of a triangle is twice as large as another angle.
_____. Complete the statement of this problem so that its answer is 40° .

Interesting Variations on a Basic Problem

Goldenberg & Walter

Find the mean of $\{7, 4, 7, 6, 3, 8, 7\}$.

1. What if only five of the seven data are given? Can we determine the missing data if we know the mean of the original seven?
2. What if we know the mean but none of the data? What, if anything, could we say about the data? What possible sets of data would fit?
3. What if the numbers are all different? Can we find seven different integers whose mean is 6? What if we also had to make the median 7 or the range 5?
4. What if the highest number is 18, not 8? Which of mean, mode, median, or range changes?
5. What if two sets of data are given instead of one? Suppose the second set is $\{2, 6, 7, 9\}$. Can we calculate the mean of the combined sets (all eleven numbers) by combining the means of the two sets in some way? Would averaging the means do that job?
6. What if we know only that the means of two sets of numbers are 7 and 10? What, if anything, can we say about the mean of the combined set?

Another variation: Find a subset of a. five b. six different positive integers from $\{1, 2, \dots, 99, 100\}$ so that the non-negative difference between the mean and median of the set is as

a) small b) large as possible.

Does either problem have more than one solution?

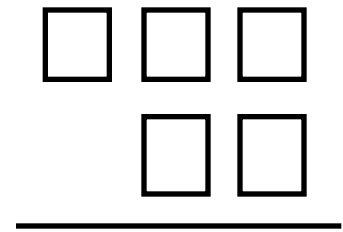
Constructing examples as a problem.

Modify algorithmic exercises.

Kindergarten Problem: After reading the Chinese Folktale "Two of Everything" where a poor farmer find a 'magic pot' that doubles anything he puts into it, the teacher asks: "What if our class fell into this pot?" [visual image, whimsy].

- a. Find the sum $40 + 41 + 42 + 43 + 44$.
- b. Write 210 as the sum of consecutive integers in as many ways as you can. "Reversal" Problems

- What is the largest and smallest product of a 2-digit number and a 3-digit number formed using each of the digits 2, 3, 5, 7, 8, exactly once?



Examine "extreme cases"

- Draw and label a figure for which the following computations give the area:

a. $A = 6 \cdot 4 + \frac{1}{2} \cdot 6 \cdot 2$ b. $A = 6 \cdot 4 - 3 \cdot 2$ c. $A = 10^2 - \pi(5^2)$

Comment: Some choices have more than one answer; others do not. Can use the same idea for volume, surface area and other measurements.

- Draw and label a figure for which the formula $P = 2a + 2b$ can be used to find its perimeter.

Algebra for Everyone:

- Find three pairs of algebraic expressions whose product is a^5 . [The third example forces consideration of a new concept: a^0 .]
- Can the product of two binomials be a binomial?
- In each case, there are three examples of a pattern. Find two more examples. Use an algebraic statement to describe the pattern. Is your statement true for all values of the variable(s) you used?

$$c. \left(\frac{1}{3}\right)^2 + \frac{2}{3} = \frac{1}{3} + \left(\frac{2}{3}\right)^2$$

$$\left(\frac{1}{5}\right)^2 + \frac{4}{5} = \frac{1}{5} + \left(\frac{4}{5}\right)^2$$

$$\left(\frac{2}{7}\right)^2 + \frac{5}{7} = \frac{2}{7} + \left(\frac{5}{7}\right)^2$$

$$(.23)^2 + .77 = .23 + (.77)^2$$

Make a decision

Perplexing Percents If an item is on sale for 20% off, most sales people will first deduct 20% of the price and then add the 8% sales tax. Another way is to compute the final price is to first add the 8% sales tax to the original price and then deduct 20% of this total. Which of these methods would you prefer if you were the

a. customer

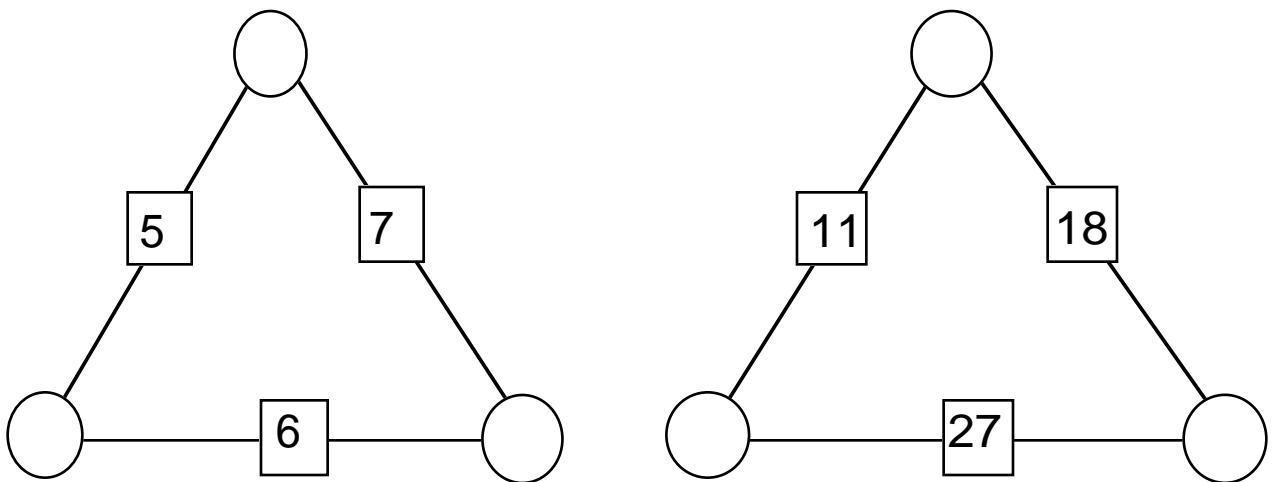
b. merchant

c. tax collector ?

- Take any two positive integers [possibly equal]. Add them. Multiply them. Find the sum of the resulting sum and product. Which positive integers **cannot** be obtained using this process?

Arithmogons [Example for Teachers]

One useful teaching strategy is to begin a unit with a problem. As you work through this activity, think about what concepts, ideas, etc. could be motivated by the problem.



A "secret number" is assigned to each vertex of a triangle. On each side is written the sum of the secret number at its ends. How can you find the secret numbers? Is there only one answer? Try some examples. Try to explain any patterns that you find. Think about other questions that could be asked and variations of the problem. Tackle the questions on the variation.

Examples:

- What numbers can occur as secret numbers?
- How do the secret numbers depend on the sums?
- What if the arithmetic operation is subtraction?

Triangle Measurements Two sides of a triangle are 6 cm and 8 cm. Find the perimeter and the area of the triangle.

Comment: The initial reaction is usually - "I need more information to solve the problem". Two options: a. Ask the students what additional piece of information they would like to know so the problem has exactly one solution. Choices include: included angle [90° is good first choice]; length of third side; etc. b. Ask for the range of possible values for the perimeter and the area using only the given information.

Fencing a Garden: Karen has 24 meters of fencing to use for her garden. She plans to use the garage as one side of her garden.

- If the garden is rectangular, what is the area of the largest garden she can build?
- If the garden has four sides [including the garage as one side], what is the area of the largest garden she can build?
- What is the area of the largest garden she can build?



Missing Hypotheses Supply a suitable hypothesis.

- If _____, then ABCD is a parallelogram.
- If _____, the $ax^2 + bx + c = 0$ has two real zeroes.

Big Problems [to open a unit]

Win the Lottery: Suppose you win \$1,000,000 in the state lottery. You can either have your prize paid to you in 25 annual installments of \$40,000 with the first installment payable immediately or you can choose to receive your winnings in a "lump sum" payable immediately. What would be a fair amount for this "lump sum"?

ALPHABET SOUP

IN JOHN'S BOX IS AN
AWFULLY UNUSUAL
GROUP OF WORDS. CAN
YOU FORM AN OPINION
ON WHY SUCH A BUNCH
OF WORDS IS SO
UNCOMMON?

Family Tree Roger Miller, the legendary "country singer" once claimed he was $\frac{1}{3}$ Cherokee. Is that possible? What could his family tree look like?

Fair Dice: Which 3-D shapes [polyhedrons] could be used as a fair die?

What is Open-Ended Problem Solving?

- problem will have multiple possible answers that can be derived by multiple solution methods. The focus is not on the answer to the problem, but on the methods for arriving at an answer.
- students are responsible for making many of the decisions that, in the past, have been the responsibility of teachers and textbooks. To decide which method, or procedure, to undertake to solve an open-ended problem, a student will draw on her previous knowledge and experience with related problems. She might construct her own procedure, trying this and that, before arriving at a solution.
- she will then reflect on and explain to others her problem-solving experience, tracing her thinking process and reviewing the strategies she attempted, determining why some worked and others didn't. This period of reflection deepens her understanding of the problem and helps to clarify her thinking about effective solution methods, and how the problem and methods she used relate to other problems or areas of mathematics.

- Teacher's key responsibilities are selecting and presenting "good" problem tasks. By choosing good problems, the teacher sets up optimal conditions for her students to be engaged in meaningful problem solving. This means that the problem will:

1. Be open-ended, in that it presents multiple solution methods and answers
2. Address important mathematical concepts
3. Challenge and interest students
4. Connect to students' previous learning

Research Says: Learning mathematics by struggling with open-ended and challenging problems accommodates diverse learning styles. The active and varied nature of problem solving helps students with diverse learning styles to develop and demonstrate mathematical understanding.

Research Says: When students encounter mathematical ideas that interest and challenge them in an open-ended problem solving context, they are more likely to experience the kinds of internal rewards that keep them engaged. Students who must resort to memorizing will lack understanding and will likely feel little sense of satisfaction, perhaps withdrawing from learning altogether. In fact, he says, evidence suggests that if students memorize and practice procedures repeatedly in a rote fashion, it's difficult for them to go back later and gain a deeper understanding of the mathematical concepts underlying those procedures.

Challenges of Adopting a Problem Solving Philosophy of Teaching

- Must believe mathematics is an active, creative endeavor involving inquiry and discovery; not simply involving correct answers and infallible procedures consisting of arithmetic operations, algebraic procedures, and geometric terms and theorems, etc.
- Must view her role as a facilitator, challenging students to think and to question their findings and assumptions; not a presenter of mathematical concepts, procedures, facts, and theorems with a focus on student practice and memorization where the meaning and context associated with many of these theorems and procedures may be relegated to the fringes.
- Must deal with the fact that most students believe that all problems have an answer; that there is only one right answer and one correct solution method; and that ordinary students cannot expect to *understand* mathematics but can merely memorize and apply mathematical procedures in a mechanical fashion.
- **Teaching nonroutine problem solving is difficult.** True problem solving is as demanding on the teacher as it is on the students. The art of teaching mathematical problem solving is best mastered over a long period of time.

Teachers must have the **mathematical expertise** to understand the different approaches that students might take to a problem and how promising those approaches will be.

Pedagogically, teachers must make complex decisions about the level of difficulty of the problems assigned, when to give help, and how to give assistance that supports students' success while ensuring that they retain ownership of their solution strategies.

Personally, teachers will sometimes find themselves in the uncomfortable position of not knowing the solution. Letting go of the "expert" role teachers have traditionally played requires experience, confidence, and self-awareness. Often, teachers are asked to teach mathematics they never encountered in school and in a way that differs from how *they* were taught. For these reasons, teachers may need additional training in mathematical content and theory, as well as in methods for teaching problem solving.

- Even if they encountered problem solving in their college methods courses, once in the classroom, they often feel they must conform to the conventional methods that hold sway in most schools.
- Being an agent of change, when one is surrounded by deeply ingrained beliefs about teaching and learning, is a difficult task.
- Teachers today are often caught between daily pressure from colleagues, parents, and others to uphold tradition in the classroom, and pressure from policy makers to employ standards-based practices (with the conflicting expectation that students will perform highly on standardized tests that measure basic skills, not performance of standards-based material).

Assessment Issues:

Research Studies of Reform Curricula [Connected Mathematics; Core Plus Mathematics, and several others] have measured learning through students' performance on standardized tests that target procedural fluency and other assessments that probe their conceptual understanding and problem-solving abilities.

Those in reform curricula

- exhibited greater conceptual understanding and performed at considerably higher levels on problem solving tasks.
- performed at the "same" level [as those taking standard curricula] on standardized tests that assessed skills and procedures. "same" = sometimes a little better, sometimes a little worse.
- students taught using problem-solving approaches are more likely to have a positive attitude about mathematics. [Historically most students regard mathematics as a meaningless collection of rules that need to be memorized.]
- It is time to reconceptualize standardized testing, decide what, as researchers and classroom teachers, we really want to test.

What do findings from research suggest about the feasibility and efficacy of teaching mathematics through problem solving?

Support both feasibility and efficacy.

"Traditional methods may appear to be easier to implement, but their consequences for students are well-documented - too many students are left out, disengaged, and unable to use much of the mathematics they learn. Evidence suggests that the extra time, effort, and resources required to teach through problem solving are well worth the effort if one's goals are students who understand mathematical concepts, are willing to tackle challenging problems, and see themselves as capable of learning mathematics."

Favorite Quotes:

"Thinking is driven by questions, not answers."

"Mathematicians can best contribute to the K-12 standards movement by offering problems - problems that provoke, problems that surprise, problems that expand minds.

- What problems would encourage an 8th grade student to take more mathematics?
- What problems must a high school graduate be able to solve to succeed in the workplace?
- What problems will make teachers into better teachers?"

Lynn Steen former president of Mathematical Association of America.

TEN COMMANDMENTS FOR TEACHERS

[George Polya, 1959].

- 1. Be interested in your subject.**
- 2. Know your subject.**
- 3. The best way to learn anything is to discover it by yourself. This applies to you as well as your students.**
- 4. Try to read the faces of your students, try to see their expectations and difficulties; put yourself in their place.**
- 5. Give them not only information, but mental attitudes, "know-how".**
- 6. Let them learn guessing.**
- 7. Let them learn proving (giving convincing arguments).**
- 8. Look out for such features of the problems at hand as may be useful in solving the problems to come - try to disclose the general pattern that lies behind the present concrete situation.**
- 9. Do not give the whole secret at once - let the students guess before you tell - let them find out for themselves as much as possible.**
- 10. Suggest it - do not force it down their throats.**

[from Mathematical Discovery, Polya,G., Wiley, 1965, Ch. 14.]

