TABLE 9 **Table for the Bit Operators *OR*, *AND*, and *XOR*.**

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Information is often represented using bit strings, which are lists of zeros and ones. When this is done, operations on the bit strings can be used to manipulate this information.

DEFINITION 7 A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

EXAMPLE 12 101010011 is a bit string of length nine. ◀

We can extend bit operations to bit strings. We define the **bitwise *OR***, **bitwise *AND***, and **bitwise *XOR*** of two strings of the same length to be the strings that have as their bits the *OR*, *AND*, and *XOR* of the corresponding bits in the two strings, respectively. We use the symbols $\vee$, $\wedge$, and $\oplus$ to represent the bitwise *OR*, bitwise *AND*, and bitwise *XOR* operations, respectively. We illustrate bitwise operations on bit strings with Example 13.

EXAMPLE 13 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

Solution: The bitwise *OR*, bitwise *AND*, and bitwise *XOR* of these strings are obtained by taking the *OR*, *AND*, and *XOR* of the corresponding bits, respectively. This gives us

```
  01 1011 0110
  11 0001 1101
  ------------
  11 1011 1111   bitwise OR
  01 0001 0100   bitwise AND
  10 1010 1011   bitwise XOR
```

◀

Exercises Chap 1.1

1. Which of these sentences are propositions? What are the truth values of those that are propositions?
   a) Boston is the capital of Massachusetts.
   b) Miami is the capital of Florida.
   c) $2 + 3 = 5$.
   d) $5 + 7 = 10$.
   e) $x + 2 = 11$.
   f) Answer this question.
2. Which of these are propositions? What are the truth values of those that are propositions?
   a) Do not pass go.
   b) What time is it?
   c) There are no black flies in Maine.
   d) $4 + x = 5$.
   e) The moon is made of green cheese.
   f) $2^n \geq 100$.
3. What is the negation of each of these propositions?
   a) Mei has an MP3 player.
   b) There is no pollution in New Jersey.
   c) $2 + 1 = 3$.
   d) The summer in Maine is hot and sunny.
4. What is the negation of each of these propositions?
   a) Jennifer and Teja are friends.
   b) There are 13 items in a baker's dozen.
   c) Abby sent more than 100 text messages every day.
   d) 121 is a perfect square.

5. What is the negation of each of these propositions?
   a) Steve has more than 100 GB free disk space on his laptop.
   b) Zach blocks e-mails and texts from Jennifer.
   c) $7 \cdot 11 \cdot 13 = 999$.
   d) Diane rode her bicycle 100 miles on Sunday.

6. Suppose that Smartphone A has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone B has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; and Smartphone C has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.
   a) Smartphone B has the most RAM of these three smartphones.
   b) Smartphone C has more ROM or a higher resolution camera than Smartphone B.
   c) Smartphone B has more RAM, more ROM, and a higher resolution camera than Smartphone A.
   d) If Smartphone B has more RAM and more ROM than Smartphone C, then it also has a higher resolution camera.
   e) Smartphone A has more RAM than Smartphone B if and only if Smartphone B has more RAM than Smartphone A.

7. Suppose that during the most recent fiscal year, the annual revenue of Acme Computer was 138 billion dollars and its net profit was 8 billion dollars, the annual revenue of Nadir Software was 87 billion dollars and its net profit was 5 billion dollars, and the annual revenue of Quixote Media was 111 billion dollars and its net profit was 13 billion dollars. Determine the truth value of each of these propositions for the most recent fiscal year.
   a) Quixote Media had the largest annual revenue.
   b) Nadir Software had the lowest net profit and Acme Computer had the largest annual revenue.
   c) Acme Computer had the largest net profit or Quixote Media had the largest net profit.
   d) If Quixote Media had the smallest net profit, then Acme Computer had the largest annual revenue.
   e) Nadir Software had the smallest net profit if and only if Acme Computer had the largest annual revenue.

8. Let p and q be the propositions
   p : I bought a lottery ticket this week.
   q : I won the million dollar jackpot.
   Express each of these propositions as an English sentence.
   a) $\neg p$ b) $p \vee q$ c) $p \rightarrow q$
   d) $p \wedge q$ e) $p \leftrightarrow q$ f) $\neg p \rightarrow \neg q$
   g) $\neg p \wedge \neg q$ h) $\neg p \vee (p \wedge q)$

9. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.
   a) $\neg q$ b) $p \wedge q$ c) $\neg p \vee q$
   d) $p \rightarrow \neg q$ e) $\neg q \rightarrow p$ f) $\neg p \rightarrow \neg q$
   g) $p \leftrightarrow \neg q$ h) $\neg p \wedge (p \vee \neg q)$

10. Let p and q be the propositions "The election is decided" and "The votes have been counted," respectively. Express each of these compound propositions as an English sentence.
    a) $\neg p$ b) $p \vee q$
    c) $\neg p \wedge q$ d) $q \rightarrow p$
    e) $\neg q \rightarrow \neg p$ f) $\neg p \rightarrow \neg q$
    g) $p \leftrightarrow q$ h) $\neg q \vee (\neg p \wedge q)$

11. Let p and q be the propositions
    p : It is below freezing.
    q : It is snowing.
    Write these propositions using p and q and logical connectives (including negations).
    a) It is below freezing and snowing.
    b) It is below freezing but not snowing.
    c) It is not below freezing and it is not snowing.
    d) It is either snowing or below freezing (or both).
    e) If it is below freezing, it is also snowing.
    f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.
    g) That it is below freezing is necessary and sufficient for it to be snowing.

12. Let p, q, and r be the propositions
    p : You have the flu.
    q : You miss the final examination.
    r : You pass the course.
    Express each of these propositions as an English sentence.
    a) $p \rightarrow q$ b) $\neg q \leftrightarrow r$
    c) $q \rightarrow \neg r$ d) $p \vee q \vee r$
    e) $(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$
    f) $(p \wedge q) \vee (\neg q \wedge r)$

13. Let p and q be the propositions
    p : You drive over 65 miles per hour.
    q : You get a speeding ticket.
    Write these propositions using p and q and logical connectives (including negations).
    a) You do not drive over 65 miles per hour.
    b) You drive over 65 miles per hour, but you do not get a speeding ticket.
    c) You will get a speeding ticket if you drive over 65 miles per hour.
    d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.
    e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.
    f) You get a speeding ticket, but you do not drive over 65 miles per hour.
    g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

14. Let p, q, and r be the propositions
    p : You get an A on the final exam.
    q : You do every exercise in this book.
    r : You get an A in this class.
    Write these propositions using p, q, and r and logical connectives (including negations).

a) You get an A in this class, but you do not do every exercise in this book.
b) You get an A on the final, you do every exercise in this book, and you get an A in this class.
c) To get an A in this class, it is necessary for you to get an A on the final.
d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.
e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.
f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

15. Let p, q, and r be the propositions

p : Grizzly bears have been seen in the area.
q : Hiking is safe on the trail.
r : Berries are ripe along the trail.

Write these propositions using p, q, and r and logical connectives (including negations).

a) Berries are ripe along the trail, but grizzly bears have not been seen in the area.
b) Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.
c) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.
d) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe.
e) For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.
f) Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

16. Determine whether these biconditionals are true or false.

a) $2 + 2 = 4$ if and only if $1 + 1 = 2$.
b) $1 + 1 = 2$ if and only if $2 + 3 = 4$.
c) $1 + 1 = 3$ if and only if monkeys can fly.
d) $0 > 1$ if and only if $2 > 1$.

17. Determine whether each of these conditional statements is true or false.

a) If $1 + 1 = 2$, then $2 + 2 = 5$.
b) If $1 + 1 = 3$, then $2 + 2 = 4$.
c) If $1 + 1 = 3$, then $2 + 2 = 5$.
d) If monkeys can fly, then $1 + 1 = 3$.

18. Determine whether each of these conditional statements is true or false.

a) If $1 + 1 = 3$, then unicorns exist.
b) If $1 + 1 = 3$, then dogs can fly.
c) If $1 + 1 = 2$, then dogs can fly.
d) If $2 + 2 = 4$, then $1 + 2 = 3$.

19. For each of these sentences, determine whether an inclusive or, or an exclusive or, is intended. Explain your answer.

a) Coffee or tea comes with dinner.
b) A password must have at least three digits or be at least eight characters long.
c) The prerequisite for the course is a course in number theory or a course in cryptography.
d) You can pay using U.S. dollars or euros.

20. For each of these sentences, determine whether an inclusive or, or an exclusive or, is intended. Explain your answer.

a) Experience with C++ or Java is required.
b) Lunch includes soup or salad.
c) To enter the country you need a passport or a voter registration card.
d) Publish or perish.

21. For each of these sentences, state what the sentence means if the logical connective or is an inclusive or (that is, a disjunction) versus an exclusive or. Which of these meanings of or do you think is intended?

a) To take discrete mathematics, you must have taken calculus or a course in computer science.
b) When you buy a new car from Acme Motor Company, you get $2000 back in cash or a 2% car loan.
c) Dinner for two includes two items from column A or three items from column B.
d) School is closed if more than 2 feet of snow falls or if the wind chill is below -100.

22. Write each of these statements in the form "if p, then q" in English. [*Hint:* Refer to the list of common ways to express conditional statements provided in this section.]

a) It is necessary to wash the boss's car to get promoted.
b) Winds from the south imply a spring thaw.
c) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.
d) Willy gets caught whenever he cheats.
e) You can access the website only if you pay a subscription fee.
f) Getting elected follows from knowing the right people.
g) Carol gets seasick whenever she is on a boat.

23. Write each of these statements in the form "if p, then q" in English. [*Hint:* Refer to the list of common ways to express conditional statements.]

a) It snows whenever the wind blows from the northeast.
b) The apple trees will bloom if it stays warm for a week.
c) That the Pistons win the championship implies that they beat the Lakers.
d) It is necessary to walk 8 miles to get to the top of Long's Peak.
e) To get tenure as a professor, it is sufficient to be world-famous.
f) If you drive more than 400 miles, you will need to buy gasoline.
g) Your guarantee is good only if you bought your CD player less than 90 days ago.
h) Jan will go swimming unless the water is too cold.

24. Write each of these statements in the form "if p, then q" in English. [*Hint:* Refer to the list of common ways to express conditional statements provided in this section.]

a) I will remember to send you the address only if you send me an e-mail message.

b) To be a citizen of this country, it is sufficient that you were born in the United States.

c) If you keep your textbook, it will be a useful reference in your future courses.

d) The Red Wings will win the Stanley Cup if their goalie plays well.

e) That you get the job implies that you had the best credentials.

f) The beach erodes whenever there is a storm.

g) It is necessary to have a valid password to log on to the server.

h) You will reach the summit unless you begin your climb too late.

25. Write each of these propositions in the form "p if and only if q" in English.

a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.

b) For you to win the contest it is necessary and sufficient that you have the only winning ticket.

c) You get promoted only if you have connections, and you have connections only if you get promoted.

d) If you watch television your mind will decay, and conversely.

e) The trains run late on exactly those days when I take it.

26. Write each of these propositions in the form "p if and only if q" in English.

a) For you to get an A in this course, it is necessary and sufficient that you learn how to solve discrete mathematics problems.

b) If you read the newspaper every day, you will be informed, and conversely.

c) It rains if it is a weekend day, and it is a weekend day if it rains.

d) You can see the wizard only if the wizard is not in, and the wizard is not in only if you can see him.

27. State the converse, contrapositive, and inverse of each of these conditional statements.

a) If it snows today, I will ski tomorrow.

b) I come to class whenever there is going to be a quiz.

c) A positive integer is a prime only if it has no divisors other than 1 and itself.

28. State the converse, contrapositive, and inverse of each of these conditional statements.

a) If it snows tonight, then I will stay at home.

b) I go to the beach whenever it is a sunny summer day.

c) When I stay up late, it is necessary that I sleep until noon.

29. How many rows appear in a truth table for each of these compound propositions?

a) $p \rightarrow \neg p$

b) $(p \vee \neg r) \wedge (q \vee \neg s)$

c) $q \vee p \vee \neg s \vee \neg r \vee \neg t \vee u$

d) $(p \wedge r \wedge t) \leftrightarrow (q \wedge t)$

30. How many rows appear in a truth table for each of these compound propositions?

a) $(q \rightarrow \neg p) \vee (\neg p \rightarrow \neg q)$

b) $(p \vee \neg t) \wedge (p \vee \neg s)$

c) $(p \rightarrow r) \vee (\neg s \rightarrow \neg t) \vee (\neg u \rightarrow v)$

d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

31. Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$ **b)** $p \vee \neg p$

c) $(p \vee \neg q) \rightarrow q$ **d)** $(p \vee q) \rightarrow (p \wedge q)$

e) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

f) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

32. Construct a truth table for each of these compound propositions.

a) $p \rightarrow \neg p$ **b)** $p \leftrightarrow \neg p$

c) $p \oplus (p \vee q)$ **d)** $(p \wedge q) \rightarrow (p \vee q)$

e) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

f) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

33. Construct a truth table for each of these compound propositions.

a) $(p \vee q) \rightarrow (p \oplus q)$ **b)** $(p \oplus q) \rightarrow (p \wedge q)$

c) $(p \vee q) \oplus (p \wedge q)$ **d)** $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

e) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$

f) $(p \oplus q) \rightarrow (p \oplus \neg q)$

34. Construct a truth table for each of these compound propositions.

a) $p \oplus p$ **b)** $p \oplus \neg p$

c) $p \oplus \neg q$ **d)** $\neg p \oplus \neg q$

e) $(p \oplus q) \vee (p \oplus \neg q)$ **f)** $(p \oplus q) \wedge (p \oplus \neg q)$

35. Construct a truth table for each of these compound propositions.

a) $p \rightarrow \neg q$ **b)** $\neg p \leftrightarrow q$

c) $(p \rightarrow q) \vee (\neg p \rightarrow q)$ **d)** $(p \rightarrow q) \wedge (\neg p \rightarrow q)$

e) $(p \leftrightarrow q) \vee (\neg p \leftrightarrow q)$

f) $(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$

36. Construct a truth table for each of these compound propositions.

a) $(p \vee q) \vee r$ **b)** $(p \vee q) \wedge r$

c) $(p \wedge q) \vee r$ **d)** $(p \wedge q) \wedge r$

e) $(p \vee q) \wedge \neg r$ **f)** $(p \wedge q) \vee \neg r$

37. Construct a truth table for each of these compound propositions.

a) $p \rightarrow (\neg q \vee r)$

b) $\neg p \rightarrow (q \rightarrow r)$

c) $(p \rightarrow q) \vee (\neg p \rightarrow r)$

d) $(p \rightarrow q) \wedge (\neg p \rightarrow r)$

e) $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$

f) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

38. Construct a truth table for $((p \rightarrow q) \rightarrow r) \rightarrow s$.

39. Construct a truth table for $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s)$.

☞ **40.** Explain, without using a truth table, why $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when p, q, and r have the same truth value and it is false otherwise.

☞ **41.** Explain, without using a truth table, why $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of p, q, and r is true and at least one is false, but is false when all three variables have the same truth value.

42. What is the value of x after each of these statements is encountered in a computer program, if $x = 1$ before the statement is reached?

a) **if** $x + 2 = 3$ **then** $x := x + 1$
b) **if** $(x + 1 = 3)$ *OR* $(2x + 2 = 3)$ **then** $x := x + 1$
c) **if** $(2x + 3 = 5)$ *AND* $(3x + 4 = 7)$ **then** $x := x + 1$
d) **if** $(x + 1 = 2)$ *XOR* $(x + 2 = 3)$ **then** $x := x + 1$
e) **if** $x < 2$ **then** $x := x + 1$

43. Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of each of these pairs of bit strings.

a) 101 1110, 010 0001
b) 1111 0000, 1010 1010
c) 00 0111 0001, 10 0100 1000
d) 11 1111 1111, 00 0000 0000

44. Evaluate each of these expressions.

a) $1\ 1000 \wedge (0\ 1011 \vee 1\ 1011)$
b) $(0\ 1111 \wedge 1\ 0101) \vee 0\ 1000$
c) $(0\ 1010 \oplus 1\ 1011) \oplus 0\ 1000$
d) $(1\ 1011 \vee 0\ 1010) \wedge (1\ 0001 \vee 1\ 1011)$

Fuzzy logic is used in artificial intelligence. In fuzzy logic, a proposition has a truth value that is a number between 0 and 1, inclusive. A proposition with a truth value of 0 is false and one with a truth value of 1 is true. Truth values that are between 0 and 1 indicate varying degrees of truth. For instance, the truth value 0.8 can be assigned to the statement "Fred is happy," because Fred is happy most of the time, and the truth value 0.4 can be assigned to the statement "John is happy," because John is happy slightly less than half the time. Use these truth values to solve Exercises 45–47.

45. The truth value of the negation of a proposition in fuzzy logic is 1 minus the truth value of the proposition. What are the truth values of the statements "Fred is not happy" and "John is not happy?"

46. The truth value of the conjunction of two propositions in fuzzy logic is the minimum of the truth values of the two propositions. What are the truth values of the statements "Fred and John are happy" and "Neither Fred nor John is happy?"

47. The truth value of the disjunction of two propositions in fuzzy logic is the maximum of the truth values of the two propositions. What are the truth values of the statements "Fred is happy, or John is happy" and "Fred is not happy, or John is not happy?"

***48.** Is the assertion "This statement is false" a proposition?

***49.** The nth statement in a list of 100 statements is "Exactly n of the statements in this list are false."

a) What conclusions can you draw from these statements?
b) Answer part (a) if the nth statement is "At least n of the statements in this list are false."
c) Answer part (b) assuming that the list contains 99 statements.

50. An ancient Sicilian legend says that the barber in a remote town who can be reached only by traveling a dangerous mountain road shaves those people, and only those people, who do not shave themselves. Can there be such a barber?

1.2 Applications of Propositional Logic

Introduction

Logic has many important applications to mathematics, computer science, and numerous other disciplines. Statements in mathematics and the sciences and in natural language often are imprecise or ambiguous. To make such statements precise, they can be translated into the language of logic. For example, logic is used in the specification of software and hardware, because these specifications need to be precise before development begins. Furthermore, propositional logic and its rules can be used to design computer circuits, to construct computer programs, to verify the correctness of programs, and to build expert systems. Logic can be used to analyze and solve many familiar puzzles. Software systems based on the rules of logic have been developed for constructing some, but not all, types of proofs automatically. We will discuss some of these applications of propositional logic in this section and in later chapters.

Translating English Sentences

There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives. In particular, English (and every other human language) is

Solving Satisfiability Problems

A truth table can be used to determine whether a compound proposition is satisfiable, or equivalently, whether its negation is a tautology (see Exercise 60). This can be done by hand for a compound proposition with a small number of variables, but when the number of variables grows, this becomes impractical. For instance, there are $2^{20} = 1{,}048{,}576$ rows in the truth table for a compound proposition with 20 variables. Clearly, you need a computer to help you determine, in this way, whether a compound proposition in 20 variables is satisfiable.

When many applications are modeled, questions concerning the satisfiability of compound propositions with hundreds, thousands, or millions of variables arise. Note, for example, that when there are 1000 variables, checking every one of the 2^{1000} (a number with more than 300 decimal digits) possible combinations of truth values of the variables in a compound proposition cannot be done by a computer in even trillions of years. No procedure is known that a computer can follow to determine in a reasonable amount of time whether an arbitrary compound proposition in such a large number of variables is satisfiable. However, progress has been made developing methods for solving the satisfiability problem for the particular types of compound propositions that arise in practical applications, such as for the solution of Sudoku puzzles. Many computer programs have been developed for solving satisfiability problems which have practical use. In our discussion of the subject of algorithms in Chapter 3, we will discuss this question further. In particular, we will explain the important role the propositional satisfiability problem plays in the study of the complexity of algorithms.

Exercises Chap 1.3

1. Use truth tables to verify these equivalences.
   a) $p \wedge \mathbf{T} \equiv p$ b) $p \vee \mathbf{F} \equiv p$
   c) $p \wedge \mathbf{F} \equiv \mathbf{F}$ d) $p \vee \mathbf{T} \equiv \mathbf{T}$
   e) $p \vee p \equiv p$ f) $p \wedge p \equiv p$
2. Show that $\neg(\neg p)$ and p are logically equivalent.
3. Use truth tables to verify the commutative laws
   a) $p \vee q \equiv q \vee p$. b) $p \wedge q \equiv q \wedge p$.
4. Use truth tables to verify the associative laws
   a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
   b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.
5. Use a truth table to verify the distributive law $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.
6. Use a truth table to verify the first De Morgan law $\neg(p \wedge q) \equiv \neg p \vee \neg q$.
7. Use De Morgan's laws to find the negation of each of the following statements.
   a) Jan is rich and happy.
   b) Carlos will bicycle or run tomorrow.

HENRY MAURICE SHEFFER (1883–1964) Henry Maurice Sheffer, born to Jewish parents in the western Ukraine, emigrated to the United States in 1892 with his parents and six siblings. He studied at the Boston Latin School before entering Harvard, where he completed his undergraduate degree in 1905, his master's in 1907, and his Ph.D. in philosophy in 1908. After holding a postdoctoral position at Harvard, Henry traveled to Europe on a fellowship. Upon returning to the United States, he became an academic nomad, spending one year each at the University of Washington, Cornell, the University of Minnesota, the University of Missouri, and City College in New York. In 1916 he returned to Harvard as a faculty member in the philosophy department. He remained at Harvard until his retirement in 1952.

Sheffer introduced what is now known as the Sheffer stroke in 1913; it became well known only after its use in the 1925 edition of Whitehead and Russell's *Principia Mathematica*. In this same edition Russell wrote that Sheffer had invented a powerful method that could be used to simplify the *Principia*. Because of this comment, Sheffer was something of a mystery man to logicians, especially because Sheffer, who published little in his career, never published the details of this method, only describing it in mimeographed notes and in a brief published abstract.

Sheffer was a dedicated teacher of mathematical logic. He liked his classes to be small and did not like auditors. When strangers appeared in his classroom, Sheffer would order them to leave, even his colleagues or distinguished guests visiting Harvard. Sheffer was barely five feet tall; he was noted for his wit and vigor, as well as for his nervousness and irritability. Although widely liked, he was quite lonely. He is noted for a quip he spoke at his retirement: "Old professors never die, they just become emeriti." Sheffer is also credited with coining the term "Boolean algebra" (the subject of Chapter 12 of this text). Sheffer was briefly married and lived most of his later life in small rooms at a hotel packed with his logic books and vast files of slips of paper he used to jot down his ideas. Unfortunately, Sheffer suffered from severe depression during the last two decades of his life.

c) Mei walks or takes the bus to class.
d) Ibrahim is smart and hard working.

8. Use De Morgan's laws to find the negation of each of the following statements.
a) Kwame will take a job in industry or go to graduate school.
b) Yoshiko knows Java and calculus.
c) James is young and strong.
d) Rita will move to Oregon or Washington.

9. Show that each of these conditional statements is a tautology by using truth tables.
a) $(p \wedge q) \rightarrow p$ b) $p \rightarrow (p \vee q)$
c) $\neg p \rightarrow (p \rightarrow q)$ d) $(p \wedge q) \rightarrow (p \rightarrow q)$
e) $\neg(p \rightarrow q) \rightarrow p$ f) $\neg(p \rightarrow q) \rightarrow \neg q$

10. Show that each of these conditional statements is a tautology by using truth tables.
a) $[\neg p \wedge (p \vee q)] \rightarrow q$
b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
c) $[p \wedge (p \rightarrow q)] \rightarrow q$
d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

11. Show that each conditional statement in Exercise 9 is a tautology without using truth tables.

12. Show that each conditional statement in Exercise 10 is a tautology without using truth tables.

13. Use truth tables to verify the absorption laws.
a) $p \vee (p \wedge q) \equiv p$ b) $p \wedge (p \vee q) \equiv p$

14. Determine whether $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is a tautology.

15. Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology.

Each of Exercises 16–28 asks you to show that two compound propositions are logically equivalent. To do this, either show that both sides are true, or that both sides are false, for exactly the same combinations of truth values of the propositional variables in these expressions (whichever is easier).

16. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.

17. Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

18. Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.

19. Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent.

20. Show that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.

21. Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.

22. Show that $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (q \wedge r)$ are logically equivalent.

23. Show that $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent.

24. Show that $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent.

25. Show that $(p \rightarrow r) \vee (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$ are logically equivalent.

26. Show that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent.

27. Show that $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent.

28. Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent.

29. Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.

30. Show that $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology.

31. Show that $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent.

32. Show that $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are not logically equivalent.

33. Show that $(p \rightarrow q) \rightarrow (r \rightarrow s)$ and $(p \rightarrow r) \rightarrow (q \rightarrow s)$ are not logically equivalent.

The **dual** of a compound proposition that contains only the logical operators $\vee$, $\wedge$, and $\neg$ is the compound proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each $\mathbf{T}$ by $\mathbf{F}$, and each $\mathbf{F}$ by $\mathbf{T}$. The dual of s is denoted by s^*.

34. Find the dual of each of these compound propositions.
a) $p \vee \neg q$ b) $p \wedge (q \vee (r \wedge \mathbf{T}))$
c) $(p \wedge \neg q) \vee (q \wedge \mathbf{F})$

35. Find the dual of each of these compound propositions.
a) $p \wedge \neg q \wedge \neg r$ b) $(p \wedge q \wedge r) \vee s$
c) $(p \vee \mathbf{F}) \wedge (q \vee \mathbf{T})$

36. When does $s^* = s$, where s is a compound proposition?

37. Show that $(s^*)^* = s$ when s is a compound proposition.

38. Show that the logical equivalences in Table 6, except for the double negation law, come in pairs, where each pair contains compound propositions that are duals of each other.

****39.** Why are the duals of two equivalent compound propositions also equivalent, where these compound propositions contain only the operators $\wedge$, $\vee$, and $\neg$?

40. Find a compound proposition involving the propositional variables p, q, and r that is true when p and q are true and r is false, but is false otherwise. [*Hint:* Use a conjunction of each propositional variable or its negation.]

41. Find a compound proposition involving the propositional variables p, q, and r that is true when exactly two of p, q, and r are true and is false otherwise. [*Hint:* Form a disjunction of conjunctions. Include a conjunction for each combination of values for which the compound proposition is true. Each conjunction should include each of the three propositional variables or its negations.]

42. Suppose that a truth table in n propositional variables is specified. Show that a compound proposition with this truth table can be formed by taking the disjunction of conjunctions of the variables or their negations, with one conjunction included for each combination of values for which the compound proposition is true. The resulting compound proposition is said to be in **disjunctive normal form**.

A collection of logical operators is called **functionally complete** if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

43. Show that $\neg$, $\wedge$, and $\vee$ form a functionally complete collection of logical operators. [*Hint:* Use the fact that every compound proposition is logically equivalent to one in disjunctive normal form, as shown in Exercise 42.]

*44. Show that $\neg$ and $\wedge$ form a functionally complete collection of logical operators. [*Hint:* First use a De Morgan law to show that $p \vee q$ is logically equivalent to $\neg(\neg p \wedge \neg q)$.]

*45. Show that $\neg$ and $\vee$ form a functionally complete collection of logical operators.

The following exercises involve the logical operators *NAND* and *NOR*. The proposition p *NAND* q is true when either p or q, or both, are false; and it is false when both p and q are true. The proposition p *NOR* q is true when both p and q are false, and it is false otherwise. The propositions p *NAND* q and p *NOR* q are denoted by $p \mid q$ and $p \downarrow q$, respectively. (The operators $\mid$ and $\downarrow$ are called the **Sheffer stroke** and the **Peirce arrow** after H. M. Sheffer and C. S. Peirce, respectively.)

46. Construct a truth table for the logical operator *NAND*.

47. Show that $p \mid q$ is logically equivalent to $\neg(p \wedge q)$.

48. Construct a truth table for the logical operator *NOR*.

49. Show that $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$.

50. In this exercise we will show that $\{\downarrow\}$ is a functionally complete collection of logical operators.
   a) Show that $p \downarrow p$ is logically equivalent to $\neg p$.
   b) Show that $(p \downarrow q) \downarrow (p \downarrow q)$ is logically equivalent to $p \vee q$.
   c) Conclude from parts (a) and (b), and Exercise 49, that $\{\downarrow\}$ is a functionally complete collection of logical operators.

*51. Find a compound proposition logically equivalent to $p \rightarrow q$ using only the logical operator $\downarrow$.

52. Show that $\{\mid\}$ is a functionally complete collection of logical operators.

53. Show that $p \mid q$ and $q \mid p$ are equivalent.

54. Show that $p \mid (q \mid r)$ and $(p \mid q) \mid r$ are not equivalent, so that the logical operator $\mid$ is not associative.

*55. How many different truth tables of compound propositions are there that involve the propositional variables p and q?

56. Show that if p, q, and r are compound propositions such that p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent.

57. The following sentence is taken from the specification of a telephone system: "If the directory database is opened, then the monitor is put in a closed state, if the system is not in its initial state." This specification is hard to understand because it involves two conditional statements. Find an equivalent, easier-to-understand specification that involves disjunctions and negations but not conditional statements.

58. How many of the disjunctions $p \vee \neg q$, $\neg p \vee q$, $q \vee r$, $q \vee \neg r$, and $\neg q \vee \neg r$ can be made simultaneously true by an assignment of truth values to p, q, and r?

59. How many of the disjunctions $p \vee \neg q \vee s$, $\neg p \vee \neg r \vee s$, $\neg p \vee \neg r \vee \neg s$, $\neg p \vee q \vee \neg s$, $q \vee r \vee \neg s$, $q \vee \neg r \vee \neg s$, $\neg p \vee \neg q \vee \neg s$, $p \vee r \vee s$, and $p \vee r \vee \neg s$ can be made simultaneously true by an assignment of truth values to p, q, r, and s?

60. Show that the negation of an unsatisfiable compound proposition is a tautology and the negation of a compound proposition that is a tautology is unsatisfiable.

61. Determine whether each of these compound propositions is satisfiable.
   a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$
   b) $(p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$
   c) $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$

62. Determine whether each of these compound propositions is satisfiable.
   a) $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg s) \wedge (p \vee \neg r \vee \neg s) \wedge (\neg p \vee \neg q \vee \neg s) \wedge (p \vee q \vee \neg s)$
   b) $(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg s) \wedge (\neg p \vee \neg r \vee \neg s) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg r \vee \neg s)$
   c) $(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg s) \wedge (q \vee \neg r \vee s) \wedge (\neg p \vee r \vee s) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee s) \wedge (\neg p \vee \neg r \vee \neg s)$

63. Show how the solution of a given 4×4 Sudoku puzzle can be found by solving a satisfiability problem.

64. Construct a compound proposition that asserts that every cell of a 9×9 Sudoku puzzle contains at least one number.

65. Explain the steps in the construction of the compound proposition given in the text that asserts that every column of a 9×9 Sudoku puzzle contains every number.

*66. Explain the steps in the construction of the compound proposition given in the text that asserts that each of the nine 3×3 blocks of a 9×9 Sudoku puzzle contains every number.

1.4 Predicates and Quantifiers

Introduction

Propositional logic, studied in Sections 1.1–1.3, cannot adequately express the meaning of all statements in mathematics and in natural language. For example, suppose that we know that

"Every computer connected to the university network is functioning properly."

Exercises Chap 1.4.

1. Let $P(x)$ denote the statement "$x \leq 4$." What are these truth values?
   a) $P(0)$ **b)** $P(4)$ **c)** $P(6)$
2. Let $P(x)$ be the statement "the word x contains the letter a." What are these truth values?
   a) $P(\text{orange})$ **b)** $P(\text{lemon})$
   c) $P(\text{true})$ **d)** $P(\text{false})$
3. Let $Q(x, y)$ denote the statement "x is the capital of y." What are these truth values?
   a) $Q(\text{Denver, Colorado})$
   b) $Q(\text{Detroit, Michigan})$
   c) $Q(\text{Massachusetts, Boston})$
   d) $Q(\text{New York, New York})$
4. State the value of x after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement "$x > 1$," if the value of x when this statement is reached is
   a) $x = 0$. **b)** $x = 1$.
   c) $x = 2$.
5. Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.
   a) $\exists x P(x)$ **b)** $\forall x P(x)$
   c) $\exists x \neg P(x)$ **d)** $\forall x \neg P(x)$
6. Let $N(x)$ be the statement "x has visited North Dakota," where the domain consists of the students in your school. Express each of these quantifications in English.
   a) $\exists x N(x)$ **b)** $\forall x N(x)$ **c)** $\neg \exists x N(x)$
   d) $\exists x \neg N(x)$ **e)** $\neg \forall x N(x)$ **f)** $\forall x \neg N(x)$
7. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.
   a) $\forall x(C(x) \rightarrow F(x))$ **b)** $\forall x(C(x) \wedge F(x))$
   c) $\exists x(C(x) \rightarrow F(x))$ **d)** $\exists x(C(x) \wedge F(x))$
8. Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.
   a) $\forall x(R(x) \rightarrow H(x))$ **b)** $\forall x(R(x) \wedge H(x))$
   c) $\exists x(R(x) \rightarrow H(x))$ **d)** $\exists x(R(x) \wedge H(x))$
9. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.
   a) There is a student at your school who can speak Russian and who knows C++.
   b) There is a student at your school who can speak Russian but who doesn't know C++.
   c) Every student at your school either can speak Russian or knows C++.
   d) No student at your school can speak Russian or knows C++.
10. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.
    a) A student in your class has a cat, a dog, and a ferret.
    b) All students in your class have a cat, a dog, or a ferret.
    c) Some student in your class has a cat and a ferret, but not a dog.
    d) No student in your class has a cat, a dog, and a ferret.
    e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.
11. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?
    a) $P(0)$ **b)** $P(1)$ **c)** $P(2)$
    d) $P(-1)$ **e)** $\exists x P(x)$ **f)** $\forall x P(x)$
12. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?
    a) $Q(0)$ **b)** $Q(-1)$ **c)** $Q(1)$
    d) $\exists x Q(x)$ **e)** $\forall x Q(x)$ **f)** $\exists x \neg Q(x)$
    g) $\forall x \neg Q(x)$
13. Determine the truth value of each of these statements if the domain consists of all integers.
    a) $\forall n(n + 1 > n)$ **b)** $\exists n(2n = 3n)$
    c) $\exists n(n = -n)$ **d)** $\forall n(3n \leq 4n)$
14. Determine the truth value of each of these statements if the domain consists of all real numbers.
    a) $\exists x(x^3 = -1)$ **b)** $\exists x(x^4 < x^2)$
    c) $\forall x((-x)^2 = x^2)$ **d)** $\forall x(2x > x)$
15. Determine the truth value of each of these statements if the domain for all variables consists of all integers.
    a) $\forall n(n^2 \geq 0)$ **b)** $\exists n(n^2 = 2)$
    c) $\forall n(n^2 \geq n)$ **d)** $\exists n(n^2 < 0)$
16. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.
    a) $\exists x(x^2 = 2)$ **b)** $\exists x(x^2 = -1)$
    c) $\forall x(x^2 + 2 \geq 1)$ **d)** $\forall x(x^2 \neq x)$
17. Suppose that the domain of the propositional function $P(x)$ consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.
    a) $\exists x P(x)$ **b)** $\forall x P(x)$ **c)** $\exists x \neg P(x)$
    d) $\forall x \neg P(x)$ **e)** $\neg \exists x P(x)$ **f)** $\neg \forall x P(x)$
18. Suppose that the domain of the propositional function $P(x)$ consists of the integers -2, -1, 0, 1, and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.
    a) $\exists x P(x)$ **b)** $\forall x P(x)$ **c)** $\exists x \neg P(x)$
    d) $\forall x \neg P(x)$ **e)** $\neg \exists x P(x)$ **f)** $\neg \forall x P(x)$

19. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ **b)** $\forall x P(x)$
c) $\neg\exists x P(x)$ **d)** $\neg\forall x P(x)$
e) $\forall x((x \neq 3) \rightarrow P(x)) \vee \exists x \neg P(x)$

20. Suppose that the domain of the propositional function $P(x)$ consists of -5, -3, -1, 1, 3, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ **b)** $\forall x P(x)$
c) $\forall x((x \neq 1) \rightarrow P(x))$
d) $\exists x((x \geq 0) \wedge P(x))$
e) $\exists x(\neg P(x)) \wedge \forall x((x < 0) \rightarrow P(x))$

21. For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone is studying discrete mathematics.
b) Everyone is older than 21 years.
c) Every two people have the same mother.
d) No two different people have the same grandmother.

22. For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone speaks Hindi.
b) There is someone older than 21 years.
c) Every two people have the same first name.
d) Someone knows more than two other people.

23. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.

a) Someone in your class can speak Hindi.
b) Everyone in your class is friendly.
c) There is a person in your class who was not born in California.
d) A student in your class has been in a movie.
e) No student in your class has taken a course in logic programming.

24. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.

a) Everyone in your class has a cellular phone.
b) Somebody in your class has seen a foreign movie.
c) There is a person in your class who cannot swim.
d) All students in your class can solve quadratic equations.
e) Some student in your class does not want to be rich.

25. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friends is perfect.
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

26. Translate each of these statements into logical expressions in three different ways by varying the domain and by using predicates with one and with two variables.

a) Someone in your school has visited Uzbekistan.
b) Everyone in your class has studied calculus and C++.
c) No one in your school owns both a bicycle and a motorcycle.
d) There is a person in your school who is not happy.
e) Everyone in your school was born in the twentieth century.

27. Translate each of these statements into logical expressions in three different ways by varying the domain and by using predicates with one and with two variables.

a) A student in your school has lived in Vietnam.
b) There is a student in your school who cannot speak Hindi.
c) A student in your school knows Java, Prolog, and C++.
d) Everyone in your class enjoys Thai food.
e) Someone in your class does not play hockey.

28. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) Something is not in the correct place.
b) All tools are in the correct place and are in excellent condition.
c) Everything is in the correct place and in excellent condition.
d) Nothing is in the correct place and is in excellent condition.
e) One of your tools is not in the correct place, but it is in excellent condition.

29. Express each of these statements using logical operators, predicates, and quantifiers.

a) Some propositions are tautologies.
b) The negation of a contradiction is a tautology.
c) The disjunction of two contingencies can be a tautology.
d) The conjunction of two tautologies is a tautology.

30. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.

a) $\exists x\, P(x, 3)$ **b)** $\forall y\, P(1, y)$
c) $\exists y \neg P(2, y)$ **d)** $\forall x \neg P(x, 2)$

31. Suppose that the domain of $Q(x, y, z)$ consists of triples x, y, z, where $x = 0$, 1, or 2, $y = 0$ or 1, and $z = 0$ or 1. Write out these propositions using disjunctions and conjunctions.

a) $\forall y Q(0, y, 0)$ **b)** $\exists x Q(x, 1, 1)$
c) $\exists z \neg Q(0, 0, z)$ **d)** $\exists x \neg Q(x, 0, 1)$

32. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

a) All dogs have fleas.

b) There is a horse that can add.

c) Every koala can climb.

d) No monkey can speak French.

e) There exists a pig that can swim and catch fish.

33. Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

a) Some old dogs can learn new tricks.

b) No rabbit knows calculus.

c) Every bird can fly.

d) There is no dog that can talk.

e) There is no one in this class who knows French and Russian.

34. Express the negation of these propositions using quantifiers, and then express the negation in English.

a) Some drivers do not obey the speed limit.

b) All Swedish movies are serious.

c) No one can keep a secret.

d) There is someone in this class who does not have a good attitude.

35. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x(x^2 \geq x)$

b) $\forall x(x > 0 \vee x < 0)$

c) $\forall x(x = 1)$

36. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.

a) $\forall x(x^2 \neq x)$ **b)** $\forall x(x^2 \neq 2)$

c) $\forall x(|x| > 0)$

37. Express each of these statements using predicates and quantifiers.

a) A passenger on an airline qualifies as an elite flyer if the passenger flies more than 25,000 miles in a year or takes more than 25 flights during that year.

b) A man qualifies for the marathon if his best previous time is less than 3 hours and a woman qualifies for the marathon if her best previous time is less than 3.5 hours.

c) A student must take at least 60 course hours, or at least 45 course hours and write a master's thesis, and receive a grade no lower than a B in all required courses, to receive a master's degree.

d) There is a student who has taken more than 21 credit hours in a semester and received all A's.

Exercises 38–42 deal with the translation between system specification and logical expressions involving quantifiers.

38. Translate these system specifications into English where the predicate $S(x, y)$ is "x is in state y" and where the domain for x and y consists of all systems and all possible states, respectively.

a) $\exists x S(x, \text{open})$

b) $\forall x(S(x, \text{malfunctioning}) \vee S(x, \text{diagnostic}))$

c) $\exists x S(x, \text{open}) \vee \exists x S(x, \text{diagnostic})$

d) $\exists x \neg S(x, \text{available})$

e) $\forall x \neg S(x, \text{working})$

39. Translate these specifications into English where $F(p)$ is "Printer p is out of service," $B(p)$ is "Printer p is busy," $L(j)$ is "Print job j is lost," and $Q(j)$ is "Print job j is queued."

a) $\exists p(F(p) \wedge B(p)) \rightarrow \exists j L(j)$

b) $\forall p B(p) \rightarrow \exists j Q(j)$

c) $\exists j(Q(j) \wedge L(j)) \rightarrow \exists p F(p)$

d) $(\forall p B(p) \wedge \forall j Q(j)) \rightarrow \exists j L(j)$

40. Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) When there is less than 30 megabytes free on the hard disk, a warning message is sent to all users.

b) No directories in the file system can be opened and no files can be closed when system errors have been detected.

c) The file system cannot be backed up if there is a user currently logged on.

d) Video on demand can be delivered when there are at least 8 megabytes of memory available and the connection speed is at least 56 kilobits per second.

41. Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) At least one mail message, among the nonempty set of messages, can be saved if there is a disk with more than 10 kilobytes of free space.

b) Whenever there is an active alert, all queued messages are transmitted.

c) The diagnostic monitor tracks the status of all systems except the main console.

d) Each participant on the conference call whom the host of the call did not put on a special list was billed.

42. Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) Every user has access to an electronic mailbox.

b) The system mailbox can be accessed by everyone in the group if the file system is locked.

c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.

d) At least one router is functioning normally if the throughput is between 100 kbps and 500 kbps and the proxy server is not in diagnostic mode.

43. Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

44. Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x\ P(x) \leftrightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

45. Show that $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ are logically equivalent.

Exercises 46–49 establish rules for **null quantification** that we can use when a quantified variable does not appear in part of a statement.

46. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $(\forall x P(x)) \vee A \equiv \forall x(P(x) \vee A)$

b) $(\exists x P(x)) \vee A \equiv \exists x(P(x) \vee A)$

47. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $(\forall x P(x)) \wedge A \equiv \forall x(P(x) \wedge A)$

b) $(\exists x P(x)) \wedge A \equiv \exists x(P(x) \wedge A)$

48. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(A \rightarrow P(x)) \equiv A \rightarrow \forall x P(x)$

b) $\exists x(A \rightarrow P(x)) \equiv A \rightarrow \exists x P(x)$

49. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(P(x) \rightarrow A) \equiv \exists x P(x) \rightarrow A$

b) $\exists x(P(x) \rightarrow A) \equiv \forall x P(x) \rightarrow A$

50. Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent.

51. Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x(P(x) \wedge Q(x))$ are not logically equivalent.

52. As mentioned in the text, the notation $\exists! x P(x)$ denotes

"There exists a unique x such that $P(x)$ is true."

If the domain consists of all integers, what are the truth values of these statements?

a) $\exists! x(x > 1)$ **b)** $\exists! x(x^2 = 1)$

c) $\exists! x(x + 3 = 2x)$ **d)** $\exists! x(x = x + 1)$

53. What are the truth values of these statements?

a) $\exists! x P(x) \rightarrow \exists x P(x)$

b) $\forall x P(x) \rightarrow \exists! x P(x)$

c) $\exists! x \neg P(x) \rightarrow \neg \forall x P(x)$

54. Write out $\exists! x P(x)$, where the domain consists of the integers 1, 2, and 3, in terms of negations, conjunctions, and disjunctions.

55. Given the Prolog facts in Example 28, what would Prolog return given these queries?

a) `?instructor(chan,math273)`

b) `?instructor(patel,cs301)`

c) `?enrolled(X,cs301)`

d) `?enrolled(kiko,Y)`

e) `?teaches(grossman,Y)`

56. Given the Prolog facts in Example 28, what would Prolog return when given these queries?

a) `?enrolled(kevin,ee222)`

b) `?enrolled(kiko,math273)`

c) `?instructor(grossman,X)`

d) `?instructor(X,cs301)`

e) `?teaches(X,kevin)`

57. Suppose that Prolog facts are used to define the predicates *mother*(M, Y) and *father*(F, X), which represent that M is the mother of Y and F is the father of X, respectively. Give a Prolog rule to define the predicate *sibling*(X, Y), which represents that X and Y are siblings (that is, have the same mother and the same father).

58. Suppose that Prolog facts are used to define the predicates *mother*(M, Y) and *father*(F, X), which represent that M is the mother of Y and F is the father of X, respectively. Give a Prolog rule to define the predicate *grandfather*(X, Y), which represents that X is the grandfather of Y. [*Hint:* You can write a disjunction in Prolog either by using a semicolon to separate predicates or by putting these predicates on separate lines.]

Exercises 59–62 are based on questions found in the book *Symbolic Logic* by Lewis Carroll.

59. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a professor," "x is ignorant," and "x is vain," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$, where the domain consists of all people.

a) No professors are ignorant.

b) All ignorant people are vain.

c) No professors are vain.

d) Does (c) follow from (a) and (b)?

60. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.

a) All clear explanations are satisfactory.

b) Some excuses are unsatisfactory.

c) Some excuses are not clear explanations.

***d)** Does (c) follow from (a) and (b)?

61. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a baby," "x is logical," "x is able to manage a crocodile," and "x is despised," respectively. Suppose that the domain consists of all people. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

a) Babies are illogical.

b) Nobody is despised who can manage a crocodile.

c) Illogical persons are despised.

d) Babies cannot manage crocodiles.

***e)** Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

62. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a duck," "x is one of my poultry," "x is an officer," and "x is willing to waltz," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

a) No ducks are willing to waltz.

b) No officers ever decline to waltz.

c) All my poultry are ducks.

d) My poultry are not officers.

***e)** Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

1.5 Nested Quantifiers

Introduction

In Section 1.4 we defined the existential and universal quantifiers and showed how they can be used to represent mathematical statements. We also explained how they can be used to translate English sentences into logical expressions. However, in Section 1.4 we avoided **nested quantifiers**, where one quantifier is within the scope of another, such as

$$\forall x \exists y (x + y = 0).$$

Note that everything within the scope of a quantifier can be thought of as a propositional function. For example,

$$\forall x \exists y (x + y = 0)$$

is the same thing as $\forall x\, Q(x)$, where $Q(x)$ is $\exists y\, P(x, y)$, where $P(x, y)$ is $x + y = 0$.

Nested quantifiers commonly occur in mathematics and computer science. Although nested quantifiers can sometimes be difficult to understand, the rules we have already studied in Section 1.4 can help us use them. In this section we will gain experience working with nested quantifiers. We will see how to use nested quantifiers to express mathematical statements such as "The sum of two positive integers is always positive." We will show how nested quantifiers can be used to translate English sentences such as "Everyone has exactly one best friend" into logical statements. Moreover, we will gain experience working with the negations of statements involving nested quantifiers.

Understanding Statements Involving Nested Quantifiers

To understand statements involving nested quantifiers, we need to unravel what the quantifiers and predicates that appear mean. This is illustrated in Examples 1 and 2.

EXAMPLE 1 Assume that the domain for the variables x and y consists of all real numbers. The statement

$$\forall x \forall y (x + y = y + x)$$

Extra Examples

says that $x + y = y + x$ for all real numbers x and y. This is the commutative law for addition of real numbers. Likewise, the statement

$$\forall x \exists y (x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an additive inverse. Similarly, the statement

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

is the associative law for addition of real numbers. ◀

EXAMPLE 16 (*Requires calculus*) Use quantifiers and predicates to express the fact that $\lim_{x \to a} f(x)$ does not exist where $f(x)$ is a real-valued function of a real variable x and a belongs to the domain of f.

Solution: To say that $\lim_{x \to a} f(x)$ does not exist means that for all real numbers L, $\lim_{x \to a} f(x) \neq L$. By using Example 8, the statement $\lim_{x \to a} f(x) \neq L$ can be expressed as

$$\neg \forall \epsilon > 0 \, \exists \delta > 0 \, \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon).$$

Successively applying the rules for negating quantified expressions, we construct this sequence of equivalent statements

$$\begin{aligned}
&\neg \forall \epsilon > 0 \, \exists \delta > 0 \, \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
&\quad \equiv \exists \epsilon > 0 \, \neg \exists \delta > 0 \, \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
&\quad \equiv \exists \epsilon > 0 \, \forall \delta > 0 \, \neg \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
&\quad \equiv \exists \epsilon > 0 \, \forall \delta > 0 \, \exists x \, \neg (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
&\quad \equiv \exists \epsilon > 0 \, \forall \delta > 0 \, \exists x (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).
\end{aligned}$$

In the last step we used the equivalence $\neg(p \rightarrow q) \equiv p \wedge \neg q$, which follows from the fifth equivalence in Table 7 of Section 1.3.

Because the statement "$\lim_{x \to a} f(x)$ does not exist" means for all real numbers L, $\lim_{x \to a} f(x) \neq L$, this can be expressed as

$$\forall L \exists \epsilon > 0 \, \forall \delta > 0 \, \exists x (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).$$

This last statement says that for every real number L there is a real number $\epsilon > 0$ such that for every real number $\delta > 0$, there exists a real number x such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$. ◀

Exercises Chap 1.5

1. Translate these statements into English, where the domain for each variable consists of all real numbers.

a) $\forall x \exists y (x < y)$

b) $\forall x \forall y (((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$

c) $\forall x \forall y \exists z (xy = z)$

2. Translate these statements into English, where the domain for each variable consists of all real numbers.

a) $\exists x \forall y (xy = y)$

b) $\forall x \forall y (((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$

c) $\forall x \forall y \exists z (x = y + z)$

3. Let $Q(x, y)$ be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

a) $\exists x \exists y Q(x, y)$ **b)** $\exists x \forall y Q(x, y)$

c) $\forall x \exists y Q(x, y)$ **d)** $\exists y \forall x Q(x, y)$

e) $\forall y \exists x Q(x, y)$ **f)** $\forall x \forall y Q(x, y)$

4. Let $P(x, y)$ be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

a) $\exists x \exists y P(x, y)$ **b)** $\exists x \forall y P(x, y)$

c) $\forall x \exists y P(x, y)$ **d)** $\exists y \forall x P(x, y)$

e) $\forall y \exists x P(x, y)$ **f)** $\forall x \forall y P(x, y)$

5. Let $W(x, y)$ mean that student x has visited website y, where the domain for x consists of all students in your school and the domain for y consists of all websites. Express each of these statements by a simple English sentence.

a) W(Sarah Smith, www.att.com)

b) $\exists x W(x,$ www.imdb.org)

c) $\exists y W($José Orez, $y)$

d) $\exists y (W($Ashok Puri, $y) \wedge W($Cindy Yoon, $y))$

e) $\exists y \forall z (y \neq$ (David Belcher) $\wedge$ (W(David Belcher, z) $\rightarrow W(y,z)$))

f) $\exists x \exists y \forall z ((x \neq y) \wedge (W(x, z) \leftrightarrow W(y, z)))$

6. Let $C(x, y)$ mean that student x is enrolled in class y, where the domain for x consists of all students in your school and the domain for y consists of all classes being

given at your school. Express each of these statements by a simple English sentence.

a) C(Randy Goldberg, CS 252)
b) $\exists x C(x$, Math 695)
c) $\exists y C$(Carol Sitea, y)
d) $\exists x(C(x$, Math 222) $\wedge C(x$, CS 252))
e) $\exists x \exists y \forall z((x \neq y) \wedge (C(x, z) \rightarrow C(y, z)))$
f) $\exists x \exists y \forall z((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$

7. Let $T(x, y)$ mean that student x likes cuisine y, where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T$(Abdallah Hussein, Japanese)
b) $\exists x T(x$, Korean) $\wedge \forall x T(x$, Mexican)
c) $\exists y(T$(Monique Arsenault, $y) \vee$ T(Jay Johnson, $y))$
d) $\forall x \forall z \exists y((x \neq z) \rightarrow \neg(T(x, y) \wedge T(z, y)))$
e) $\exists x \exists z \forall y(T(x, y) \leftrightarrow T(z, y))$
f) $\forall x \forall z \exists y(T(x, y) \leftrightarrow T(z, y))$

8. Let $Q(x, y)$ be the statement "student x has been a contestant on quiz show y." Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.

a) There is a student at your school who has been a contestant on a television quiz show.
b) No student at your school has ever been a contestant on a television quiz show.
c) There is a student at your school who has been a contestant on *Jeopardy* and on *Wheel of Fortune*.
d) Every television quiz show has had a student from your school as a contestant.
e) At least two students from your school have been contestants on *Jeopardy*.

9. Let $L(x, y)$ be the statement "x loves y," where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry.
b) Everybody loves somebody.
c) There is somebody whom everybody loves.
d) Nobody loves everybody.
e) There is somebody whom Lydia does not love.
f) There is somebody whom no one loves.
g) There is exactly one person whom everybody loves.
h) There are exactly two people whom Lynn loves.
i) Everyone loves himself or herself.
j) There is someone who loves no one besides himself or herself.

10. Let $F(x, y)$ be the statement "x can fool y," where the domain consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody can fool Fred.
b) Evelyn can fool everybody.
c) Everybody can fool somebody.
d) There is no one who can fool everybody.
e) Everyone can be fooled by somebody.
f) No one can fool both Fred and Jerry.
g) Nancy can fool exactly two people.
h) There is exactly one person whom everybody can fool.
i) No one can fool himself or herself.
j) There is someone who can fool exactly one person besides himself or herself.

11. Let $S(x)$ be the predicate "x is a student," $F(x)$ the predicate "x is a faculty member," and $A(x, y)$ the predicate "x has asked y a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

a) Lois has asked Professor Michaels a question.
b) Every student has asked Professor Gross a question.
c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
d) Some student has not asked any faculty member a question.
e) There is a faculty member who has never been asked a question by a student.
f) Some student has asked every faculty member a question.
g) There is a faculty member who has asked every other faculty member a question.
h) Some student has never been asked a question by a faculty member.

12. Let $I(x)$ be the statement "x has an Internet connection" and $C(x, y)$ be the statement "x and y have chatted over the Internet," where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.

a) Jerry does not have an Internet connection.
b) Rachel has not chatted over the Internet with Chelsea.
c) Jan and Sharon have never chatted over the Internet.
d) No one in the class has chatted with Bob.
e) Sanjay has chatted with everyone except Joseph.
f) Someone in your class does not have an Internet connection.
g) Not everyone in your class has an Internet connection.
h) Exactly one student in your class has an Internet connection.
i) Everyone except one student in your class has an Internet connection.
j) Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.
k) Someone in your class has an Internet connection but has not chatted with anyone else in your class.
l) There are two students in your class who have not chatted with each other over the Internet.
m) There is a student in your class who has chatted with everyone in your class over the Internet.
n) There are at least two students in your class who have not chatted with the same person in your class.
o) There are two students in the class who between them have chatted with everyone else in the class.

13. Let $M(x, y)$ be "x has sent y an e-mail message" and $T(x, y)$ be "x has telephoned y," where the domain consists of all students in your class. Use quantifiers to express each of these statements. (Assume that all e-mail messages that were sent are received, which is not the way things often work.)

a) Chou has never sent an e-mail message to Koko.

b) Arlene has never sent an e-mail message to or telephoned Sarah.

c) José has never received an e-mail message from Deborah.

d) Every student in your class has sent an e-mail message to Ken.

e) No one in your class has telephoned Nina.

f) Everyone in your class has either telephoned Avi or sent him an e-mail message.

g) There is a student in your class who has sent everyone else in your class an e-mail message.

h) There is someone in your class who has either sent an e-mail message or telephoned everyone else in your class.

i) There are two different students in your class who have sent each other e-mail messages.

j) There is a student who has sent himself or herself an e-mail message.

k) There is a student in your class who has not received an e-mail message from anyone else in the class and who has not been called by any other student in the class.

l) Every student in the class has either received an e-mail message or received a telephone call from another student in the class.

m) There are at least two students in your class such that one student has sent the other e-mail and the second student has telephoned the first student.

n) There are two different students in your class who between them have sent an e-mail message to or telephoned everyone else in the class.

14. Use quantifiers and predicates with more than one variable to express these statements.

a) There is a student in this class who can speak Hindi.

b) Every student in this class plays some sport.

c) Some student in this class has visited Alaska but has not visited Hawaii.

d) All students in this class have learned at least one programming language.

e) There is a student in this class who has taken every course offered by one of the departments in this school.

f) Some student in this class grew up in the same town as exactly one other student in this class.

g) Every student in this class has chatted with at least one other student in at least one chat group.

15. Use quantifiers and predicates with more than one variable to express these statements.

a) Every computer science student needs a course in discrete mathematics.

b) There is a student in this class who owns a personal computer.

c) Every student in this class has taken at least one computer science course.

d) There is a student in this class who has taken at least one course in computer science.

e) Every student in this class has been in every building on campus.

f) There is a student in this class who has been in every room of at least one building on campus.

g) Every student in this class has been in at least one room of every building on campus.

16. A discrete mathematics class contains 1 mathematics major who is a freshman, 12 mathematics majors who are sophomores, 15 computer science majors who are sophomores, 2 mathematics majors who are juniors, 2 computer science majors who are juniors, and 1 computer science major who is a senior. Express each of these statements in terms of quantifiers and then determine its truth value.

a) There is a student in the class who is a junior.

b) Every student in the class is a computer science major.

c) There is a student in the class who is neither a mathematics major nor a junior.

d) Every student in the class is either a sophomore or a computer science major.

e) There is a major such that there is a student in the class in every year of study with that major.

17. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

a) Every user has access to exactly one mailbox.

b) There is a process that continues to run during all error conditions only if the kernel is working correctly.

c) All users on the campus network can access all websites whose url has a .edu extension.

***d)** There are exactly two systems that monitor every remote server.

18. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

a) At least one console must be accessible during every fault condition.

b) The e-mail address of every user can be retrieved whenever the archive contains at least one message sent by every user on the system.

c) For every security breach there is at least one mechanism that can detect that breach if and only if there is a process that has not been compromised.

d) There are at least two paths connecting every two distinct endpoints on the network.

e) No one knows the password of every user on the system except for the system administrator, who knows all passwords.[

19. Express each of these statements using mathematical and logical operators, predicates, and quantifiers, where the domain consists of all integers.

a) The sum of two negative integers is negative.

b) The difference of two positive integers is not necessarily positive.

c) The sum of the squares of two integers is greater than or equal to the square of their sum.
d) The absolute value of the product of two integers is the product of their absolute values.

20. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.
a) The product of two negative integers is positive.
b) The average of two positive integers is positive.
c) The difference of two negative integers is not necessarily negative.
d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

21. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that every positive integer is the sum of the squares of four integers.

22. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that there is a positive integer that is not the sum of three squares.

23. Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.
a) The product of two negative real numbers is positive.
b) The difference of a real number and itself is zero.
c) Every positive real number has exactly two square roots.
d) A negative real number does not have a square root that is a real number.

24. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.
a) $\exists x \forall y(x + y = y)$
b) $\forall x \forall y(((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$
c) $\exists x \exists y(((x \leq 0) \wedge (y \leq 0)) \wedge (x - y > 0))$
d) $\forall x \forall y((x \neq 0) \wedge (y \neq 0) \leftrightarrow (xy \neq 0))$

25. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.
a) $\exists x \forall y(xy = y)$
b) $\forall x \forall y(((x < 0) \wedge (y < 0)) \rightarrow (xy > 0))$
c) $\exists x \exists y((x^2 > y) \wedge (x < y))$
d) $\forall x \forall y \exists z(x + y = z)$

26. Let $Q(x, y)$ be the statement "$x + y = x - y$." If the domain for both variables consists of all integers, what are the truth values?
a) $Q(1, 1)$ b) $Q(2, 0)$
c) $\forall y Q(1, y)$ d) $\exists x Q(x, 2)$
e) $\exists x \exists y Q(x, y)$ f) $\forall x \exists y Q(x, y)$
g) $\exists y \forall x Q(x, y)$ h) $\forall y \exists x Q(x, y)$
i) $\forall x \forall y Q(x, y)$

27. Determine the truth value of each of these statements if the domain for all variables consists of all integers.
a) $\forall n \exists m(n^2 < m)$ b) $\exists n \forall m(n < m^2)$
c) $\forall n \exists m(n + m = 0)$ d) $\exists n \forall m(nm = m)$
e) $\exists n \exists m(n^2 + m^2 = 5)$ f) $\exists n \exists m(n^2 + m^2 = 6)$
g) $\exists n \exists m(n + m = 4 \wedge n - m = 1)$
h) $\exists n \exists m(n + m = 4 \wedge n - m = 2)$
i) $\forall n \forall m \exists p(p = (m + n)/2)$

28. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.
a) $\forall x \exists y(x^2 = y)$ b) $\forall x \exists y(x = y^2)$
c) $\exists x \forall y(xy = 0)$ d) $\exists x \exists y(x + y \neq y + x)$
e) $\forall x(x \neq 0 \rightarrow \exists y(xy = 1))$
f) $\exists x \forall y(y \neq 0 \rightarrow xy = 1)$
g) $\forall x \exists y(x + y = 1)$
h) $\exists x \exists y(x + 2y = 2 \wedge 2x + 4y = 5)$
i) $\forall x \exists y(x + y = 2 \wedge 2x - y = 1)$
j) $\forall x \forall y \exists z(z = (x + y)/2)$

29. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.
a) $\forall x \forall y P(x, y)$ b) $\exists x \exists y P(x, y)$
c) $\exists x \forall y P(x, y)$ d) $\forall y \exists x P(x, y)$

30. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).
a) $\neg \exists y \exists x P(x, y)$ b) $\neg \forall x \exists y P(x, y)$
c) $\neg \exists y(Q(y) \wedge \forall x \neg R(x, y))$
d) $\neg \exists y(\exists x R(x, y) \vee \forall x S(x, y))$
e) $\neg \exists y(\forall x \exists z T(x, y, z) \vee \exists x \forall z U(x, y, z))$

31. Express the negations of each of these statements so that all negation symbols immediately precede predicates.
a) $\forall x \exists y \forall z T(x, y, z)$
b) $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$
c) $\forall x \exists y(P(x, y) \wedge \exists z R(x, y, z))$
d) $\forall x \exists y(P(x, y) \rightarrow Q(x, y))$

32. Express the negations of each of these statements so that all negation symbols immediately precede predicates.
a) $\exists z \forall y \forall x T(x, y, z)$
b) $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$
c) $\exists x \exists y(Q(x, y) \leftrightarrow Q(y, x))$
d) $\forall y \exists x \exists z(T(x, y, z) \vee Q(x, y))$

33. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).
a) $\neg \forall x \forall y P(x, y)$ b) $\neg \forall y \exists x P(x, y)$
c) $\neg \forall y \forall x(P(x, y) \vee Q(x, y))$
d) $\neg(\exists x \exists y \neg P(x, y) \wedge \forall x \forall y Q(x, y))$
e) $\neg \forall x(\exists y \forall z P(x, y, z) \wedge \exists z \forall y P(x, y, z))$

34. Find a common domain for the variables x, y, and z for which the statement $\forall x \forall y((x \neq y) \rightarrow \forall z((z = x) \vee (z = y)))$ is true and another domain for which it is false.

35. Find a common domain for the variables x, y, z, and w for which the statement $\forall x \forall y \forall z \exists w((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ is true and another common domain for these variables for which it is false.

36. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

a) No one has lost more than one thousand dollars playing the lottery.
b) There is a student in this class who has chatted with exactly one other student.
c) No student in this class has sent e-mail to exactly two other students in this class.
d) Some student has solved every exercise in this book.
e) No student has solved at least one exercise in every section of this book.

37. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

a) Every student in this class has taken exactly two mathematics classes at this school.
b) Someone has visited every country in the world except Libya.
c) No one has climbed every mountain in the Himalayas.
d) Every movie actor has either been in a movie with Kevin Bacon or has been in a movie with someone who has been in a movie with Kevin Bacon.

38. Express the negations of these propositions using quantifiers, and in English.

a) Every student in this class likes mathematics.
b) There is a student in this class who has never seen a computer.
c) There is a student in this class who has taken every mathematics course offered at this school.
d) There is a student in this class who has been in at least one room of every building on campus.

39. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x \forall y(x^2 = y^2 \rightarrow x = y)$
b) $\forall x \exists y(y^2 = x)$
c) $\forall x \forall y(xy \geq x)$

40. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x \exists y(x = 1/y)$
b) $\forall x \exists y(y^2 - x < 100)$
c) $\forall x \forall y(x^2 \neq y^3)$

41. Use quantifiers to express the associative law for multiplication of real numbers.

42. Use quantifiers to express the distributive laws of multiplication over addition for real numbers.

43. Use quantifiers and logical connectives to express the fact that every linear polynomial (that is, polynomial of degree 1) with real coefficients and where the coefficient of x is nonzero, has exactly one real root.

44. Use quantifiers and logical connectives to express the fact that a quadratic polynomial with real number coefficients has at most two real roots.

45. Determine the truth value of the statement $\forall x \exists y(xy = 1)$ if the domain for the variables consists of

a) the nonzero real numbers.
b) the nonzero integers.
c) the positive real numbers.

46. Determine the truth value of the statement $\exists x \forall y(x \leq y^2)$ if the domain for the variables consists of

a) the positive real numbers.
b) the integers.
c) the nonzero real numbers.

47. Show that the two statements $\neg \exists x \forall y P(x, y)$ and $\forall x \exists y \neg P(x, y)$, where both quantifiers over the first variable in $P(x, y)$ have the same domain, and both quantifiers over the second variable in $P(x, y)$ have the same domain, are logically equivalent.

***48.** Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x \forall y(P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain, are logically equivalent. (The new variable y is used to combine the quantifications correctly.)

***49.** **a)** Show that $\forall x P(x) \wedge \exists x Q(x)$ is logically equivalent to $\forall x \exists y\ (P(x) \wedge Q(y))$, where all quantifiers have the same nonempty domain.
b) Show that $\forall x P(x) \vee \exists x Q(x)$ is equivalent to $\forall x \exists y\ (P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain.

A statement is in **prenex normal form (PNF)** if and only if it is of the form

$$Q_1 x_1 Q_2 x_2 \cdots Q_k x_k P(x_1, x_2, \ldots, x_k),$$

where each Q_i, $i = 1, 2, \ldots, k$, is either the existential quantifier or the universal quantifier, and $P(x_1, \ldots, x_k)$ is a predicate involving no quantifiers. For example, $\exists x \forall y(P(x, y) \wedge Q(y))$ is in prenex normal form, whereas $\exists x P(x) \vee \forall x Q(x)$ is not (because the quantifiers do not all occur first).

Every statement formed from propositional variables, predicates, **T**, and **F** using logical connectives and quantifiers is equivalent to a statement in prenex normal form. Exercise 51 asks for a proof of this fact.

***50.** Put these statements in prenex normal form. [*Hint:* Use logical equivalence from Tables 6 and 7 in Section 1.3, Table 2 in Section 1.4, Example 19 in Section 1.4, Exercises 45 and 46 in Section 1.4, and Exercises 48 and 49.]

a) $\exists x P(x) \vee \exists x Q(x) \vee A$, where A is a proposition not involving any quantifiers.
b) $\neg(\forall x P(x) \vee \forall x Q(x))$
c) $\exists x P(x) \rightarrow \exists x Q(x)$

****51.** Show how to transform an arbitrary statement to a statement in prenex normal form that is equivalent to the given statement. (Note: A formal solution of this exercise requires use of structural induction, covered in Section 5.3.)

***52.** Express the quantification $\exists! x P(x)$, introduced in Section 1.4, using universal quantifications, existential quantifications, and logical operators.

Exercises Chap 1.7

1. Use a direct proof to show that the sum of two odd integers is even.
2. Use a direct proof to show that the sum of two even integers is even.
3. Show that the square of an even number is an even number using a direct proof.
4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.
5. Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even. What kind of proof did you use?
6. Use a direct proof to show that the product of two odd numbers is odd.
7. Use a direct proof to show that every odd integer is the difference of two squares.
8. Prove that if n is a perfect square, then $n + 2$ is not a perfect square.
9. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.
10. Use a direct proof to show that the product of two rational numbers is rational.
11. Prove or disprove that the product of two irrational numbers is irrational.
12. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.
13. Prove that if x is irrational, then $1/x$ is irrational.
14. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.
15. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.
16. Prove that if m and n are integers and mn is even, then m is even or n is even.
17. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using
    a) a proof by contraposition.
    b) a proof by contradiction.
18. Prove that if n is an integer and $3n + 2$ is even, then n is even using
    a) a proof by contraposition.
    b) a proof by contradiction.
19. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?
20. Prove the proposition $P(1)$, where $P(n)$ is the proposition "If n is a positive integer, then $n^2 \geq n$." What kind of proof did you use?
21. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a + b)^n \geq a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?
22. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.
23. Show that at least ten of any 64 days chosen must fall on the same day of the week.
24. Show that at least three of any 25 days chosen must fall in the same month of the year.
25. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$. [*Hint:* Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]
26. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.
27. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.
28. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.
29. Prove or disprove that if m and n are integers such that $mn = 1$, then either $m = 1$ and $n = 1$, or else $m = -1$ and $n = -1$.
30. Show that these three statements are equivalent, where a and b are real numbers: (*i*) a is less than b, (*ii*) the average of a and b is greater than a, and (*iii*) the average of a and b is less than b.
31. Show that these statements about the integer x are equivalent: (*i*) $3x + 2$ is even, (*ii*) $x + 5$ is odd, (*iii*) x^2 is even.
32. Show that these statements about the real number x are equivalent: (*i*) x is rational, (*ii*) $x/2$ is rational, (*iii*) $3x - 1$ is rational.
33. Show that these statements about the real number x are equivalent: (*i*) x is irrational, (*ii*) $3x + 2$ is irrational, (*iii*) $x/2$ is irrational.
34. Is this reasoning for finding the solutions of the equation $\sqrt{2x^2 - 1} = x$ correct? (*1*) $\sqrt{2x^2 - 1} = x$ is given; (2) $2x^2 - 1 = x^2$, obtained by squaring both sides of (1); (*3*) $x^2 - 1 = 0$, obtained by subtracting x^2 from both sides of (2); (*4*) $(x - 1)(x + 1) = 0$, obtained by factoring the left-hand side of $x^2 - 1$; (*5*) $x = 1$ or $x = -1$, which follows because $ab = 0$ implies that $a = 0$ or $b = 0$.
35. Are these steps for finding the solutions of $\sqrt{x + 3} = 3 - x$ correct? (*1*) $\sqrt{x + 3} = 3 - x$ is given; (2) $x + 3 = x^2 - 6x + 9$, obtained by squaring both sides of (1); (*3*) $0 = x^2 - 7x + 6$, obtained by subtracting $x + 3$ from both sides of (2); (*4*) $0 = (x - 1)(x - 6)$, obtained by factoring the right-hand side of (3); (*5*) $x = 1$ or $x = 6$, which follows from (4) because $ab = 0$ implies that $a = 0$ or $b = 0$.
36. Show that the propositions p_1, p_2, p_3, and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4$, $p_2 \leftrightarrow p_3$, and $p_1 \leftrightarrow p_3$.
37. Show that the propositions p_1, p_2, p_3, p_4, and p_5 can be shown to be equivalent by proving that the conditional statements $p_1 \rightarrow p_4$, $p_3 \rightarrow p_1$, $p_4 \rightarrow p_2$, $p_2 \rightarrow p_5$, and $p_5 \rightarrow p_3$ are true.

38. Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

39. Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

40. Use Exercise 39 to show that if the first 10 positive integers are placed around a circle, in any order, there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

41. Prove that if n is an integer, these four statements are equivalent: (*i*) n is even, (*ii*) $n + 1$ is odd, (*iii*) $3n + 1$ is odd, (*iv*) $3n$ is even.

42. Prove that these four statements about the integer n are equivalent: (*i*) n^2 is odd, (*ii*) $1 - n$ is even, (*iii*) n^3 is odd, (*iv*) $n^2 + 1$ is even.

1.8 Proof Methods and Strategy

Introduction

In Section 1.7 we introduced many methods of proof and illustrated how each method can be used. In this section we continue this effort. We will introduce several other commonly used proof methods, including the method of proving a theorem by considering different cases separately. We will also discuss proofs where we prove the existence of objects with desired properties.

In Section 1.7 we briefly discussed the strategy behind constructing proofs. This strategy includes selecting a proof method and then successfully constructing an argument step by step, based on this method. In this section, after we have developed a versatile arsenal of proof methods, we will study some aspects of the art and science of proofs. We will provide advice on how to find a proof of a theorem. We will describe some tricks of the trade, including how proofs can be found by working backward and by adapting existing proofs.

When mathematicians work, they formulate conjectures and attempt to prove or disprove them. We will briefly describe this process here by proving results about tiling checkerboards with dominoes and other types of pieces. Looking at tilings of this kind, we will be able to quickly formulate conjectures and prove theorems without first developing a theory.

We will conclude the section by discussing the role of open questions. In particular, we will discuss some interesting problems either that have been solved after remaining open for hundreds of years or that still remain open.

Exhaustive Proof and Proof by Cases

Sometimes we cannot prove a theorem using a single argument that holds for all possible cases. We now introduce a method that can be used to prove a theorem, by considering different cases separately. This method is based on a rule of inference that we will now introduce. To prove a conditional statement of the form

$$(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q$$

the tautology

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)]$$

can be used as a rule of inference. This shows that the original conditional statement with a hypothesis made up of a disjunction of the propositions $p_1, p_2, \ldots, p_n$ can be proved by proving each of the n conditional statements $p_i \rightarrow q$, $i = 1, 2, \ldots, n$, individually. Such an argument is called a **proof by cases**. Sometimes to prove that a conditional statement $p \rightarrow q$ is true, it is convenient to use a disjunction $p_1 \vee p_2 \vee \cdots \vee p_n$ instead of p as the hypothesis of the conditional statement, where p and $p_1 \vee p_2 \vee \cdots \vee p_n$ are equivalent.