3. Use mathematical induction to show that $1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots + n \cdot 2^{n-1} = (n-1) \cdot 2^n + 1$ whenever n is a positive integer.

4. Use mathematical induction to show that

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

whenever n is a positive integer.

5. Show that

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

whenever n is a positive integer.

6. Use mathematical induction to show that $2^n > n^2 + n$ whenever n is an integer greater than 4.

7. Use mathematical induction to show that $2^n > n^3$ whenever n is an integer greater than 9.

8. Find an integer N such that $2^n > n^4$ whenever n is greater than N. Prove that your result is correct using mathematical induction.

9. Use mathematical induction to prove that $a - b$ is a factor of $a^n - b^n$ whenever n is a positive integer.

10. Use mathematical induction to prove that 9 divides $n^3 + (n+1)^3 + (n+2)^3$ whenever n is a nonnegative integer.

11. Use mathematical induction to prove this formula for the sum of the terms of an arithmetic progression.

$$a + (a+d) + \cdots + (a+nd) = (n+1)(2a+nd)/2$$

12. Suppose that $a_j \equiv b_j \pmod{m}$ for $j = 1, 2, \ldots, n$. Use mathematical induction to prove that

a) $\sum_{j=1}^{n} a_j \equiv \sum_{j=1}^{n} b_j \pmod{m}$.

b) $\prod_{j=1}^{n} a_j \equiv \prod_{j=1}^{n} b_j \pmod{m}$.

13. Show that if n is a positive integer, then

$$\sum_{k=1}^{n} \frac{k+4}{k(k+1)(k+2)} = \frac{n(3n+7)}{2(n+1)(n+2)}.$$

14. For which positive integers n is $n + 6 < (n^2 - 8n)/16$? Prove your answer using mathematical induction.

15. (*Requires calculus*) Suppose that $f(x) = e^x$ and $g(x) = xe^x$. Use mathematical induction together with the product rule and the fact that $f'(x) = e^x$ to prove that $g^{(n)}(x) = (x+n)e^x$ whenever n is a positive integer.

16. (*Requires calculus*) Suppose that $f(x) = e^x$ and $g(x) = e^{cx}$, where c is a constant. Use mathematical induction together with the chain rule and the fact that $f'(x) = e^x$ to prove that $g^{(n)} = c^n e^{cx}$ whenever n is a positive integer.

***17.** Determine which Fibonacci numbers are even, and use a form of mathematical induction to prove your conjecture.

***18.** Determine which Fibonacci numbers are divisible by 3. Use a form of mathematical induction to prove your conjecture.

***19.** Prove that $f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1}$ for all nonnegative integers n, where k is a nonnegative integer and f_i denotes the ith Fibonacci number.

The sequence of **Lucas numbers** is defined by $l_0 = 2$, $l_1 = 1$, and $l_n = l_{n-1} + l_{n-2}$ for $n = 2, 3, 4, \ldots$.

20. Show that $f_n + f_{n+2} = l_{n+1}$ whenever n is a positive integer, where f_i and l_i are the ith Fibonacci number and ith Lucas number, respectively.

21. Show that $l_0^2 + l_1^2 + \cdots + l_n^2 = l_n l_{n+1} + 2$ whenever n is a nonnegative integer and l_i is the ith Lucas number.

***22.** Use mathematical induction to show that the product of any n consecutive positive integers is divisible by $n!$. [*Hint:* Use the identity $m(m+1)\cdots(m+n-1)/n! = (m-1)m(m+1)\cdots(m+n-2)/n! + m(m+1)\cdots(m+n-2)/(n-1)!$.]

23. Use mathematical induction to show that $(\cos x + i \sin x)^n = \cos nx + i \sin nx$ whenever n is a positive integer. (Here i is the square root of -1.) [*Hint:* Use the identities $\cos(a+b) = \cos a \cos b - \sin a \sin b$ and $\sin(a+b) = \sin a \cos b + \cos a \sin b$.]

***24.** Use mathematical induction to show that $\sum_{j=1}^{n} \cos jx = \cos[(n+1)x/2] \sin(nx/2)/\sin(x/2)$ whenever n is a positive integer and $\sin(x/2) \neq 0$.

25. Use mathematical induction to prove that $\sum_{j=1}^{n} j^2 2^j = n^2 2^{n+1} - n2^{n+2} + 3 \cdot 2^{n+1} - 6$ for every positive integer n.

26. (*Requires calculus*) Suppose that the sequence $x_1, x_2, \ldots, x_n, \ldots$ is recursively defined by $x_1 = 0$ and $x_{n+1} = \sqrt{x_n + 6}$.

a) Use mathematical induction to show that $x_1 < x_2 < \cdots < x_n < \cdots$, that is, the sequence $\{x_n\}$ is monotonically increasing.

b) Use mathematical induction to prove that $x_n < 3$ for $n = 1, 2, \ldots$.

c) Show that $\lim_{n \to \infty} x_n = 3$.

27. Show if n is a positive integer with $n \geq 2$, then

$$\sum_{j=2}^{n} \frac{1}{j^2 - 1} = \frac{(n-1)(3n+2)}{4n(n+1)}.$$

28. Use mathematical induction to prove Theorem 1 in Section 3.6, that is, show if b is a positive integer, $b > 1$, and n is a positive integer, then n can be expressed uniquely in the form $n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$.

***29.** A **lattice point** in the plane is a point (x, y) where both x and y are integers. Use mathematical induction to show that at least $n + 1$ straight lines are needed to ensure that every lattice point (x, y) with $x \geq 0$, $y \geq 0$, and $x + y \leq n$ lies on one of these lines.

30. (*Requires calculus*) Use mathematical induction and the product rule to show that if n is a positive integer and