On the sensitivity of the Black capital asset pricing model to the market portfolio

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Abstract. We show that Black Capital Asset Pricing Model (Black CAPM) is extremely sensitive to the choice of the market portfolio and becomes unstable as market portfolios approach the Global Minimum-Variance portfolio. When market portfolios approach the minimum-variance portfolio, the expected return on the zero beta asset approaches negative infinity and its variance increases rapidly. Moreover, expected return on a fixed portfolio becomes indefinite (i.e., takes infinitely many values), and betas of all portfolios approach one. Unlike the Sharpe–Lintner CAPM, the market risk premium in the Black CAPM always has a positive minimum, while beta may have a negative minimum value, dependent on the underlying covariance matrix.

Keywords: Asset pricing, Black CAPM, global minimum variance asset, zero-beta asset

1. Introduction

The capital asset pricing model (CAPM) seeks to explain the relation between risk and return in a rational market. The Sharpe [7] and Lintner [4] version of the CAPM is an extension of the Markowitz [5] mean–variance efficient portfolio. Markowitz argues that investors are risk averse, so they choose portfolios that maximize expected return at a given level of risk, or that minimize the variance of the portfolio’s return. The Sharpe–Lintner CAPM builds on Markowitz [5,6] by also assuming that investors not only agree on the joint probability distribution of asset returns, but that they can unrestrictedly borrow and lend at a given riskless rate of interest.

The Sharpe–Lintner extension implies that investors combine the riskless asset with the best attainable mean–variance efficient portfolio, MVEP (i.e., tangency portfolio) depending on their risk–return preferences [8]. Under the Sharpe–Lintner CAPM, the expected return $E R_i$ on any asset $i$, can be expressed as:

$$E R_i = R_f + \beta_i E(R_m - R_f),$$

where $R_f$ represents the riskless interest rate, $\beta_i$ symbolizes asset $i$’s market beta and $E(R_m - R_f)$ denotes the market risk premium (per unit of beta risk). This market risk premium is sometimes negative. Asset $i$’s market beta represents the market risk-adjusted covariance of the return on the asset and the return of the mean–variance efficient market portfolio, and is explicitly given as

$$\beta_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}.$$

Since the CAPM provides an intuitive mechanism to price risk, it is widely used by analysts, corporations and investors to estimate the cost of capital and to evaluate performance. The 2010–2011 sovereign debt crisis has provoked debate concerning the applications of the CAPM, as hardly any asset actually appears to be risk free. Due to the nature of sovereign debt, practitioners usually consider short-term government bonds to be riskless. However, in wake of rising national debt levels, a crisis of confidence has emerged in sovereign bonds markets around the world. Rating agencies have cast doubt on the credit rating of the United States, the United Kingdom, Austria, France, and other countries, suggesting that riskless rates are “illusions”.

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Market participants in global finance have long struggled to apply the CAPM, as there is no appropriate risk free rate. Seeing that many sovereign bonds no longer appear to be “riskless” owing to recent downgrades, practitioners in the domestic setting have to now question their reference risk free rate. Yet, the CAPM assumes that a riskless asset can be specified. Because financial decisions are more complicated when there is uncertainty, we examine whether Black’s approach to the CAPM is an appropriate alternative.

Black [1] relaxes the assumption that investors can borrow and lend at the riskless rate and develops a version of the CAPM where all assets are risky, provided that there is a zero-beta asset (a portfolio constructed to have zero systematic risk). The Black CAPM gives the expected return on an asset or portfolio \( i \), with return \( R_i \), in terms of the expected return on the zero-beta asset \( z_p \), with return \( R_{zp} \), which is uniquely determined from any mean–variance efficient “market” portfolio \( p \), with return \( R_p \). Consequently, the expected return on any portfolio \( i \), is given by the simple relationship:

\[
E[R_i - R_{zp}] = \beta_{ip}E[R_p - R_{zp}],
\]

(1)

\[
\beta_{ip} = \frac{\text{Cov}(R_i, R_p)}{\text{Var}(R_p)},
\]

(2)

where \( \beta_{ip} \) is the market risk-adjusted covariance of the return on the asset, \( R_i \), with the return of the mean–variance efficient portfolio or asset, \( R_p \). Define the expected return of asset “a” by \( \mu_a = E[R_a] \), where \( a \in \{ i, g, p, z, p, z \} \). Then

\[
\mu_i = (1 - \beta_{ip})\mu_{zp} + \beta_{ip}\mu_p.
\]

(3)

Hence, Black [1] implies that the CAPM does not strictly require the existence of a riskless asset. In this paper, we show that Eq. (3) holds only if \( \mu_{zp} \) is not very large in absolute value. That is, if \( p \), the efficient market portfolio, is not close to the global minimum variance portfolio, \( g \). We demonstrate that the Black CAPM becomes unstable as portfolios approach the global minimum variance portfolio on the mean–variance efficient frontier. The results indicate that as the market portfolio moves towards the global minimum variance portfolio, the expected return on the zero-beta asset approaches negative infinity, while its variance approaches positive infinity. Therefore, unlike the Sharpe–Lintner CAPM, with a risk free (zero variance) asset, the unique zero-beta asset of the Black CAPM has a variance that may become infinite.

In addition, expected returns on a fixed portfolio becomes indefinite (that is, takes infinitely many values), and betas of all portfolios approach 1. We also show that, unlike the Sharpe–Lintner CAPM, the market risk premium in the Black CAPM always has a positive minimum value which varies with the underlying covariance matrix. Consequently, betas of portfolios are sometimes negative, a situation that does not generally occur in the Sharpe–Lintner CAPM.

We also show that given the mean–variance efficient frontier (MVEF), the zero-beta asset uniquely determines the market portfolio on the MVEF. We then give a simplified Black CAPM that has a mean zero-beta return of zero, and a mean market return of \( \frac{A}{B} \), where \( A \) and \( B \) are extracted from the underlying covariance matrix. The betas of assets can become negative, and have a minimum value of \( -\frac{A}{B} \). The Security Market Line has zero intercept, and all portfolios have cost of capital dependent only on the MVEP with mean \( \frac{B}{A} \).

2. The Black mean–variance frontier

In this section, we briefly review the construction and properties of the Black [1] mean–variance frontier. Recall that there is no risk-free asset in the Black framework. Rather, Black posits that the return on the zero-beta asset is uncorrelated with the return of the efficient market portfolio.

Consider \( n \) risky assets, with random returns \( R_1, R_2, \ldots, R_n \) and return vector \( \bar{R} = (R_1, R_2, \ldots, R_n)' \), and mean return vector \( \mu = E[R] = (E[R_1], E[R_2], \ldots, E[R_n])' \). Black [1] assumes that no linear combination of these risky assets have zero variance. That is, the covariance matrix of returns, denoted \( \Omega = \text{Cov}(\bar{R}, \bar{R}) \), is non-singular. Let \( R_p \) be the return on a portfolio built from these \( n \) assets, with weight vector \( \bar{w} = (w_1, w_2, \ldots, w_n)' \). Since there are no risk-free assets, the portfolio \( p \) is fully invested in the assets 1, 2, \ldots, \( n \), with \( R_p = \bar{w}R_\bar{R}, \mu_p = \bar{w}\mu, w_\bar{I}' = 1 \) and \( I \) is a column of ones. The mean–variance frontier is the locus of \( (\sigma_p, \mu_p) \), where

\[
\sigma_p^2 = \min_w \bar{w}'\Omega\bar{w},
\]

subject to \( \mu_p = \bar{w}\mu', w_\bar{I}' = 1 \).

Thus, for a given mean return \( \mu_p \), the minimum variance of the return on the portfolio is \( \sigma_p^2 \). One can easily
show that the weight vector $w_p$ is a linear function of the expected return $\mu_p$ of the portfolio (see [2]). That is,

$$w_p = g + h \mu_p,$$

(4)

where the constant vectors $g$ and $h$ are built from the (inverse) covariance matrix $\Omega$, the mean return vector $\mu$ and $I$ as follows:

$$g = \frac{1}{D}[B(\Omega^{-1}I) - A(\Omega^{-1}\mu)],$$

(5)

$$h = \frac{1}{D}[C(\Omega^{-1}\mu) - A(\Omega^{-1}I)].$$

(6)

The parameters $A$, $B$, $C$ and $D$ are constants constructed from the inverse covariance matrix of the underlying base portfolio from which all portfolios are constructed. Explicitly,

$$A = \mu^T \Omega^{-1}I,$$

$$B = \mu^T \Omega^{-1}\mu,$$

$$C = I^T \Omega^{-1}I,$$

$$D = BC - A^2.$$  

(7)

We now give an explicit formula for the minimum variance conditioned on the expected mean return on the portfolio. The reader is directed to [2] for proofs of the following proposition and its corollary.

**Proposition 1.** The mean–variance frontier is the locus of $(\sigma_p, \mu_p)$, where

$$\sigma_p^2 = \text{Var} R_p = \frac{C}{D} \left( \frac{\mu_p - A}{C} \right)^2 + \frac{1}{C}.$$  

Moreover, if $R_p$ and $R_q$ are mean–variance portfolios, then

$$\text{Cov}(R_p, R_q) = \frac{C}{D} \left( \frac{\mu_p - A}{C} \right) \left( \frac{\mu_q - A}{C} \right) + \frac{1}{C}.$$  

(8)

2.1. The global minimum mean–variance portfolio (GMVP)

It follows immediately from Proposition 1 that the variance, $\sigma_p^2$, has a global minimum of $\frac{1}{C}$ when $\mu_p = \frac{A}{C}$. The unique portfolio with this mean and variance is called the Global Minimum Mean–Variance Portfolio (GMVP). We reserve the subscript "g" to reference GMVP. Figure 1 plots the mean–variance frontier with the corresponding mean return for each zero-beta portfolio. The graph emphasizes that as the market portfolio approaches the global minimum variance portfolio, the mean return of the zero-beta portfolio approaches negative infinity.

![Mean variance frontier with market/tangency portfolios and their corresponding zero–beta portfolios](image)

**Fig. 1.** $\mu_p$ versus $\sigma_p$ is the mean–variance frontier with the corresponding mean return for each zero-beta portfolios. The graph emphasizes that as the market portfolio approaches the global minimum variance portfolio, the mean of the zero-beta portfolio approaches negative infinity. The graph is based on the 6 Fama-French Size and Book-to-Market portfolios. We collect data for 60 months from July 2002 to June 2007 from the Center for Research in Security Prices (CRSP).
Corollary 1. The Global Minimum Portfolio $g$, with return $R_g$, has mean, variance and weight vector given by
\[
\mu_g = \frac{A}{C},
\]
\[
\sigma_g^2 = \text{Var} \ R_g = \frac{1}{C},
\]
\[
w_g = \frac{1}{C} \Omega^{-1} = \frac{1}{C} \Omega^{-1} \sigma_g^2.
\]
Moreover, the covariance between $R_g$ and the return on any other mean–variance portfolio (MVP) $R_p$, is equal to the global minimum variance, $\frac{1}{C}$.

3. The zero-beta portfolio

From Proposition 1, one can obtain the mean return of a given MVP in terms of the mean return of any other MVP provided that the covariance between them is known. We offer these results in this section.

Theorem 1. Let $R_q$ and $R_p$ be returns of MVPs, with expected returns, $\mu_q$ and $\mu_p$, respectively and covariance $\gamma_{pq}$. Then provided $\mu_p \neq \frac{A}{C}$,
\[
\mu_q = \frac{A}{C} + \frac{(D/C)(\gamma_{pq} - 1/C)}{\mu_p - A/C}.
\]
Moreover, $\gamma_{pq} = \frac{1}{C}$ if and only if $q$ is the GMVP.

Proof. Rewriting Eq. (8) in terms of $\gamma_q$ yields, $\gamma_q = \frac{C}{C_D}(\mu_p - \frac{A}{C})(\mu_q - \frac{A}{C})$, whence $\mu_q - \frac{A}{C} = \frac{\mu_p - A/C}{\mu_p - A/C}$, which gives the result.

If $\gamma_{pq} = \frac{1}{C}$, then $\mu_q = \frac{A}{C}$, which is the mean return of the GMVP. Conversely, if $q$ is the GMVP, then from Corollary 1, $\mu_q = \frac{A}{C}$, whence $\gamma_{pq} - \frac{1}{C} = 0$. □

Equation (9) indicates that, given the mean return of an MVP, we can find the mean return of any other MVP provided we know how both covary. Since covariance is a proxy for correlation, we can find the mean of any other MVP, given the mean and correlation coefficient, $\rho$, of some fixed MVP. We explicitly give this relationship in the following corollary.

Corollary 2. Let $R_q$ and $R_p$ be returns of MVPs, with expected returns and volatilities, $\mu_q, \mu_p$ and $\sigma_p, \sigma_q$, respectively, and correlation coefficient $\rho_{pq}$. Then, provided $\mu_p \neq \frac{A}{C}$,
\[
\mu_q = \frac{A}{C} + \frac{(D/C)(\rho_{pq} \sigma_p \sigma_q - 1/C)}{\mu_p - A/C}.
\]
Moreover, $\rho_{pq} = \frac{\sigma_p \sigma_q}{\sigma_q}$ if and only if $q$ is the GMVP.

This representation is a non-linear equation in $\mu_q$ since $\sigma_q^2$ can be written as a quadratic function thereof. It is fairly complicated since it requires more inputs that just correlation; e.g., the variances of both portfolios are also required, in addition to the standard input $\mu_p$, the mean market return. We now provide another simple consequence of Proposition 1.

Corollary 3. Let $R_q$ and $R_p$ be returns of MVPs, with expected returns and volatilities $\mu_q, \mu_p$ and $\sigma_p, \sigma_q$, respectively, and correlation coefficient $\rho_{pq}$. Then provided $\mu_p \neq \frac{A}{C}$, the portfolios are always positively correlated, with
\[
\rho_{pq} \geq \frac{1}{\sigma_p \sigma_q C} = \frac{\sigma_g^2}{\sigma_p \sigma_q}.
\]

There is a lower bound on $C$, given by $C \geq \frac{1}{\sigma_p \sigma_q}$.

Proof. Equation (11) follows from (10) by setting $\gamma_{pq} = \rho_{pq} \sigma_p \sigma_q$. Since $\rho_{pq} \leq 1$, and the portfolios are MVPs, then $\mu_p > \frac{A}{C}$. Thus $\rho_{pq} \sigma_p \sigma_q - \frac{1}{C} > 0$, whence $\rho_{pq} \geq \frac{1}{\sigma_p \sigma_q C}$. The other representation follows from the fact that $\sigma_g^2 = \frac{1}{C}$. Since $\rho_{pq} \leq 1$, then $1 \geq \frac{1}{\sigma_p \sigma_q C}$, from which the result follows. Alternatively, the minimum variance portfolio has variance $\frac{1}{\sigma_g^2} = \sigma_g^2 = \sigma_g \sigma_q \leq \sigma_p \sigma_q$, for any MVPs, $p$ and $q$. □

Thus, correlation between MVPs is bounded below by the inverse of the product of $C$ and the volatilities, $\sigma_p$ and $\sigma_q$.

We now introduce the zero-beta portfolio $zp$, for each mean–variance market portfolio $p$. This is the MVP with return $R_{zp}$, which has minimum variance and zero covariance with the MVP portfolio $p$ having return $R_p$. Thus $\gamma_{zp,p} = 0$. Unlike the risk-free asset, the zero-beta asset has mean and variance that are tied to the mean and variance of the market return $R_p$. This portfolio is unique. We give these measures in the following theorem.

Theorem 2. Let \( R_p \) be the return of a MVP, with mean \( \mu_p \) and volatility \( \sigma_p \). Provided \( p \) is not the global minimum variance portfolio GMVP, there exists a unique zero-beta portfolio \( z_p \), with return \( R_{zp} \), that is also MVP, with mean \( \mu_{zp} \) and volatility \( \sigma_{zp} \) given by

\[
\begin{align*}
\mu_{zp} &= \frac{A}{C} - \frac{D/C^2}{\mu_p - A/C}, \\
\sigma_{zp}^2 &= \frac{1}{C^2} \frac{1}{\sigma_p^2} - \frac{1}{1} + \frac{1}{C}.
\end{align*}
\]

(12)

Remark 1. (i) The mean of the zero-beta asset is a constant minus a positive component that is inversely related to the excess of the mean return of \( p \) and the global minimum variance portfolio, \( g \).

(ii) The variance of the return on \( z_p \) is a constant plus a positive component that is inversely related to the excess variance of the market portfolio over the GMVP.

Proof. Equation (12) comes directly from setting \( \gamma_{zp,p} = 0 \) in Theorem 1. Therefore \( \mu_{zp} = \frac{A}{C} = \mu_{zp} - \mu_g \) and \( \sigma_{zp}^2 = -\frac{D/C^2}{\mu_p - A/C} = -\frac{D/C^2}{\mu_p - \mu_g} \). Since \( R_{zp} \) is MVP, then from Proposition 1, it follows that \( \text{Var} R_{zp} = \sigma_{zp}^2 = \text{Cov}(R_{zp}, R_p) = \frac{C}{D}(\mu_{zp} - \mu_g)^2 + \frac{1}{C} = \frac{C}{D}(\mu_{zp} - \mu_g)^2 + \frac{1}{C} = \frac{C}{D} \frac{1}{(\mu_p - \mu_g)^2} + \frac{1}{C} \). However, \( \sigma_p^2 = \frac{C}{D}(\mu_p - \mu_g)^2 + \frac{1}{C} \). Thus \( \sigma_{zp}^2 = \frac{1}{C} \sigma_p^2 = \sigma_g^2 + \frac{1}{C} = \sigma_g^2 + \frac{1}{C} \). Therefore, \( \sigma_{zp}^2 = \frac{D}{C^2} \frac{1}{(\mu_p - \mu_g)^2} + \frac{1}{C} = \frac{1}{C^2} \sigma_p^2 - \sigma_g^2 + \frac{1}{C} = \frac{1}{C^2} \sigma_p^2 - \sigma_g^2 + 1 = \frac{\sigma_g^2}{\sigma_p^2 - \sigma_g^2} + 1 = \frac{\sigma_p^2 \sigma_g^2}{(\sigma_p^2 - \sigma_g^2)}.
\]

Observe from the proof that the excess volatility of the zero-beta portfolio over the global minimum variance portfolio is inversely proportional to the excess volatility of the market portfolio over the global minimum variance portfolio. That is, excess variances and hence, volatilities covary negatively.

\[
\sigma_{zp}^2 - \sigma_g^2 = \frac{1}{C^2} \left( \frac{1}{\sigma_p^2} - \frac{1}{\sigma_g^2} \right).
\]

The mean returns and variances of the GMVP and zero-beta portfolio are linked as follows.

Corollary 4. Let \( \mu_g = \frac{A}{C} \) be the mean and \( \text{Var} R_g = \frac{1}{C} \) be the variance of the GMVP. Then provided \( \mu_p \neq \mu_g \), the mean and variance of the unique zero-beta portfolio \( z_p \), of the market portfolio \( p \), with return \( R_p \), are given by:

\[
\begin{align*}
\mu_{zp} &= \mu_g - \frac{D/C^2}{\mu_p - \mu_g}, \\
\text{Var} R_p &= \frac{D}{C} (\mu_p - \mu_g)^2 + \text{Var} R_g, \\
\text{Var} R_{zp} &= \frac{1}{C^2} \left( \frac{1}{C} \right) \text{Var} R_p - \text{Var} R_g + \text{Var} R_g \\
&= \frac{\sigma_p^2 \sigma_g^2}{\sigma_p^2 - \sigma_g^2}.
\end{align*}
\]

(13)

Proof. The results follow when we set \( \mu_g = \frac{A}{C} \). \( \text{Var} R_g = \frac{1}{C} = \sigma_g^2 \) and \( \text{Var} R_p = \sigma_p^2 \) in Theorem 2 and Proposition 1. In addition,

\[
\sigma_{zp}^2 = \frac{1}{C^2} \left( \frac{1}{\sigma_p^2} - \sigma_g^2 \right) + \frac{1}{C} = \frac{\sigma_g^2}{\sigma_p^2 - \sigma_g^2} + 1 = \frac{\sigma_p^2 \sigma_g^2}{\sigma_p^2 - \sigma_g^2}.
\]

Figure 2 shows that the mean zero-beta return \( \mu_{zp} \), is a strictly increasing function of the mean market return \( \mu_p \), with \( \mu_{zp} \) being strictly negative if the mean market return is less than \( \frac{A}{C} = 0.016353 \). In addition, the mean zero-beta return \( \mu_{zp} \), is a strictly decreasing function of \( \mu_g \), which is the mean global minimum variance portfolio as shown in Fig. 3. It approaches negative infinity as \( \mu_g \) approaches the vertical asymptote, \( \frac{A}{C} = 0.016353 \). Furthermore, as shown in Fig. 4, the volatility of the zero-beta portfolio, \( \sigma_{zp} \), is a strictly decreasing function of \( \sigma_p \), the market return volatility. It becomes infinitely large when \( \sigma_p \) approaches the volatility \( \sigma_g \), of the global minimum variance portfolio, and is bounded below by it.

4. Minimum risk premia

This section shows that the expected excess return of the market portfolio \( p \), over the return of the zero-beta asset \( z_p \), has a positive minimum of \( \frac{\mu_g}{\sqrt{D}} \). This is a surprising result, since in the Sharpe–Lintner CAPM, this can be negative. We formalize this result in the following theorem.

Theorem 3. Let \( p \) be the market portfolio and \( z_p \) its zero-beta portfolio, with mean returns \( \mu_p \) and \( \mu_{zp} \), respectively. Let \( g \) be the GMVP, with mean return \( \mu_g \),
Fig. 2. The mean zero-beta return $\mu_{zp}$ is a strictly increasing function of the mean market return $\mu_p$, with $\mu_{zp}$ being strictly negative if the mean market return is less than $\frac{B}{A} = 0.016353$. The graph is based on the 6 Fama-French Size and Book-to-Market portfolios. We collect data for 60 months from July 2002 to June 2007 from the Center for Research in Security Prices (CRSP).

Fig. 3. The mean zero-beta return $\mu_{zp}$ is a strictly decreasing function of $\mu_g$, the mean global minimum variance portfolio. It becomes infinitely negatively large as $\mu_g$ approach the vertical asymptote, $\frac{B}{A} = 0.016353$. The graph is based on the 6 Fama-French Size and Book-to-Market portfolios. We collect data for 60 months from July 2002 to June 2007 from the Center for Research in Security Prices (CRSP).

Then the excess expected return of the market over the zero-beta portfolio is at least $\frac{2\sqrt{D}}{C}$; that is

$$\mu_p - \mu_{zp} \geq \frac{2\sqrt{D}}{C}.$$  

In particular, the minimum (equality) is achieved when

$$\mu_p = \mu_g + \frac{\sqrt{D}}{C} = \frac{A + \sqrt{D}}{C},$$  

where $A$, $C$ and $D$ are defined by (7).
Fig. 4. The volatility of the zero-beta portfolio $\sigma_{zp}$ is a strictly decreasing function of $\sigma_p$, the market return volatility. It becomes infinitely large when $\sigma_p$ approaches the volatility $\sigma_g$ of the global minimum variance portfolio, and is bounded below by it. The graph is based on the 6 Fama-French Size and Book-to-Market portfolios. We collect data for 60 months from July 2002 to June 2007 from the Center for Research in Security Prices (CRSP).

**Proof.** From Corollary 4, $\mu_{zp} = \mu_g - \frac{D/C^2}{\mu_p - \mu_g}$, whence

$$y \equiv \mu_p - \mu_{zp} = \mu_p - \mu_g + \frac{D/C^2}{\mu_p - \mu_g} = x + \frac{D/C^2}{x} \equiv f(x),$$

where $x \equiv \mu_p - \mu_g$. The market risk premium $y$ has a turning point at $f'(x) = 1 - \frac{D/C^2}{x^2} = 0$. Thus $x = \sqrt{\frac{D}{C}}$. At this value of $x$, $f''(x) = \frac{2D/C^2}{x^3} > 0$, which confirms a minimum value of $y$. Thus

$$\mu_p - \mu_{zp} \geq f\left(\frac{\sqrt{D}}{C}\right) = \frac{\sqrt{D}}{C} + \frac{D/C^2}{\sqrt{D}/C} = 2\sqrt{\frac{D}{C}},$$

at $x = \mu_p - \mu_g = \frac{\sqrt{D}}{C}$. Since $\mu_g = \frac{A}{C}$, then $\mu_p = \mu_g + \frac{\sqrt{D}}{C} = \frac{A + \sqrt{D}}{C}$. $\square$

An immediate consequence of Theorem 3 is that the expected return on any arbitrary portfolio $i$, also has a guaranteed minimum that is a function of its beta. We formalize this in the following corollary.

**Corollary 5.** Let $p$ be the market portfolio and $zp$ its zero-beta portfolio, with mean returns $\mu_p$ and $\mu_{zp}$, respectively. Let $g$ be the GMVP, with mean return $\mu_g$ and let $i$ be any arbitrary portfolio with mean $\mu_i$ and beta $\beta_{ip}$.

(i) If $\beta_{ip} \geq 0$, then the excess expected return of the portfolio over the zero-beta portfolio is at least $\beta_{ip} 2\sqrt{\frac{D}{C}}$; that is

$$\mu_i - \mu_{zp} \geq \beta_{ip} \frac{2\sqrt{D}}{C}.$$  

In particular, the minimum expected return on portfolio $i$, is

$$\min \mu_i = \mu_{zp} + \beta_{ip} \frac{2\sqrt{D}}{C} = \frac{A}{C} + \frac{\sqrt{D}}{C} (2\beta_{ip} - 1).$$

(ii) If $\beta_{ip} < 0$, then the excess expected return of the portfolio over the zero-beta portfolio is at most $\beta_{ip} 2\sqrt{\frac{D}{C}}$; that is

$$\mu_i - \mu_{zp} \leq \beta_{ip} \frac{2\sqrt{D}}{C}. $$
In particular, the maximum expected return on portfolio \( i \), is

\[
\begin{align*}
\max \mu_i &= \mu_{zp} + \beta_{ip} \frac{2\sqrt{D}}{C} \\
&= A + \frac{\sqrt{D}}{C}(2\beta_{ip} - 1).
\end{align*}
\]

For both cases, optimality is achieved when \( \mu_{zp} = \mu_g - \frac{\sqrt{D}}{C} \), or equivalently, when \( \mu_p = \mu_g + \frac{\sqrt{D}}{C} \), where \( A, C \) and \( D \) are positive numbers defined by (7).

**Proof.** By the Black CAPM formula (1), and Theorem 3, \( \min(\mu_i - \mu_{zp}) = \min \mathbb{E}[R_i - R_{zp}] = \beta_{ip} \mathbb{E}[R_p - R_{zp}] = \beta_{ip} \frac{2\sqrt{D}}{C} \), which is achieved when \( \mu_p = \mu_g + \frac{\sqrt{D}}{C} = \frac{A+\sqrt{D}}{C} \). But from Corollary 4, \( \mu_{zp} = \mu_g - \frac{D/C^2}{\mu_p-\mu_g} = \frac{A}{C} - \frac{\sqrt{D}}{C} = \frac{A-\sqrt{D}}{2\sqrt{D}} \), which gives the result. Since \( \mu_i \geq -1 \), then its minimum value must be greater than \(-1\). Therefore, \( \frac{A}{C} + \frac{\sqrt{D}}{C}(2\beta_{ip} - 1) \geq -1 \). Re-arranging yields, \( \beta_{ip} \geq \frac{\sqrt{D}}{2\sqrt{D}C^2} \), Part (ii) follows similarly using the fact that for any function \( f(x) \), \( \max f(x) = -\min(-f(x)) \). \( \Box \)

Note that the minimum return on a portfolio \( i \), is positive only if the mean zero-beta return is greater than the negative of the product of its beta and the minimum market risk premium. Likewise, the maximum return is negative if the betas are less than or equal to \( \frac{\sqrt{D}}{2\sqrt{D}C^2} \), which is a function of the underlying covariance matrix. Let us now examine the trade-off between the expected return on the zero-beta asset, and the expected returns on the GMVP \( g \), and the market portfolio, \( p \).

### 4.1. Sensitivity of \( \mu_{zp} \) to \( \mu_g \)

Differentiating \( \mu_{zp} = \mu_g - \frac{D/C^2}{\mu_p-\mu_g} \) partially with respect to \( \mu_g \), yields

\[
\frac{\partial \mu_{zp}}{\partial \mu_g} = 1 - \frac{D}{C^2(\mu_p - \mu_g)^2}.
\]

Clearly, \( \frac{\partial \mu_{zp}}{\partial \mu_g} \geq 0 \) if only if \( \mu_p - \mu_g \geq \frac{\sqrt{D}}{C} \). It is zero at \( \mu_p - \mu_g = \frac{\sqrt{D}}{C} \), which, from Theorem 3, is the point where the minimum expected return of the market over the zero-beta portfolio is attained. Thus, for fixed \( \mu_p \), the mean zero-beta return increases with the mean global minimum return when \( \mu_g < \mu_p - \frac{\sqrt{D}}{C} \) and decreases with the mean global minimum return when \( \mu_g > \mu_p - \frac{\sqrt{D}}{C} \).

#### 4.2. Sensitivity of \( \mu_{zp} \) to \( \mu_p \)

Fixing \( \mu_{zp} \) and differentiating \( \mu_{zp} = \mu_g - \frac{D/C^2}{\mu_p-\mu_g} \) partially with respect to \( \mu_p \), yield

\[
\frac{\partial \mu_{zp}}{\partial \mu_p} = \frac{D}{C^2(\mu_p - \mu_g)^2} > 0,
\]

which confirms that the mean zero-beta return increases monotonically with the mean market return. We now give the limiting behaviour of the mean and variance of the zero-beta asset \( zp \), when the underlying portfolio \( p \) gets close to the global minimum asset, \( g \).

**Theorem 4.** Let \( g \) be the GMVP. For any choice of the market portfolio \( p \) on the MV efficient frontier, \( \mu_{zp} \) is monotonic increasing in \( \mu_p \). In particular:

(i) \( \mu_{zp} < \mu_g \) and \( \sigma_{zp}^2 \geq \sigma_g^2 \);

(ii) \( \mu_{zp} \uparrow \mu_g \) and \( \sigma_{zp}^2 \downarrow \sigma_g^2 \) as \( \mu_p \to \infty \);

(iii) \( \mu_{zp} \downarrow -\infty \) and \( \sigma_{zp}^2 \uparrow \infty \) as \( \mu_p \to \mu_g \) that is as \( p \to q \).

**Proof.** (i) On the mean–variance efficient frontier, \( \mu_p \geq \mu_g \), whence from Eq. (13) of Corollary 4, \( \mu_{zp} = \mu_g - \frac{D/C^2}{\mu_p-\mu_g} < \mu_g \), since \( D \) is always positive.

(ii) Since \( \frac{\partial \mu_{zp}}{\partial \mu_p} \) is always positive then \( \mu_{zp} \) is an strictly increasing function of \( \mu_p \). Thus \( \mu_{zp} \) increases monotonically to \( \mu_g \). Similarly, from Corollary 4, if \( \mu_p \) explodes then \( \mu_{zp} \) converges to the GMVP, whence \( \sigma_{zp}^2 \uparrow \sigma_g^2 \).

(iii) If \( \mu_p \to \mu_g \), then by Eq. (13) of Corollary 4, \( \mu_{zp} = \mu_g - \frac{D/C^2}{\mu_p-\mu_g} \downarrow -\infty \). Since \( \sigma_{zp}^2 \downarrow \sigma_g^2 \) then by the variance formula \( \sigma_{zp}^2 = \frac{\sigma_g^2}{\sigma_p^2 - \sigma_g^2} + 1 \uparrow \infty \). \( \Box \)
Fig. 5. The excess mean market return over the zero-beta portfolio is a strictly positive function of the excess mean market return over the global minimum variance portfolio. It decreases strictly from positive infinity to a positive minimum of $2\sqrt{DC} = 0.015678$ when the mean excess market return over the GMVP is $\sqrt{DC} = 0.0078$. It then increases strictly at a positive rate to infinity. The graph is based on the 6 Fama-French Size and Book-to-Market portfolios. We collect data for 60 months from July 2002 to June 2007 from the Center for Research in Security Prices (CRSP).

Figure 5 shows that the excess mean market return over the return of the zero-beta portfolio is a strictly positive function of the excess mean market return over the global minimum variance portfolio. It decreases strictly from positive infinity to a positive minimum of $2\sqrt{DC} = 0.015678$ when the mean excess market return over the GMVP is $\sqrt{DC} = 0.0078$. It then increases strictly at a positive rate to infinity. In addition, Fig. 6 shows that $\mu_{zp}$ is a strictly increasing function of $\mu_p$, which increases at a decreasing rate as $\mu_p$, the mean market return, increases. $\mu_{zp}$ is also a strictly decreasing function of $\mu_g$, the mean global minimum variance portfolio, but does so at an increasing rate as $\mu_g$ increases.

5. Instability of Black CAPM near the GMVP

In this section, we prove that the Black CAPM fails to give stable cost of capital whenever the underlying efficient market portfolio is close to the global minimum portfolio. Assume that $\mu_{zp} \neq 0$ and the returns on market $p$ and an arbitrary asset $i$, are finite and fixed.

From Eq. (2), the beta of the CAPM equation is given by

$$\beta_{ip} = \frac{\text{Cov}(R_i, R_p)}{\text{Var}(R_p)} = \frac{\mathbf{E}[R_i - R_{zp}]}{\mathbf{E}[R_p - R_{zp}]}$$

$$= \frac{\mu_i - \mu_{zp}}{\mu_p - \mu_{zp}} = \frac{\mu_i/\mu_{zp} - 1}{\mu_p/\mu_{zp} - 1}. \quad (14)$$

Now let the market portfolio $p$ converge to the global minimum portfolio $g$. Then from Theorem 4, $\mu_{zp} \downarrow -\infty$ which forces the ratios $\frac{\mu_i}{\mu_{zp}}$ and $\frac{\mu_p}{\mu_{zp}}$ to zero, and hence, by Eq. (14), the asset’s beta $\beta_{ip}$ goes to one! In this situation, an infinite number of expected returns satisfy Eq. (14); that is, $\mu_i \in \{x : x < -\infty\}$ and the return on the asset becomes arbitrary! We subsequently show that given the mean–variance (MV) frontier, a zero-beta asset $z$ of finite expected return, $\mu_z$, uniquely determines the expected market portfolio return, $\mu_{pz}$, and hence the market portfolio, $pz$. Using a similar argument, we see that if $z$, which lives on the lower half of the MV frontier, gets close to the global minimum portfolio $g$, then $\mu_{pz} \uparrow \infty$ and the ratios $\frac{\mu_i}{\mu_{pz}}$
and $\frac{\mu_z}{\mu_{pz}}$ go to zero, whence from Eq. (14),

$$\beta_{i,pz} = \frac{\mu_i/\mu_{pz} - \mu_z/\mu_{pz}}{1 - \mu_z/\mu_{pz}} \rightarrow 0.$$  

In this case, asset $i$ degenerates to an asset that has mean return equal to that of the GMVP. That is, $\mu_i = \mu_g$ for any arbitrary asset $i$. Thus, the model now considers all asset to be the GMVP, a clearly unreasonable and impractical state of affair. That is, the model can no longer distinguish between assets! In fact, this situation implies that even the MV portfolios have the same mean, and hence the same variance, thereby forcing the MV frontier to be a single point at the GMVP. We encapsulate these findings in the following theorem.

Theorem 5. (a) Let $g$ the GMVP, and $p$ any MV portfolio. Let $i$ be any fixed portfolio. If $p \downarrow g$, then $\beta_{i,p} \rightarrow 1$ and $\mu_i$ is any value in the set $\{ x : x < \infty \}$.

(b) Let $i$ be any fixed arbitrary portfolio. Given the mean–variance (MV) frontier, if $z$ is any zero-beta portfolio, with finite mean $\mu_z$, and matching MV efficient portfolio $pz$, then $\beta_{i,pz} \rightarrow 0$ and $\mu_i = \mu_g$ as $z \rightarrow g$. That is, all arbitrary portfolios have the same expected mean return as the mean of the zero-beta portfolio approaches the mean of the GMVP.

Remark. In the sequel we simply refer to $pz$ as $p$ and $z$ as $zp$, since they are obviously the same objects.

6. A zero-beta asset determines a unique market portfolio

Given the MV frontier and a zero-beta asset $z$, of finite expected return $\mu_z$, we now show that this portfolio or rather mean, uniquely determines the (mean of) MV efficient portfolio $pz$.

Proposition 2. Let $z$ be any arbitrary zero-beta asset with finite mean $\mu_z$. Then there exists an MV efficient portfolio $pz$, with mean $\mu_{pz}$, given by

$$\mu_{pz} = \mu_g - \frac{D/C^2}{\mu_z - \mu_g}. \quad (15)$$

Proof. Let $\mu_z$ be the given zero-beta portfolio. Construct or choose a portfolio $p^t$, with mean return $\mu_{p^t}$, on the MV efficient frontier, such that its unique zero-beta portfolio has mean $\mu_{zp^t} = \mu_z$. Then from Theorem 2, $\mu_z = \mu_g - \frac{D/C^2}{\mu_{p^t} - \mu_g}$. Solving for $\mu_{p^t}$ yields $\mu_{p^t} = \mu_g - \frac{D/C^2}{\mu_z - \mu_g}$. Taking $\mu_{pz} = \mu_{p^t}$ yields the desired result. \hfill $\square$

Replacing $pz$ by $p$ and $z$ by $zp$, we can now rewrite Eq. (15) in the usual notation.

$$\mu_p = \mu_{g} - \frac{D/C^2}{\mu_{zp} - \mu_{g}}. \quad (16)$$
with \( \mu_p \uparrow \) as \( \mu_{zp} \uparrow \). Equation (16) is striking! We can now increased our degree of freedom in the use of Black CAPM, in the sense that we can now choose returns of mean–variance efficient portfolios (MVEP), given any zero-beta expected return that is less than the mean return of the GMVP. That is, we can start with a zero-beta mean return and work backwards to find the mean market return via Eq. (16). This leads to considerable simplification of the Black CAPM, in the sense that we can now choose a mean-zero-beta return that is actually zero! We explore this concept with the help of the following proposition.

**Proposition 3.** Let \( \mu_{zp} \) be the mean of the zero-beta asset corresponding to the MVEP \( p \), with mean \( \mu_p \). Then \( \mu_p = \frac{B}{A} \) if \( \mu_{zp} = 0 \), with variance \( \sigma_p^2 = \frac{B}{A} \), and \( \mu_p > \frac{B}{A} \) iff \( \mu_{zp} > 0 \).

**Proof.** Recall that if \( \mu_{zp} = 0 \) then \( \mu_p = \mu_g - \frac{D/C}{AC} = \frac{A^2 - D}{AC} = -\frac{B/C}{AC} = \frac{B}{A} \). If \( \mu_p > \frac{B}{A} \) then from Eq. (16) \( \mu_p - \mu_g = -\frac{D/C}{AC} > \frac{B}{A} - \frac{A}{C} = \frac{BC - A^2}{AC} = \frac{D}{AC} \). Thus \( \frac{1/C}{\mu_g - \mu_p} > \frac{1}{A} \), whence \( (\mu_{zp} - \mu_g) < \frac{A}{C} \), which yields, \( \mu_{zp} > 0 \). The converse follows similarly!

From Proposition 1, if \( \mu_p = \frac{B}{A} \), then \( \sigma_p^2 = \frac{A}{C} \left( \frac{B}{A} - \frac{A}{C} \right)^2 + \frac{1}{C} \) = \( \frac{D}{A^2C} \left( \frac{BC - A^2}{AC} \right)^2 + \frac{1}{C} = \frac{D^2}{AC^2} + \frac{1}{C} = \frac{D}{AC} \). \( \Box \)

**Remark 2.** It follows immediately from the last result and Theorem 3, that \( A, B, C, D \) must satisfy the condition, \( \frac{B}{A} \geq 2\sqrt{\frac{A}{C}} \). This can also be proven directly by algebraic manipulation of \( (BC - A^2)^2 \geq 0 \).

Since we are now free to choose the mean return of the zero-beta asset, this leads to a zero-intercept Black CAPM.

**Theorem 6.** Let \( R_i \) be the return on any asset with finite mean \( \mu_i \). There exists \( \beta_i \) such that

\[
\mu_i = \beta_i A \frac{B}{C},
\]

where \( \beta_{i,AB} = \frac{A^2 \text{Cov}(R_i, R_{A,B})}{B} \) is greater than or equal to \( -\frac{A}{C} \) and \( R_{A,B} \) is the zero-market MVEP with mean return \( \frac{B}{A} \) and weight vector \( w_{A,B} = h + g \frac{B}{A} \).

**Proof.** Choosing \( \mu_{zp} = 0 \), yields \( \mu_p = \frac{B}{A} \). Substituting these values in Black CAPM yields, \( \mu_i = (1 - \beta_{ip}) \mu_{zp} + \beta_{ip}\mu_p = \beta_{ip}\mu_p = \beta_{i,AB} A \frac{B}{C} \). Since \( \mu_i \geq -1 \) then \( \beta_{ip} B \frac{A}{C} \geq -1 \). Therefore, \( \beta_{i,AB} \geq -\frac{A}{B} \).

The weight vector for the zero-market MVEP \( R_{A,B} \) follows from Eqs (4)–(7). \( \Box \)

Thus, the expected return on any asset is positive or negative if it covaries positively or negatively with the zero-market MVEP \( R_{A,B} \), which has mean return \( \frac{B}{A} \). Moreover, unlike the non-negative betas in the Sharpe–Lintner CAPM model, the betas in the Black CAPM model can be negative, with a lower bound of \( -\frac{A}{B} \).

**7. Example**

We use the six Fama-French Size and Book-to-Market portfolios to generate the mean–variance frontier. We collect data for 60 months from July 2002 to June 2007. We obtain

\[
\begin{align*}
A &= 16.29556, & B &= 0.266481, \\
C &= 1551.746, & D &= 147.9653.
\end{align*}
\]

The mean–variance frontier is generated by the equation:

\[
\sigma^2 = \frac{C}{D}(\mu - \mu_g)^2 + \frac{1}{C} = 10.48723(\mu - 0.010501)^2 + 0.000644.
\]

In Fig. 7, we plot a scaled version of the mean–variance frontier of the 6 Fama-French size and book-to-market portfolios.

The zero-beta portfolio has mean return \( \mu_{zp} = 0 \), with variance \( \sigma_{zp}^2 = 0.00179758 \).

The zero-market MVEP has mean return \( \mu_p = \frac{B}{A} = 0.016353 \), variance \( \sigma_p^2 = 0.00100352 \) and volatility \( \sigma_p = 0.03167838 \).

The betas for the Black CAPM model must satisfy the condition \( \beta_i \geq \frac{A}{B} = -61.151 \).

The GMVP has mean return \( \mu_g = 0.010501 \) and variance \( \sigma_g^2 = \frac{1}{C} = 0.000644 \).

In Fig. 8, we plot the Security Market Line of the Black CAPM model. This is given as

\[
\mu_i = 0.016353\beta_i, \quad \text{where } \beta_i \geq -61.151.
\]
Fig. 7. This is a scaled version of the mean–variance frontier of the 6 Fama-French size and Book-to-Market portfolios. The data are obtained from the Center for Research in Security Prices (CRSP). We collect data for the 60 months from July 2002 to June 2007.

Fig. 8. The security Market Line $\mu = 0.016353\beta$ for $\beta$ in the range $-61$ to $100$. The graph is based on the 6 Fama-French Size and Book-to-Market portfolios. We collect data for 60 months from July 2002 to June 2007 from the Center for Research in Security Prices (CRSP).

Thus, the expected return of assets increases by 16.353 basis points for each positive unit change in beta. In addition, betas can take negative values greater than $-61.151$.

The Black CAPM model breaks down in the general case if $\mu_p \downarrow 0.010501$.

The mean-zero-beta return for each corresponding expected return of a MVEP is

$$\mu_{zp} = \mu_g - \frac{D}{C^2} \frac{1}{\mu_p - \mu_g} \mu_p - 0.010501 = 0.010501 - \frac{6.14495}{\mu_p - 0.010501}.$$
with variance
\[
\sigma_p^2 = \frac{\sigma_p^2 \sigma_g^2}{\sigma_p^2 - \sigma_g^2} = \frac{0.000644 \sigma_p^2}{\sigma_p^2 - 0.000644}.
\]

The minimum market risk premium is
\[
\min_{\mu \in MVEP} [\mu_p - \mu_g] = 2 \sqrt{D} = 0.015678.
\]

8. Conclusion

We show that the Black CAPM [1] is extremely sensitive to the choice of the "market" portfolio and becomes unstable as portfolios approach the Global Minimum (GMVP) portfolio on the mean–variance efficient frontier. Unlike the Sharpe [7] and Lintner [4] versions of CAPM with a risk free (zero variance) asset, the unique zero beta asset of Black CAPM has variance that may become extremely large or infinite. As the market portfolio approaches the GMVP, the expected return on the zero beta asset approaches negative infinity while its variance approaches positive infinity. In addition, expected returns on portfolios become indefinite (that is, have infinitely many values), and betas of portfolios approach one. We also show that unlike the Sharpe–Lintner CAPM, the market risk premium in the Black CAPM always has a positive minimum value dependent on the underlying covariance matrix. We show that given the mean–variance frontier, the zero-beta asset uniquely determines the market portfolio on the MVEF. We then give a simplified Black CAPM model that has a mean zero-beta return of zero and a mean zero-market return of \(\frac{B}{A}\), where \(A, B\) are constants extracted from the underlying covariance matrix. Unlike the Sharpe–Lintner CAPM, the beta of a portfolio can be negative, and takes a minimum value of \(\frac{A}{\sqrt{B}}\), in accordance to how it covaries with the zero-market efficient portfolio. The Security Market Line has zero intercept, and all portfolios have cost of capital dependent only on the zero-market MVEP.

We believe that our findings may encourage investors to empirically test the Black CAPM to see whether it generates portfolios near the GMVP before they use it for their investment decisions. In addition, when the market portfolios degenerates to the GMVP, the instability in the model could possibly lead to numerical instability and large rounding errors when practically applying the Black CAPM. It is also argued in [3] that equity investors should invest in the GMVP. However, our results show that it might not be optimal to all the investors to invest in the GMVP.

Our main results can also persuade scholars to further investigate other applications of the relative pricing concept in the finance literature. In particular, we would like to bring the following analogous application of the relative pricing concept used in the complete markets. First, we recall that CAPM extends the Markowitz model by introducing a risk free asset and hence uses a relative pricing concept. The Black CAPM replaces the risk-free asset (or the numeraire) with zero-beta asset which is independent of the market portfolio. Arrow’s and Debreu’s state preference theory provides the fundamental framework for defining a relative pricing concept which uses an appropriate martingale measure. However, the martingale approach in the complete market can possibly lead to a martingale which is different from the risk neutral one. Hence, the relative pricing concept used in the complete market could employ a numeraire which is different from the risk free numeraire which may be analogous to the zero-beta asset in the Black CAPM.

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