The Production-Planning Problem, Revisited

Consider the production-planning problem discussed in Section 2.1. In addition to the standard linear-programming formulation, we observed in Exercise 8.1-9 that a problem of this type can also be formulated and solved as a transportation problem. Our aim here is to show that a third approach to the solution of this problem is via dynamic programming.

We shall not repeat the statement of the problem here. It is therefore a good idea to quickly review the previous description. The notation will also stay the same. To demonstrate the flexibility of a dynamic-programming formulation, we will, however, relax the assumption of having no setup/fixed costs for production. More specifically, let

$c_k(x)$ = the total cost of producing x units of the product in month k, for $k = 1, 2, \ldots, 12$;

then, the new assumption is

$$c_k(x) = \begin{cases} 0 & \text{if } x = 0, \\ s_k + c_k x & \text{if } x > 0, \end{cases}$$

where s_k is the setup cost (assumed given) for production in month k. In contrast with the original assumption of $c_k(x) = c_k x$ for all $x \geq 0$, this cost function is nonlinear. It follows that a linear-programming solution of this problem is no longer feasible.

As before, the objective is to minimize the total production and inventory-holding costs over the given planning horizon.

As a simple numerical example, suppose that: the duration of the planning horizon is reduced to 3; the demands are $d_1 = 2$, $d_2 = 4$, and $d_3 = 1$; the setup costs are $s_1 = 10$, $s_2 = 10$, and $s_3 = 15$; the "variable" production costs are $c_1 = 4$, $c_2 = 6$, and $c_3 = 6$; the inventory-holding costs are $h_1 = 1$, $h_2 = 1$, and $h_3 = 1$; and finally, the production capacity is constant, with $m_k = 10$ for $k = 1, 2, 3$. We will use this example to illustrate the calculation of total cost. Consider the production schedule $(x_1, x_2, x_3) = (5, 2, 0)$. The total cost associated with this schedule can be computed as follows. For the first month, the production cost is $s_1 + c_1 x_1 = 10 + 4 \times 5 = 30$ and, since the ending inventory is $y_1 = y_0 + x_1 - d_1 = 0 + 5 - 2 = 3$, the inventory-holding cost is $h_1 y_1 = 1 \times 3 = 3$. For the second month, the production cost is $s_2 + c_2 x_2 = 10 + 6 \times 2 = 22$ and, since the ending inventory is $y_2 = y_1 + x_2 - d_2 = 3 + 2 - 4 = 1$, the inventory-holding cost is $h_2 y_2 = 1 \times 1 = 1$. For the third month, since $x_3 = 0$, the production cost is 0 and, since the ending inventory is $y_3 = y_2 + x_3 - d_3 = 1 + 0 - 1 = 0$, the inventory-holding cost is also equal to 0. Therefore, the total cost over this three-month period is given by: $(30 + 3) + (22 + 1) + (0 + 0) = 56$.

We now describe how to derive the optimal production schedule using dynamic programming.

Stages and States

Since there is one decision for each month, we define one stage for each month.

Suppose that decisions for months 1 through $k-1$ have already been made, and that we are in charge of making the remaining decisions for months k through N, where $N \equiv 12$. A little bit of reflection should convince you that the answer to the consultant question is that we need to know the inventory level at the beginning of stage k. We will, therefore, define the inventory level as the state.

Optimal-Value Function

In the language of the present problem, let

$V_k(i)$ = the best possible total cost from the beginning of month k to the end of month N, assuming that the initial inventory level for month k is i.

The goal is to determine $V_1(0)$, since we assume the initial inventory is zero.

Recurrence Relation

Suppose we are now in stage k; and suppose further that we are in state i, which means that the current inventory level is i. We shall not belabor the precise range for i. One reason is that it is tedious to do so; another reason is that we would like to place the emphasis on the structure of the recurrence relation. For the same reasons, we also won't belabor the precise range of x_k, for a given i, except noting the obvious bounds $0 \leq x_k \leq m_k$.

That is, it is enough for now to have the intuitive understanding that both i and x_k should not be too "high", so that there is a positive inventory at the end of month N, or too "low", so that demands during the remainder of the planning horizon cannot be met even if the production levels are set at the maximum. The precise ranges will be discussed later when we consider a specific computational example.

For any given combination of state i and action x_k, the immediate one-stage cost in stage k has two components. The first is the production cost, given by $c_k(x_k)$; and the second is the inventory-holding cost, given by $h_k(i + x_k - d_k)$. Since the subsequent state in stage $k+1$ will be $i+x_k-d_k$, the best possible total cost from stage $k+1$ to the end is $V_{k+1}(i+x_k-d_k)$. It follows that if we take action x_k, i.e., if we produce x_k units of the product, then the best possible total cost from stage k to the end is equal to $c_k(x_k)+h_k(i+x_k-d_k)+V_{k+1}(i+x_k-d_k)$. Minimizing over x_k now yields the following recurrence relation:

$$V_k(i) = \min_{x_k} \left[c_k(x_k) + h_k(i + x_k - d_k) + V_{k+1}(i + x_k - d_k) \right].$$

Notice that we have left the feasible range for x_k "open", which means that it is to be determined based on problem-specific information.

We next move on to the recursive computation of the $V_k(i)$'s.

Computation

We will illustrate the computation with the simple numerical example above.

To get things started quickly, we shall introduce a fictitious fourth stage (or month) and let $V_4(0) = 0$. That the final inventory should be zero is a consequence of the assumption that left-over units have no value.

We begin with an analysis of the highest possible value for i in each stage. For stage 3, the state should not be greater than 1; this is because the demand in stage 3 is 1 (i.e., $d_3 = 1$) and we should not have any excess units at the end of that stage. Next, observe that the total demand is $d_1 + d_2 + d_3 = 7$; therefore, the production level for stage 1 should not exceed 7. Now, with $x_1 = 7$, we have $y_1 = y_0 + x_1 - d_1 = 0 + 7 - 2 = 5$. It follows that the state in stage 2 should not exceed 5. Finally, the initial state in stage 1 is of course 0.

Since the production capacity in any stage, 10, is greater than the total demand 7, there is no fear of not being able to produce enough units. It follows that the value of i can be zero in every stage.

We now consider stage 3. For state 0, since $d_3 = 1$, the only feasible action is to let $x_3 = 1$; therefore,

$$\begin{aligned} V_3(0) &= c_3(1) + h_3(0 + 1 - 1) + V_4(0 + 1 - 1) \\ &= 15 + 6 \times 1 + 1 \times 0 + 0 \\ &= 21, \end{aligned}$$

where $V_4(0) = 0$ is from the boundary condition. Similarly, the only feasible action for state 1 is to let $x_3 = 0$; therefore,

$$\begin{aligned} V_3(1) &= c_3(0) + h_3(1 + 0 - 1) + V_4(1 + 0 - 1) \\ &= 0 + 1 \times 0 + 0 \\ &= 0. \end{aligned}$$

These calculations are summarized in the stage-3 table below.

Stage 3:

i	Actions $x_3 = 0$	$x_3 = 1$	$V_3(i)$	x_3^*
0	$-$	$21 + 0$	21	1
1	$0 + 0$	$-$	0	0

Next, we consider stage 2, where the possible states are 0 through 5. For state 0, the value of x_2 can be either 4 or 5; this is because $y_2 = y_1 + x_2 - d_2 = 0 + x_2 - 4$ and y_2, being the state in stage 3, must be either 0 or 1. Therefore,

$$\begin{aligned} V_2(0) &= \min_{4 \le x_2 \le 5} [c_2(x_2) + h_2(0 + x_2 - 4) + V_3(0 + x_2 - 4)] \\ &= \min [10 + 6 \times 4 + 1 \times 0 + 21,\ 10 + 6 \times 5 + 1 \times 1 + 0] \\ &= \min [34 + 0 + 21,\ 40 + 1 + 0] \\ &= 41 \end{aligned}$$

and the optimal action is to let $x_2 = 5$. The calculations for the other states are similar, so we will be brief. For state 1, the value of x_2 can be either 3 or 4; therefore,

$$\begin{aligned} V_2(1) &= \min [10 + 6 \times 3 + 1 \times 0 + 21,\ 10 + 6 \times 4 + 1 \times 1 + 0] \\ &= \min [28 + 0 + 21,\ 34 + 1 + 0] \\ &= 35 \end{aligned}$$

and the optimal action is to let $x_2 = 4$. For state 2, the value of x_2 can be either 2 or 3; therefore,

$$\begin{aligned} V_2(2) &= \min [10 + 6 \times 2 + 1 \times 0 + 21,\ 10 + 6 \times 3 + 1 \times 1 + 0] \\ &= \min [22 + 0 + 21,\ 28 + 1 + 0] \\ &= 29 \end{aligned}$$

and the optimal action is to let $x_2 = 3$. For state 3, the value of x_2 can be either 1 or 2; therefore,

$$\begin{aligned} V_2(3) &= \min [10 + 6 \times 1 + 1 \times 0 + 21,\ 10 + 6 \times 2 + 1 \times 1 + 0] \\ &= \min [16 + 0 + 21,\ 22 + 1 + 0] \\ &= 23 \end{aligned}$$

and the optimal action is to let $x_2 = 2$. For state 4, the value of x_2 can be either 0 or 1; therefore,

$$\begin{aligned} V_2(4) &= \min [0 + 1 \times 0 + 21,\ 10 + 6 \times 1 + 1 \times 1 + 0] \\ &= \min [0 + 0 + 21,\ 16 + 1 + 0] \\ &= 17 \end{aligned}$$

and the optimal action is to let $x_2 = 1$. Finally, for state 5, the only feasible action is to let $x_2 = 0$; therefore,

$$\begin{aligned} V_2(5) &= 0 + 1 \times 1 + 0 \\ &= 1. \end{aligned}$$

These calculations are summarized in the stage-2 table below.

Stage 2:

	Actions							
i	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$x_2 = 5$	$V_2(i)$	x_2^*
0	$-$	$-$	$-$	$-$	$34 + 21$	$41 + 0$	41	5
1	$-$	$-$	$-$	$28 + 21$	$35 + 0$	$-$	35	4
2	$-$	$-$	$22 + 21$	$29 + 0$	$-$	$-$	29	3
3	$-$	$16 + 21$	$23 + 0$	$-$	$-$	$-$	23	2
4	$0 + 21$	$17 + 0$	$-$	$-$	$-$	$-$	17	1
5	$1 + 0$	$-$	$-$	$-$	$-$	$-$	1	0

Finally, consider stage 1, where the only state is 0. Since $d_1 = 2$, the feasible values for x_1 range from 2 through 7; therefore, the optimal value for state 0 is

$$\begin{aligned} V_1(0) &= \min_{2 \le x_1 \le 7} [c_1(x_1) + h_1(0 + x_1 - 2) + V_2(0 + x_1 - 2)] \\ &= \min \begin{bmatrix} 10 + 4 \times 2 + 1 \times 0 + 41, \\ 10 + 4 \times 3 + 1 \times 1 + 35, \\ 10 + 4 \times 4 + 1 \times 2 + 29, \\ 10 + 4 \times 5 + 1 \times 3 + 23, \\ 10 + 4 \times 6 + 1 \times 4 + 17, \\ 10 + 4 \times 7 + 1 \times 5 + 1 \end{bmatrix} = \min \begin{bmatrix} 18 + 41, \\ 23 + 35, \\ 28 + 29, \\ 33 + 23, \\ 38 + 17, \\ 43 + 1 \end{bmatrix} \\ &= 44 \end{aligned}$$

and the optimal action is to let $x_1 = 7$. This yields the table below.

Stage 1:

	Actions							
i	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$x_1 = 5$	$x_1 = 6$	$x_1 = 7$	$V_1(i)$	x_1^*
0	$18 + 41$	$23 + 35$	$28 + 29$	$33 + 23$	$38 + 17$	$43 + 1$	44	7

Since $V_1(0) = 44$, we conclude that the lowest possible total cost is 44. This is achieved by the production schedule $x_1 = 7$, $x_2 = 0$, and $x_3 = 0$, which are read from the above tables in the usual manner. This completes the solution of the numerical example.

As usual, the recursive computation above can also be carried out via a network representation of the problem. You should work this out on your own as an exercise.