Sensitivity Analysis: An Example

Consider the linear program:

Maximize \( z = -5x_1 + 5x_2 + 13x_3 \)

Subject to:

\[
\begin{align*}
-x_1 + x_2 + 3x_3 & \leq 20 \quad (1) \\
12x_1 + 4x_2 + 10x_3 & \leq 90 \quad (2) \\
x_1, x_2, x_3 & \geq 0 .
\end{align*}
\]

After introducing two slack variables \( s_1 \) and \( s_2 \) and executing the Simplex algorithm to optimality, we obtain the following final set of equations:

\[
\begin{align*}
z + 2x_3 + 5s_1 &= 100 , \quad (0) \\
-x_1 + x_2 + 3x_3 + s_1 &= 20 , \quad (1) \\
16x_1 - 2x_3 - 4s_1 + s_2 &= 10 . \quad (2)
\end{align*}
\]

Our task is to conduct sensitivity analysis by independently investigating each of a set of nine changes (detailed below) in the original problem. For each change, we will use the fundamental insight to revise the final set of equations (in tableau form) to identify a new solution and to test the new solution for feasibility and (if applicable) optimality.

We will first recast the above equation systems into the following pair of initial and final tableaus.

**Initial Tableau:**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Basic</th>
<th>( z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>5</td>
<td>-5</td>
<td>-13</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>12</td>
<td>4</td>
<td>10</td>
<td>0</td>
<td>1</td>
<td>90</td>
</tr>
</tbody>
</table>

**Final Tableau:**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Basic</th>
<th>( z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>16</td>
<td>0</td>
<td>-2</td>
<td>-4</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

The basic variables associated with this final tableau are \( x_2 \) and \( s_2 \); therefore, the current basic feasible solution is \((x_1, x_2, x_3, s_1, s_2) = (0, 20, 0, 0, 10)\), which has an objective-function value of 100.

An inspection of the initial tableau shows that the columns associated with \( z, s_1, \) and \( s_2 \) form a 3 \( \times \) 3 identity matrix. Therefore, the \( P \) matrix will come from the corresponding columns in the final tableau. That is, we have

\[
P = \begin{bmatrix}
1 & 5 & 0 \\
0 & 1 & 0 \\
0 & -4 & 1
\end{bmatrix} ;
\]
and the final tableau equals the matrix product of this $P$ and the initial tableau, i.e., $T_F = P \times T_I$.

Our basic approach for dealing with parameter changes in the original problem is in two steps. In the first step, we will revise the final tableau by multiplying the same $P$ to the new initial tableau; in other words, despite a revision in $T_I$, we intend to follow the original sequence of pivots. After producing a revised $T_F$, we will, in the second step, take the revised $T_F$ as the starting point and initiate any necessary further analysis of the revised problem.

We now begin a detailed sensitivity analysis of this problem.

(a) Change the right-hand side of constraint (1) to 30.

Denote the right-hand-side constants in the original constraints as $b_1$ and $b_2$. Then, the proposed change is to revise $b_1$ from 20 to 30, while retaining the original value of $b_2$ at 90. With this change, the RHS column in the initial tableau becomes

\[
\begin{bmatrix}
0 \\
30 \\
90
\end{bmatrix}
\]

Since the rest of the columns in the initial tableau stays the same, the only necessary revision in $T_F$ will be in the RHS column. To determine this new RHS column, we multiply $P$ to the above new column to obtain:

\[
\begin{bmatrix}
1 & 5 & 0 \\
0 & 1 & 0 \\
0 & -4 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
30 \\
90
\end{bmatrix}
= \begin{bmatrix}
150 \\
30 \\
-30
\end{bmatrix}.
\]

Since the basic variables in the final tableau are $x_2$ and $s_2$, the solution associated with the revised $T_F$ is $(x_1, x_2, x_3, s_1, s_2) = (0, 30, 0, 0, -30)$. With a negative value for $s_2$, this (basic) solution is not feasible.

Geometrically speaking, increasing the value of $b_1$ from 20 to 30 means that we are relaxing the first inequality constraint. Relaxing a constraint is tantamount to enlarging the feasible set; therefore, one would expect an improved optimal objective-function value. The fact that the revised solution above is not feasible is not a contradiction to this statement. It only means that additional work is necessary to determine the new optimal solution.

What causes the infeasibility of the new solution? Recall that the original optimal solution is $(x_1, x_2, x_3, s_1, s_2) = (0, 20, 0, 0, 10)$. Since $x_1$, $x_3$, and $s_1$ are serving as nonbasic variables, the defining equations for this solution are: $x_1 = 0$, $x_3 = 0$, and $-x_1 + x_2 + 3x_3 = 20$. Now,
imagine an attempt to increase the RHS constant of the last equation from 20 to $20 + \delta$ (say) while maintaining these three equalities. As we increase $\delta$ (from 0), we will trace out a family of solutions. That the new solution is infeasible simply means that if $\delta$ is made sufficiently large (in this case, $\delta = 10$), then this family of solutions will eventually exit the feasible set.

More formally, suppose the original RHS column is revised to

$$\begin{bmatrix}
0 \\
20 + \delta \\
90
\end{bmatrix};$$
or alternatively, to

$$\begin{bmatrix}
0 \\
20 \\
90
\end{bmatrix} + \begin{bmatrix}
\delta \\
0 \\
0
\end{bmatrix}.$$

Then, after premultiplying this new column by $P$, we obtain

$$\begin{bmatrix}
1 & 5 & 0 \\
0 & 1 & 0 \\
0 & -4 & 1
\end{bmatrix} \times \begin{bmatrix}
0 \\
20 + \delta \\
90
\end{bmatrix} = \begin{bmatrix}
1 & 5 & 0 \\
0 & 1 & 0 \\
0 & -4 & 1
\end{bmatrix} \times \left( \begin{bmatrix}
0 \\
20 \\
90
\end{bmatrix} + \begin{bmatrix}
\delta \\
0 \\
0
\end{bmatrix} \right)$$

$$= \begin{bmatrix}
100 \\
20 \\
10
\end{bmatrix} + \begin{bmatrix}
5\delta \\
\delta \\
-4\delta
\end{bmatrix}$$

Hence, with $\delta = 10$, we indeed have $s_2 = -30$, which means that the original inequality constraint $12x_1 + 4x_2 + 10x_3 \leq 90$ is violated. Moreover, this calculation also shows that in order for $10 - 4\delta$ to remain nonnegative, $\delta$ cannot exceed $5/2$. In other words, at $\delta = 5/2$, the family of solutions $(0, 20 + \delta, 0, 0, 10 - 4\delta)$ “hits” the constraint equation $12x_1 + 4x_2 + 10x_3 = 90$; and therefore, progressing further will produce solutions that are outside the feasible set.

Interestingly, our analysis above holds even if we allow $\delta$ to assume a negative value. Such a case corresponds to a tightening of the constraint $-x_1 + x_2 + 3x_3 \leq 20$. A quick inspection of

$$\begin{bmatrix}
100 + 5\delta \\
20 + \delta \\
10 - 4\delta
\end{bmatrix}$$
shows that \( x_2 \) is reduced to 0 when \( \delta \) reaches \(-20\). It follows that in order to maintain feasibility, and hence optimality (since the optimality test is not affected by a change in the RHS column), of solutions of the form \((0, 20 + \delta, 0, 0, 10 - 4\delta)\), the value of \(\delta\) must stay within the range \([-20, 5/2]\).

Another important observation regarding the above calculation is that the optimal objective-function value will increase from 100 to 100 + 5\(\delta\), provided that \(\delta\) is sufficiently small (so that we remain within the feasible set). If we interpret the value of \(b_1\) as the availability of a resource, then this observation implies that for every additional unit of this resource, the optimal objective-function value will increase by 5. Thus, from an economics viewpoint, we will be unwilling to pay more than 5 (dollars) for an additional unit of this resource. For this reason, the value 5 is called the shadow price of this resource.

It is interesting to note that the shadow price of the first resource (5, in this case) can be read directly from the top entry in the second column of \(P\).

It is possible to derive a new optimal solution for the proposed new problem with \(\delta = 10\). The standard approach for doing this is to start from the revised final tableau and apply what is called the dual Simplex algorithm. As this algorithm is more advanced, we will not attempt to solve this new problem to optimality.

(b) Change the right-hand side of constraint (2) to 70.

Since the original value of \(b_2\) is 90, this is an attempt to reduce the availability of the second resource by 20. The analysis is similar to that in part (a). Again, we will write the new RHS column in the initial tableau as

\[
\begin{bmatrix}
0 \\
20 \\
90
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\delta
\end{bmatrix},
\]

where \(\delta\) is targeted to assume the value \(-20\). After premultiplying this new column by \(P\), we obtain

\[
\begin{bmatrix}
1 & 5 & 0 \\
0 & 1 & 0 \\
0 & -4 & 1
\end{bmatrix} \times \left( \begin{bmatrix}
0 \\
20 \\
90
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\delta
\end{bmatrix} \right)
= \begin{bmatrix}
100 \\
20 \\
10 + \delta
\end{bmatrix}.
\]
Hence, for all $\delta$ within the range $[-10, \infty)$, solutions of the form $(0, 20, 0, 0, 10 + \delta)$ will remain optimal.

With the particular choice of $\delta = -20$, we have

$$\begin{bmatrix} 100 \\ 20 \\ 10 + \delta \end{bmatrix} = \begin{bmatrix} 100 \\ 20 \\ -10 \end{bmatrix}.$$  

It follows that the new solution $(0, 20, 0, 0, -10)$ is infeasible. As in part (a), we will not attempt to derive a new optimal solution.

The shadow price of the second resource can be read directly from the top entry in the third column of $P$. In this case, it is given by 0. That the shadow price of the second resource is equal to 0 is expected. It is a consequence of the fact that in the current optimal solution, we have $s_2 = 10$ and hence there is already an excess in the supply of the second resource. In fact, we will have an over supply as long as the availability of the second resource is no less than 80 (which corresponds to $\delta = -10$).

(c) Change $b_1$ and $b_2$ to 10 and 100, respectively.

Again, we will first consider a revision of the RHS column in $T_I$ of the form:

$$\begin{bmatrix} 0 \\ 20 \\ 90 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta_1 \\ \delta_2 \end{bmatrix},$$

where $\delta_1$ and $\delta_2$ are two independent changes. After premultiplying this new column by $P$, we obtain

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \left( \begin{bmatrix} 0 \\ 20 \\ 90 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta_1 \\ \delta_2 \end{bmatrix} \right) = \begin{bmatrix} 100 + 5\delta_1 \\ 20 + \delta_1 \\ 10 - 4\delta_1 + \delta_2 \end{bmatrix}.$$  

With $\delta_1 = -10$ and $\delta_2 = 10$, the new RHS column in $T_F$ is:

$$\begin{bmatrix} 50 \\ 10 \\ 60 \end{bmatrix}.$$  

Since the new solution $(x_1, x_2, x_3, s_1, s_2) = (0, 10, 0, 0, 60)$ is feasible, it is also optimal. The new optimal objective-function value is 50.

(d) Change the coefficient of $x_3$ in the objective function to $c_3 = 8$ (from $c_3 = 13$).
Consider a revision in the value of \( c_3 \) by \( \delta \); that is, let \( c_3 = 13 + \delta \). Then, the \( x_3 \)-column in \( T_I \) is revised to

\[
\begin{bmatrix}
-13 - \delta \\
3 \\
10
\end{bmatrix}; \text{ or alternatively, to } \begin{bmatrix} -13 \\ 3 \\ 10 \end{bmatrix} + \begin{bmatrix} -\delta \\ 0 \\ 0 \end{bmatrix}.
\]

From the fundamental insight, the corresponding revision in the \( x_3 \)-column in \( T_F \) is

\[
\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \times \left( \begin{bmatrix} -13 \\ 3 \\ 10 \end{bmatrix} + \begin{bmatrix} -\delta \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -\delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 - \delta \\ 3 \\ -2 \end{bmatrix}.
\]

Therefore, if \( \delta = -5 \), which corresponds to \( c_3 = 8 \), then the new \( x_3 \)-column in \( T_F \) is explicitly given by

\[
\begin{bmatrix} 2 - \delta \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 - (-5) \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}.
\]

Observe that the \( x_3 \)-column is the only column in \( T_F \) that requires a revision, the variable \( x_3 \) is nonbasic, and the coefficient of \( x_3 \) in the revised \( R_0 \) is positive (7, that is). It follows that the original optimal solution \((x_1, x_2, x_3, s_1, s_2) = (0, 20, 0, 0, 10)\) remains optimal.

More generally, an inspection of the top entry in the new \( x_3 \)-column,

\[
\begin{bmatrix} 2 - \delta \\ 3 \\ -2 \end{bmatrix},
\]

reveals that the original optimal solution will remain optimal for all \( \delta \) such that \( 2 - \delta \geq 0 \), i.e., for all \( \delta \) in the range \((-\infty, 2]\).

*(e) Change \( c_1 \) to \(-2\), \( a_{11} \) to \(0\), and \( a_{21} \) to \(5\).*

This means that the \( x_1 \)-column in \( T_I \) is revised from

\[
\begin{bmatrix} 5 \\ -1 \\ 12 \end{bmatrix} \text{ to } \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}.
\]
Since the corresponding new column in $T_F$ is
\[
\begin{bmatrix}
1 & 5 & 0 \\
0 & 1 & 0 \\
0 & -4 & 1
\end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix},
\]
where the top entry, 2, is positive, and since $x_1$ is nonbasic in $T_F$, we see that the original optimal solution remains optimal.

(f) Change $c_2$ to 6, $a_{12}$ to 2, and $a_{22}$ to 5.

This means that the $x_2$-column in $T_I$ is revised from
\[
\begin{bmatrix}
-5 \\
1 \\
4
\end{bmatrix}
\]
to
\[
\begin{bmatrix}
-6 \\
2 \\
5
\end{bmatrix}.
\]

The fundamental insight implies that the corresponding new $x_2$-column in $T_F$ is
\[
\begin{bmatrix}
1 & 5 & 0 \\
0 & 1 & 0 \\
0 & -4 & 1
\end{bmatrix} \times \begin{bmatrix} -6 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}.
\]
The fact that this new column is no longer of the form
\[
\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]
indicates that $x_2$ cannot serve as a basic variable in $R_1$. It follows that a pivot in the $x_2$-column is needed to restore $x_2$ back to the status of a basic variable. More explicitly, the revised final tableau is

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>16</td>
<td>-3</td>
<td>-2</td>
<td>-4</td>
<td>1</td>
</tr>
</tbody>
</table>

and we will execute a pivot with the $x_2$-column as the pivot column and $R_1$ as the pivot row. After this pivot, we obtain

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>-4</td>
<td>3</td>
<td>0</td>
<td>60</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>-1/2</td>
<td>1</td>
<td>3/2</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>29/2</td>
<td>0</td>
<td>5/2</td>
<td>-5/2</td>
<td>1</td>
</tr>
</tbody>
</table>

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Since $x_3$ now has a negative coefficient in $R_0$, indicating that the new solution is not optimal, the Simplex algorithm should be restarted to derive a new optimal solution (if any).

(g) Introduce a new variable $x_4$ with $c_4 = 10$, $a_{14} = 3$, and $a_{24} = 5$.

This means that we need to introduce the new $x_4$-column

\[
\begin{bmatrix}
-10 \\
3 \\
5
\end{bmatrix}
\]

into the initial tableau. (The precise location of this new column is not important.) The corresponding new column in the final tableau will be

\[
\begin{bmatrix}
1 & 5 & 0 \\
0 & 1 & 0 \\
0 & -4 & 1
\end{bmatrix} \times \begin{bmatrix}
-10 \\
3 \\
5
\end{bmatrix} = \begin{bmatrix}
5 \\
3 \\
-7
\end{bmatrix}.
\]

Since this column has a positive entry at the top and since $x_4$ is nonbasic, the current optimal solution remains optimal. In an application, this means that there is insufficient incentive to engage in the new “activity” $x_4$.

(h) Introduce a new constraint $2x_1 + 3x_2 + 5x_3 \leq 50$.

After adding a new slack variable $s_3$, this inequality constraint becomes $2x_1 + 3x_2 + 5x_3 + s_3 = 50$. Next, we incorporate this equation into the final tableau to obtain

\begin{table}[h]
\centering
\begin{tabular}{c|ccccccc}
  & $z$ & $x_1$ & $x_2$ & $x_3$ & $s_1$ & $s_2$ & $s_3$ \\
\hline
Basic Variable & 1 & 0 & 0 & 2 & 5 & 0 & 0 & 100 \\
$ -$ & 0 & -1 & 1 & 3 & 1 & 0 & 0 & 20 \\
s_2 & 0 & 16 & 0 & -2 & -4 & 1 & 0 & 10 \\
s_3 & 0 & 2 & 3 & 5 & 0 & 0 & 1 & 50 \\
\end{tabular}
\end{table}

Observe that $x_2$ participates in the new equation and, therefore, cannot serve as the basic variable for $R_1$. To rectify this situation, we will execute the row operation $(-3) \times R_1 + R_3$. This yields

\begin{table}[h]
\centering
\begin{tabular}{c|ccccccc}
  & $z$ & $x_1$ & $x_2$ & $x_3$ & $s_1$ & $s_2$ & $s_3$ \\
\hline
Basic Variable & 1 & 0 & 0 & 2 & 5 & 0 & 0 & 100 \\
x_2 & 0 & -1 & 1 & 3 & 1 & 0 & 0 & 20 \\
s_2 & 0 & 16 & 0 & -2 & -4 & 1 & 0 & 10 \\
s_3 & 0 & 5 & 0 & -4 & -3 & 0 & 1 & -10 \\
\end{tabular}
\end{table}
With $s_3 = -10$, the new basic solution is not feasible. We will not attempt to continue the solution of this new problem (as it is now necessary to apply the dual Simplex algorithm).

(i) Change constraint (2) to $10x_1 + 5x_2 + 10x_3 \leq 100$.

With this revision, the initial tableau becomes

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>10</td>
<td>5</td>
<td>10</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

After premultiplying this by $P$, we obtain the revised final tableau below.

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>14</td>
<td>1</td>
<td>-2</td>
<td>-4</td>
<td>1</td>
</tr>
</tbody>
</table>

Observe that $x_2$ participates in $R_2$ and, therefore, cannot serve as the basic variable for $R_1$. To rectify this situation, we will execute the row operation $(-1) \times R_1 + R_2$. This yields

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>15</td>
<td>0</td>
<td>-5</td>
<td>-5</td>
<td>1</td>
</tr>
</tbody>
</table>

Therefore, the new optimal solution is $(x_1, x_2, x_3, s_1, s_2) = (0, 20, 0, 0, 0)$. 

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