[a]2. A real number x is represented approximately by 0.6032, and we are told that the relative error is at most 0.1%. What is x?
Note: There are *two* answers.

[a]3. What is the relative error involved in rounding 4.9997 to 5.000?

[a]4. The value of π can be generated by the computer to nearly full machine precision by the assignment statement

$pi \leftarrow 4.0 \arctan(1.0)$

Suggest at least four other ways to compute π using basic functions on your computer system.

5. A given doubly subscripted array $(a_{ij})_{n\times n}$ can be added in any order. Write the pseudocode segments for each of the following parts. Which is best?

[a]a. $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$ b. $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}$

c. $\sum_{i=1}^{n}\left(\sum_{j=1}^{i} a_{ij} + \sum_{j=1}^{i-1} a_{ji}\right)$

[a]d. $\sum_{k=0}^{n-1}\sum_{|i-j|=k} a_{ij}$ e. $\sum_{k=2}^{2n}\sum_{i+j=k}^{n} a_{ij}$

[a]6. Count the number of operations involved in evaluating a polynomial using nested multiplication. Do *not* count subscript calculations.

7. For small x, show that $(1+x)^2$ can sometimes be more accurately computed from $(x+2)x+1$. Explain. What other expressions can be used to compute it?

8. Show how these polynomials can be efficiently evaluated:

[a]a. $p(x) = x^{32}$ b. $p(x) = 3(x-1)^5 + 7(x-1)^9$

[a]c. $p(x) = 6(x+2)^3 + 9(x+2)^7 + 3(x+2)^{15} - (x+2)^{31}$

d. $p(x) = x^{127} - 5x^{37} + 10x^{17} - 3x^7$

9. Using the exponential function $\exp(x)$, write an efficient pseudocode segment for the statement $y = 5e^{3x} + 7e^{2x} + 9e^x + 11$.

[a]10. Write a pseudocode segment to evaluate the expression

$$z = \sum_{i=1}^{n} b_i^{-1} \prod_{j=1}^{i} a_j$$

where $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are linear arrays containing given values.

11. Write segments of pseudocode to evaluate the following expressions efficiently:

a. $p(x) = \sum_{k=0}^{n-1} kx^k$

[a]b. $z = \sum_{i=1}^{n}\prod_{j=1}^{i} x^{n-j+1}$

c. $z = \prod_{i=1}^{n}\sum_{j=1}^{i} x_j$

d. $p(t) = \sum_{i=1}^{n} a_i \prod_{j=1}^{i-1}(t - x_j)$

12. Using summation and product notation, write mathematical expressions for the following pseudocode segments:

a.
```
integer i, n; real v, x
real array (a_i)0:n
v ← a0
for i = 1 to n
    v ← v + x a_i
end for
```

[a]b.
```
integer i, n; real v, x
real array (a_i)0:n
v ← a_n
for i = 1 to n
    v ← v x + a_(n−i)
end for
```

c.
```
integer i, n; real v, x
real array (a_i)0:n
v ← a0
for i = 1 to n
    v ← v x + a_i
end for
```

d.
```
integer i, n; real v, x, z
real array (a_i)0:n
v ← a0
z ← x
for i = 1 to n
    v ← v + z a_i
    z ← x z
end for
```

[a]e.
```
integer i, n; real v
real array (a_i)0:n
v ← a_n
for i = 1 to n
    v ← (v + a_(n−i)) x
end for
```

[a]13. Express in mathematical notation *without* parentheses the final value of z in the following pseudocode segment:

```
integer k, n; real z
real array (b_i)0:n
z ← b_n + 1
for k = 1 to n − 2
    z ← z b_(n−k) + 1
end for
```

[a]**14.** How many multiplications occur in executing the following pseudocode segment?

```
integer i, j, n; real x
real array (a_ij)_{0:n×0:n}, (b_ij)_{0:n×0:n}
x ← 0.0
for j = 1 to n
    for i = 1 to j
        x ← x + a_ij b_ij
    end for
end for
```

15. Criticize the following pseudocode segments and write improved versions:

a.
```
integer i, n; real x, z; real array (a_i)_{0:n}
for i = 1 to n
    x ← z^2 + 5.7
    a_i ← x/i
end for
```

[a]**b.**
```
integer i, j, n
real array (a_ij)_{0:n×0:n}
for i = 1 to n
    for j = 1 to n
        a_ij ← 1/(i + j − 1)
    end for
end for
```

c.
```
integer i, j, n; real array (a_ij)_{0:n×0:n}
for j = 1 to n
    for i = 1 to n
        a_ij ← 1/(i + j − 1)
    end for
end for
```

16. a. Solve Example 1.1.2 to full precision.

b. Repeat for this augmented matrix

$$\left[\begin{array}{cc|c} 3.5713 & 2.1426 & 7.2158 \\ 10.714 & 6.4280 & 1.3379 \end{array}\right]$$

for a system of two equations and two unknowns x and y.

c. Can small changes in the data lead to massive change in the each of these solutions?

17. A base 60 approximation (circa 1750 BCE) is

$$\sqrt{2} \approx 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$$

Determine how accurate it is. See Sauer [2006] for additional details.

18. Use Horner's algorithm to evaluate each of these polynomials at the point indicated.

a. $2x^4 + 9x^2 - 16x + 12$ at -6

b. $2x^4 - 3x^3 - 5x^2 + 3x + 8$ at 2

c. $3x^5 - 38x^3 + 5x^2 - 1$ at 4

Computer Exercises 1.1

1. Write and run a computer program that corresponds to the pseudocode program *First* described on pp. 12–13 and interpret the results.

2. (Continuation) Select a function f and a point x and carry out a computer experiment like the one given in the text. Interpret the results. Do not select too simple a function. For example, you might consider $1/x$, $\log x$, e^x, $\tan x$, $\cosh x$, or $x^3 - 23x$.

3. (Continuation) As we saw in the computer experiment *First*, the accuracy of a formula for numerical differentiation may deteriorate as the step-size h decreases. Study the following **central difference formula**:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

as $h \to 0$. We learn in Section 4.3 that the **truncation error** for this formula is $-\frac{1}{6}h^2 f'''(\xi)$ for some ξ in the interval $(x - h, x + h)$.

Modify and run the code for the experiment *First* so that approximate values for the rounding error and truncation error are computed. On the same graph, plot the rounding error, the truncation error, and the total error (sum of these two errors) using a log-scale; that is, the axes in the plot should be $-\log_{10}|\text{error}|$ versus $\log_{10} h$. Analyze these results.

[a]**4.** The limit

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

defines the number e in calculus. Estimate e by taking the value of this expression for $n = 8, 8^2, 8^3, \ldots, 8^{10}$.

17. In 1706, Machin used the formula

$$\pi = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)$$

to compute 100 digits of π. Derive this formula. Reproduce Machin's calculations by using suitable software. *Hint*: Let $\tan\theta = \frac{1}{5}$, and use standard trigonometric identities.

18. Using a symbol-manipulating program such as Maple, Mathematica, MATLAB, or Macsyma, carry out the following tasks. Record your work in some manner, for example, by using a `diary` or `script` command.

a. Find the Taylor series, up to and including the term x^{10}, for the function $(\tan x)^2$, using 0 as the point x_0.

b. Find the indefinite integral of $(\cos x)^{-4}$.

c. Find the definite integral $\int_0^1 \log|\log x|\,dx$.

d. Find the first prime number greater than 27448.

e. Obtain the numerical value of $\int_0^1 \sqrt{1+\sin^3 x}\,dx$.

f. Find the solution of the differential equation $y' + y = (1+e^x)^{-1}$.

g. Define the function $f(x, y) = 9x^4 - y^4 + 2y^2 - 1$. You want to know the value of $f(40545, 70226)$. Compute this in the straightforward way by direct substitution of $x = 40545$ and $y = 70226$ in the definition of $f(x, y)$, using first 6-decimal accuracy, then 7-, 8-, and so on up to 24-decimal digits of accuracy. Next, prove by means of elementary algebra that

$$f(x, y) = (3x^2 - y^2 + 1)(3x^2 + y^2 - 1)$$

Use this formula to compute the same value of $f(x, y)$, again using different precisions, from 6-decimal to 24-decimal. Describe what you have learned. To force the program to do floating-point operations instead of integer arithmetic, write your numbers in the form 9.0, 40545.0, and so forth.

19. Consider the following pseudocode segments:

```
a. integer i; real x, y, z
   for i = 1 to 20
       x ← 2 + 1.0/8^i
       y ← arctan(x) − arctan(2)
       z ← 8^i y
       output x, y, z
   end for
```

```
b. real epsi ← 1
   while 1 < 1 + epsi
       epsi ← epsi/2
       output epsi
   end while
```

What is the purpose of each program? Is it achieved? Explain. Code and run each one to verify your conclusions.

20. Consider some oversights involving assignment statements.

[a]**a.** What is the difference between the following two assignment statements? Write a code that contains them and illustrate with specific examples to show that sometimes $x = y$ and sometimes $x \neq y$.

```
integer m, n; real x, y
x ← real(m/n)
y ← real(m)/real(n)
output x, y
```

b. What value does n receive?

```
integer n;   real x, y
x ← 7.4
y ← 3.8
n ← x + y
output n
```

What happens when the last statement is replaced with the following?

```
n ← integer(x) + integer(y)
```

21. Write a computer code that contains the following assignment statements exactly as shown. Analyze the results.

a. Print these values first using the default format and then with an extremely large format field:

```
real p, q, u, v, w, x, y, z
x ← 0.1
y ← 0.01
z ← x − y
p ← 1.0/3.0
q ← 3.0p
u ← 7.6
v ← 2.9
w ← u − v
output x, y, z, p, q, u, v, w
```

b. What values would be computed for x, y, and z if this code is used?

Thus, it suffices to select n so that $(3n^3)^{-1} < \frac{1}{2} \times 10^{-6}$, or $n \geqq 88$. (A more sophisticated analysis improves this considerably.) ■

Summary 1.2

- Complete Horner's Algorithm:

```
for k = 0 to n − 1
    for j = n − 1 to k
        a_j ← a_j + r a_{j+1}
    end for
end for
```

- The **Taylor series expansion** about c for $f(x)$ is

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1}$$

with error term

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

A more useful form for us is the **Taylor series expansion** for $f(x+h)$, which is

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}h^k + E_{n+1}$$

with error term

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1} = \mathcal{O}(h^{n+1})$$

- An **alternating series**

$$S = \sum_{k=1}^{\infty}(-1)^{k-1}a_k$$

converges when the terms a_k converge downward to zero. Furthermore, the partial sums S_n differ from S by an amount that is bounded by

$$|S - S_n| \leqq a_{n+1}$$

Exercises 1.2

1. The Maclaurin series for $(1+x)^n$ is also known as the **binomial series**. It states that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \qquad (x^2 < 1)$$

Derive this series. Then give its particular forms in summation notation by letting $n = 2$, $n = 3$, and $n = \frac{1}{2}$. Next use the last form to compute $\sqrt{1.0001}$ correct to 15 decimal places (rounded).

2. (Continuation) Use the series in the preceding problem to obtain series (4). How could this series be used on a computing machine to produce x/y if only addition and multiplication are built-in operations?

3. (Continuation) Use the previous problem to obtain a series for $(1+x^2)^{-1}$.

4. Why do the following functions not possess Taylor series expansions at $x = 0$?

[a]**a.** $f(x) = \sqrt{x}$ [a]**b.** $f(x) = |x|$

c. $f(x) = \arcsin(x-1)$ **d.** $f(x) = \cot x$

[a]**e.** $f(x) = \log x$ **f.** $f(x) = x^{\pi}$

[a]**5.** Determine the Taylor series for $\cosh x$ about zero. Evaluate $\cosh(0.7)$ by summing four terms. Compare with the actual value.

6. Determine the first two nonzero terms of the series expansion about zero for the following:

[a]**a.** $e^{\cos x}$ [a]**b.** $\sin(\cos x)$

c. $(\cos x)^2(\sin x)$

[a]**7.** Find the smallest nonnegative integer m such that the Taylor series about m for $(x-1)^{1/2}$ exists. Determine the coefficients in the series.

[a]**8.** Determine how many terms are needed to compute e correctly to 15 decimal places (rounded) using e^x series (1) for e^x.

[a]**9.** (Continuation) If $x < 0$ in the preceding problem, what are the signs of the terms in the series? Loss of significant digits can be a serious problem in using the series. Will the formula $e^{-x} = 1/e^x$ be helpful in reducing the error? Explain. (See Section 1.5 for further discussion.) Try high-precision computer arithmetic to see how bad the floating-point errors can be.

10. Show how the simple equation $\ln 2 = \ln[e(2/e)]$ can be used to speed up the calculation of $\ln 2$ in series (10).

[a]**11.** What is the series for $\ln(1-x)$? What is the series for $\ln[(1+x)/(1-x)]$?

[a]**12.** (Continuation) In the series for $\ln[(1+x)/(1-x)]$, determine what value of x to use if we wish to compute $\ln 2$. Estimate the number of terms needed for ten digits (rounded) of accuracy. Is this method practical?

13. Use the Alternating Series Theorem to determine the number of terms in series (5) needed for computing $\ln 1.1$ with error less than $\frac{1}{2} \times 10^{-8}$.

14. Write the Taylor series for the function $f(x) = x^3 - 2x^2 + 4x - 1$, using $x = 2$ as the point of expansion; that is, write a formula for $f(2+h)$.

15. Determine the first four nonzero terms in the series expansion about zero for

[a]**a.** $f(x) = (\sin x) + (\cos x)$ and find an approximate value for $f(0.001)$.

[a]**b.** $g(x) = (\sin x)(\cos x)$ and find an approximate value for $g(0.0006)$.

Compare the accuracy of these approximations to those obtained from tables or via a calculator.

[a]**16.** Verify this Taylor series and prove that it converges on the interval $-e < x \leqq e$.

$$\ln(e+x) = 1 + \frac{x}{e} - \frac{x^2}{2e^2} + \frac{x^3}{3e^3} - \frac{x^4}{4e^4} + \cdots$$
$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{x}{e}\right)^k$$

[a]**17.** How many terms are needed in series (3) to compute $\cos x$ for $|x| < \frac{1}{2}$ accurate to 12 decimal places (rounded)?

[a]**18.** A function f is defined by the series

$$f(x) = \sum_{k=1}^{\infty}(-1)^k\left(\frac{x^k}{k^4}\right)$$

Determine the minimum number of terms needed to compute $f(1)$ with error less than 10^{-8}.

19. Verify that the partial sums $s_k = \sum_{i=0}^{k} x^i/i!$ in the series (1) for e^x can be written recursively as $s_k = s_{k-1} + t_k$, where $s_0 = 1$, $t_1 = x$, and $t_k = (x/k)t_{k-1}$.

[a]**20.** What is the fifth term in the Taylor series of $(1-2h)^{1/2}$?

21. Show that if $E = \mathcal{O}(h^n)$, then $E = \mathcal{O}(h^m)$ for any nonnegative integer $m \leqq n$. Here $h \to 0$.

22. Show how $p(x) = 6(x+3) + 9(x+3)^5 - 5(x+3)^8 - (x+3)^{11}$ can be efficiently evaluated.

[a]**23.** What is the second term in the Taylor series of $\sqrt[4]{4x-1}$ about 4.25?

[a]**24.** How would you compute a table of $\log n!$ for $1 \leqq n \leqq 1000$?

25. For small x, the approximation $\sin x \approx x$ is often used. For what range of x is this good to a *relative* accuracy of $\frac{1}{2} \times 10^{-14}$?

26. In the Taylor series for the function $3x^2 - 7 + \cos x$ (expanded in powers of x), what is the coefficient of x^2?

27. In the Taylor series (about $\pi/4$) for the function $\sin x + \cos x$, find the third nonzero term.

[a]**28.** By using Taylor's Theorem, one can be sure that for all x that satisfy $|x| < \frac{1}{2}$, $|\cos x - (1 - x^2/2)|$ is less than or equal to what numerical value?

29. Find the
$f(x) =$

30. Use Tay
proxima

31. For the
$a_0 > a$
$S_2 > S$
$0 < S_{2n}$

[a]**32.** What is
$3 + 7x -$
this func

33. In the te
e^x only

34. Determi
of h for e
where C

[a]**35.** What is t
as 3.14 (

36. Using th
mine the
uate $e^{\sin}$
constant

37. Develop
series in

[a]**38.** Determi
uate cos(
Hint: π

[a]**39.** Determi
uate sin(

40. Establish
csc($\pi/6$
curacy a

41. Establish
a. e^{x+2h}
c. ln[(x

[a]**42.** Determi
terms of

43. Given th

how man
(choppe