Practical Guide for Use of the Wilcoxon Signed Rank Test

Setting

- A continuous and symmetric parent distribution \( F \) with density function \( f(x) \).
- Target parameter: the median \( \nu \).
  (Thus, for random variable \( X \) having distribution \( F \), we have: \( P(X \leq \nu) = P(X \geq \nu) = 0.5 \) and \( P(X = \nu) = 0 \).)
- A random sample \( X_1, \ldots, X_n \) from the parent distribution \( F \).

The Point Estimator

The point estimator is the sample median, \( \hat{\nu} \).

Key property. By a version of the Central Limit Theorem, the distribution of \( \hat{\nu} \) is approximately \( N\left(\nu, \frac{1}{4(f(\nu)^2)n}\right) \). This can be used to obtain a confidence interval for \( \nu \), but it requires also estimating \( f(\nu) \), which is another problem. On the other hand, bypassing this problem, one can also obtain a C.I. for \( \nu \) using a pair of order statistics. See the handout “Inference Procedures for One Sample and Paired-Data Location Problems”.

Wilcoxon Signed Rank Test of \( H_0 : \nu = \nu_0 \) (a specified value)

Like the sign test, this test also uses the signs of the differences \( X_i - \nu_0 \), \( 1 \leq i \leq n \), but in addition it takes into account the magnitudes (absolute values) of these differences.

(a) The Wilcoxon signed rank test statistic is developed as follows.

(i) We ignore the cases where \( X_i = \nu_0 \). Let \( n^* \) be the number of sample values \( X_i \) not equal to \( \nu_0 \), and relabel these cases as \( X_1, \ldots, X_{n^*} \).

(ii) Let

\[
Y_i = \begin{cases} 
1, & X_i > \nu_0 \\
0, & X_i \leq \nu_0.
\end{cases}
\]

(iii) Let

\[
R_1 = \text{rank of } |X_1 - \nu_0| \text{ among } \{|X_1 - \nu_0|, \ldots, |X_{n^*} - \nu_0|\}
\]

\[
R_2 = \text{rank of } |X_2 - \nu_0| \text{ among } \{|X_1 - \nu_0|, \ldots, |X_{n^*} - \nu_0|\}
\]

\[
\vdots
\]

\[
R_{n^*} = \text{rank of } |X_{n^*} - \nu_0| \text{ among } \{|X_1 - \nu_0|, \ldots, |X_{n^*} - \nu_0|\}.
\]

(The largest gets rank \( n^* \), the smallest gets rank 1.)
(iv) Then the Wilcoxon signed rank test statistic is the sum of the ranks for the positive differences:

\[ W^+ = \sum_{i=1}^{n} Y_i R_i. \]

(b) The \( H_0 \)-distribution of \( W^+ \).

(i) Exact distribution of \( W^+ \). The exact distribution is found in tables for small sample sizes, in the backs of typical textbooks. The exact mean and variance of \( W^+ \) are with

\[
E_{H_0}(W^+) = \frac{1}{4} n^*(n^* + 1), \quad \text{Var}_{H_0}(W^+) = \frac{1}{24} n^*(n^* + 1)(2n^* + 1).
\]

(ii) Normal approximation to distribution of \( W^+ \): \( N\left(\frac{1}{4} n^*(n^* + 1), \frac{1}{24} n^*(n^* + 1)(2n^* + 1)\right) \).

NOTE. If \( W^+ \neq \frac{1}{4} n^*(n^* + 1) \) and there are ties among the differences \(|X_i - \nu_0|\), then the variance used in this Normal approximation is adjusted to

\[
\frac{1}{24} n^*(n^* + 1)(2n^* + 1) + \frac{1}{48} \sum_{i=1}^{g} (t_i^3 - t_i),
\]

where \( g \) is number of groups of tied differences and \( t_i \) is the number of tied differences in the \( i \)th group.

(c) Get the \( p \)-value using (b) (i) or (b) (ii).

(d) Interpret lower \( p \)-value as stronger evidence against \( H_0 \).

(e) For a test of \( H_0 \) at significance level \( \alpha \), reject \( H_0 \) if \( p \)-value \( \leq \alpha \) and otherwise accept \( H_0 \).

– RJS, 3/14/2011