The Geometric Distribution

- *Setting.* A series of independent success-failure trials, with success probability p. Target random variable:
$$Y = \text{number of trials until the 1st success.}$$

- *Probability Distribution of Y.* For possible values $k = 1, 2, \ldots$,
$$P(Y = k) = (1-p)^{k-1}p.$$
We call it the *geometric*(p) distribution.

- *Mean and Variance.* We have
$$E(Y) = \frac{1}{p}$$
and
$$\text{Var}(Y) = \frac{1-p}{p^2}.$$

- *Illustrative Application.* Suppose that a cereal manufacturer puts a special prize in 1/20 of the boxes. What is the *average number of purchases* of cereal until "success" occurs? Answer: $1/0.05 = 20$. What is the probability that *at least three purchases* are required? Answer: $1 - P(Y=1) - P(Y=2) = 1 - 0.05 - 0.95 \times 0.05 = 0.9025$.

- *The Geometric Distribution has a Memory-Free Property.* Suppose that the number of days until a device fails is *geometric*(p): $P(\text{device first fails on } k\text{th day}) = (1-p)^{k-1}p$. Then the *conditional probability* that a device which has lasted m days already will last at least k more days is
$$P(Y \geq m+k \,|\, Y > m) = \frac{P(Y \geq m+k)}{P(Y > m)} = \frac{\sum_{j=m+k}^{\infty}(1-p)^{j-1}p}{\sum_{j=m+1}^{\infty}(1-p)^{j-1}p} = \ldots = (1-p)^k,$$
for $k = 1, 2, \ldots$

  Interpretation. The future performance of the device satisfies the "*memory-free property*" that the failure probabilities for the device after m days of use are exactly the same as for starting out with a *brand new* device. That is, a used device is neither better nor worse than a new one. *Which would you rather be given – a brand new light bulb or one which has been used without failure for 10 days?*

– RJS, 2/5/08