## Roles of $\sigma$-Algebras and $\lambda$-Classes in Probability and Statistics

- $\lambda$-*class*. A nonempty class $\mathcal{C}$ of subsets of $\mathcal{X}$ containing $\mathcal{X}$ and closed under the formation of complements and of finite and countable *disjoint* unions.

- $\sigma$-*algebra*, or $\sigma$-*field*. A nonempty class $\mathcal{C}$ of subsets of $\mathcal{X}$ closed under the formation of complements and countable unions.

  COMMENT: A $\lambda$-class closed under formation of pairwise intersections is a $\sigma$-algebra. Equivalently, removing the "disjoint" restriction in the definition of $\lambda$-class yields that of a $\sigma$-algebra. Thus a $\sigma$-algebra represents an extension of a $\lambda$-class to a larger class of sets.

- *Similar events*. Let $\{P_\theta, \theta \in \Theta\}$ be a family of probability distributions. An event $A$ is called *similar* if $P_\theta(A)$ is independent of $\theta \in \Theta$. The class of similar events forms a $\lambda$-class.

  COMMENT: *Permutation tests* (including *rank tests*) are those whose critical regions belong to the $\lambda$-class of similar events. Critical regions of rank tests, however, constitute a $\sigma$-algebra.

- *Probability functions* are well-defined on $\lambda$-classes. *Conditional* probability functions and expectations, however, are defined relative to specified $\sigma$-algebras and have no counterparts for $\lambda$-classes. Thus the Neyman-Pearson Lemma, for example, would not be available.

- *Example when it is straightforward to define probability on a $\lambda$-class but not on its extension to a $\sigma$-algebra.*

  > Let event $E$ occur with probability 1/2, and let event $F$ occur independently of $E$. Let $A$ occur if $E$ occurs, and let $B$ occur if $(E \cap F) \cup (E^c \cap F^c)$ occurs. Then $P(A) = P(B) = 1/2$. But $P(A \cap B)$ is indeterminate from the given experiment.

*– RJS, 1/26/06*