

Some Basic Classes of Subsets of a Space \mathcal{X}

- *Semiring*. A nonempty class \mathcal{C} of subsets of \mathcal{X} satisfying

- (a) $A, B \in \mathcal{C}$ implies $A \cap B \in \mathcal{C}$.
- (b) For $A, B \in \mathcal{C}$ with $A \subset B$, there is a finite partition of $B - A$ in \mathcal{C} : there exist disjoint C_1, \dots, C_n in \mathcal{C} such that $B - A = \cup_{i=1}^n C_i$.

EXAMPLES: (i) The class $\emptyset \cup \{ \{x\}, x \in \mathcal{X} \}$.

(ii) The class of finite, left-closed, and right-open intervals in \mathbb{R} .

- *Ring*. A nonempty class \mathcal{C} of subsets of \mathcal{X} satisfying

- (a) $A, B \in \mathcal{C}$ implies $A - B \in \mathcal{C}$.
- (b) $A, B \in \mathcal{C}$ implies $A \cup B \in \mathcal{C}$.

FACTS: (i) The empty set \emptyset must belong to \mathcal{C} .

(ii) A semiring closed under the formation of unions is a ring.

- σ -ring. A ring closed under the formation of countable unions.

- *Monotone class*

A nonempty class \mathcal{C} of sets containing the limits of monotone sequences in \mathcal{C} : for an increasing or decreasing sequence of sets $C_i \in \mathcal{C}$, we have $\lim_n C_n \in \mathcal{C}$.

FACTS: (i) Every σ -ring is a monotone class.

(ii) Every monotone ring is a σ -ring.

- π -class. A nonempty class \mathcal{C} of subsets of \mathcal{X} closed under the formation of intersections.

- λ -class. A nonempty class \mathcal{C} of subsets of \mathcal{X} containing \mathcal{X} and closed under the formation of complements and of finite and countable *disjoint* unions.

- *Semialgebra*, or *semifield*. A semiring containing \mathcal{X} .

- *Algebra*, or *field*. A nonempty class \mathcal{C} of subsets of \mathcal{X} closed under the formation of complements and unions:

- (a) For $A \in \mathcal{C}$, the complement A^c is in \mathcal{C} .
- (b) $A, B \in \mathcal{C}$ implies $A \cup B \in \mathcal{C}$.

FACTS: (i) An algebra is a ring containing \mathcal{X} .

(ii) Every monotone class is an algebra.

- σ -algebra, or σ -field. A nonempty class \mathcal{C} of subsets of \mathcal{X} closed under the formation of complements and countable unions.

FACTS: (i) A σ -algebra is a σ -ring containing \mathcal{X} .

(ii) A σ -algebra is a special case of λ -class.

(iii) A class that is both a π -class and a λ -class is a σ -algebra.

(iv) A σ -algebra is closed under countable intersections.

(v) An algebra is a σ -algebra if and only if it is a monotone class.

- *Generated σ -algebras.* Given a class \mathcal{E} of subsets of \mathcal{X} , there exists a *smallest* σ -algebra containing \mathcal{E} . We call it the *minimal σ -algebra containing \mathcal{E}* , or the *σ -algebra generated by \mathcal{E}* , and denote it by $\sigma(\mathcal{E})$.
- *The Borel sets of \mathbb{R} .* Start with the semiring

$$\mathcal{E}_0 = \{[a, b) : -\infty < a < b < \infty\},$$

the class of finite, left-closed, and right-open intervals in the real line. The *algebra* \mathcal{E} generated by \mathcal{E}_0 consists of all finite unions of disjoint sets of the form $[a, b)$, $(-\infty, a)$, or $[b, \infty)$. The *σ -algebra* $\sigma(\mathcal{E}_0) = \sigma(\mathcal{E})$ generated by \mathcal{E} (or \mathcal{E}_0) is called the *Borel sets in \mathbb{R}* .

The semiring of left-open, right-closed intervals also generates the Borel sets. In fact, quite simply, the smallest σ -algebra containing the class of infinite intervals

$$\{(-\infty, a) : -\infty < a < \infty\}$$

is the class of Borel sets.

Note that, using the above-noted fact that an algebra is a σ -algebra if and only if it is a monotone class, one can pass from the algebra \mathcal{E} to $\sigma(\mathcal{E})$ by means of limiting processes.

To determine a probability on the Borel sets $\sigma(\mathcal{E}_0)$, it suffices to specify its values simply on the intervals $[a, b)$. This is because any such specification has a *unique extension* to a probability measure on all the Borel sets. In the case of continuous probability distributions, however, extensions beyond this class of sets need not be unique. Thus the Borel sets represent a convenient level of generality for probability modeling.

- *The Borel sets of \mathbb{R}^k .* Similarly defined, replacing intervals by rectangles

$$[\mathbf{x} = (x_1, \dots, x_k) : a_i \leq x_i < b_i, i = 1, \dots, k]$$

(sometimes called hyperrectangles) in the definition of $\sigma(\mathcal{E}_0)$. Indeed, the (countable) class of rectangles with a_i and b_i all rational generates the Borel sets of \mathbb{R}^k .

– RJS, 8/24/09