Some Basic Classes of Subsets of a Space $\mathcal{X}$

- **Semiring.** A nonempty class $\mathcal{C}$ of subsets of $\mathcal{X}$ satisfying
  
  (a) $A, B \in \mathcal{C}$ implies $A \cap B \in \mathcal{C}$.
  
  (b) For $A, B \in \mathcal{C}$ with $A \subseteq B$, there is a finite partition of $B - A$ in $\mathcal{C}$: there exist disjoint $C_1, \ldots, C_n$ in $\mathcal{C}$ such that $B - A = \bigcup_{i=1}^n C_i$.

EXAMPLS:

(i) The class $\emptyset \cup \{ \{x\}, x \in \mathcal{X}\}$.

(ii) The class of finite, left-closed, and right-open intervals in $\mathbb{R}$.

- **Ring.** A nonempty class $\mathcal{C}$ of subsets of $\mathcal{X}$ satisfying
  
  (a) $A, B \in \mathcal{C}$ implies $A - B \in \mathcal{C}$.
  
  (b) $A, B \in \mathcal{C}$ implies $A \cup B \in \mathcal{C}$.

FACTS:

(i) The empty set $\emptyset$ must belong to $\mathcal{C}$.

(ii) A semiring closed under the formation of unions is a ring.

- **$\sigma$-ring.** A ring closed under the formation of countable unions.

- **Monotone class.** A nonempty class $\mathcal{C}$ of sets containing the limits of monotone sequences in $\mathcal{C}$; for an increasing or decreasing sequence of sets $C_i \in \mathcal{C}$, we have $\lim_n C_n \in \mathcal{C}$.

FACTS:

(i) Every $\sigma$-ring is a monotone class.

(ii) Every monotone ring is a $\sigma$-ring.

(iii) A nonmonotone sequence can have a limit. Take $C_{2n} \subset C_{2n+2} \subset \cdots \uparrow C$ and Take $C_{2n+1} \supset C_{2n+3} \supset \cdots \downarrow C$.

- **$\pi$-class.** A nonempty class $\mathcal{C}$ of subsets of $\mathcal{X}$ closed under the formation of intersections.

- **$\lambda$-class (also called $d$-system).** A nonempty class $\mathcal{C}$ of subsets of $\mathcal{X}$ containing $\mathcal{X}$ and satisfying
  
  (a) $A, B \in \mathcal{C}$ with $A \subseteq B$ implies that the difference $B \setminus A \in \mathcal{C}$.
  
  (b) For an increasing sequence of sets $C_i \in \mathcal{C}$, we have $\lim_n C_n \in \mathcal{C}$.

FACTS:

(i) A $\lambda$-class is closed under the formation of complements and of finite and countable disjoint unions.

(ii) Similar events. Let $\{P_\theta, \theta \in \Theta\}$ be a family of probability distributions. An event $A$ is “similar” if $P_\theta(A)$ is independent of $\theta \in \Theta$. The class of similar events forms a $\lambda$-class.

- **Semialgebra, or semifield.** A semiring containing $\mathcal{X}$. 

• **Algebra, or field.** A nonempty class $C$ of subsets of $X$ closed under the formation of complements and unions:
  
  (a) For $A \in C$, the complement $A^c$ is in $C$.
  (b) $A, B \in C$ implies $A \cup B \in C$.

**FACTS:**

(i) An algebra is a ring containing $X$.

(ii) Every monotone class is an algebra.

• **$\sigma$-algebra, or $\sigma$-field.** A nonempty class $C$ of subsets of $X$ closed under the formation of complements and countable unions.

**FACTS:**

(i) A $\sigma$-algebra is a $\sigma$-ring containing $X$.

(ii) A $\sigma$-algebra is a special case of $\lambda$-class: namely, a $\lambda$-class closed under formation of pairwise intersections. Equivalently, removing the “disjoint” restriction in the definition of $\lambda$-class yields that of a $\sigma$-algebra.
  
  * Permutation tests (including rank tests) have critical regions belong to the $\lambda$-class of similar events. Critical regions of rank tests, however, constitute a $\sigma$-algebra.
  
  * Probability functions are well-defined on $\lambda$-classes. Conditional probability functions and expectations, however, are defined only relative to specified $\sigma$-algebras.
  
  * Example when it is straightforward to define probability on a $\lambda$-class but not on its extension to a $\sigma$-algebra. Let event $E$ occur with probability $1/2$, and let event $F$ occur independently of $E$. Let $A$ occur if $E$ occurs, and let $B$ occur if $(E \cap F) \cup (E^c \cap F^c)$ occurs. Then $P(A) = P(B) = 1/2$. But $P(A \cap B)$ is indeterminate from the given experiment.

(iii) A class that is both a $\pi$-class and a $\lambda$-class is a $\sigma$-algebra.

(iv) A $\sigma$-algebra is closed under countable intersections.

(v) An algebra is a $\sigma$-algebra if and only if it is a monotone class.

• **Generated $\sigma$-algebras.** For a class $\mathcal{E}$ of subsets of $X$, there exists a smallest (or minimal) $\sigma$-algebra containing $\mathcal{E}$, denoted $\sigma(\mathcal{E})$. It is also called the $\sigma$-algebra generated by $\mathcal{E}$.

• **The Borel sets of $\mathbb{R}$.** Start with the semiring

$$\mathcal{E}_0 = \{[a, b) : -\infty < a < b < \infty\},$$

the class of finite, left-closed, and right-open intervals in the real line. The algebra $\mathcal{E}$ generated by $\mathcal{E}_0$ consists of all finite unions of disjoint sets of the form $[a, b), (-\infty, a)$, or $[b, \infty)$. The $\sigma$-algebra $\sigma(\mathcal{E}_0) = \sigma(\mathcal{E})$ generated by $\mathcal{E}$ (or $\mathcal{E}_0$) is called the Borel sets in $\mathbb{R}$.

The semiring of left-open, right-closed intervals also generates the Borel sets. In fact, quite simply, the smallest $\sigma$-algebra containing the class of infinite intervals $\{(-\infty, a) : -\infty < a < \infty\}$ is the class of Borel sets.

Note that, using the above-noted fact that an algebra is a $\sigma$-algebra if and only if it is a monotone class, one can pass from the algebra $\mathcal{E}$ to $\sigma(\mathcal{E})$ by means of limiting processes.

To determine a probability on the Borel sets $\sigma(\mathcal{E}_0)$, it suffices to specify its values simply on the intervals $[a, b)$.

This is because any such specification has a unique extension to a
probability measure on all the Borel sets. In the case of continuous probability distributions, however, extensions beyond this class of sets need not be unique. Thus the Borel sets represent a convenient level of generality for probability modeling.

• **The Borel sets of** $\mathbb{R}^k$. Similarly defined, replacing intervals by rectangles

$$[\mathbf{x} = (x_1, \ldots, x_k) : a_i \leq x_i < b_i, i = 1, \ldots, k]$$

(sometimes called hyperrectangles) in the definition of $\sigma(\mathcal{E}_0)$. Indeed, the (countable) class of rectangles with $a_i$ and $b_i$ all rational generates the Borel sets of $\mathbb{R}^k$.

– RJS, 1/28/2012