A Gini Autocovariance Function for Time Series Modeling

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Abstract

In stationary time series modeling, the autocovariance function (ACV) through its associated autocorrelation function provides an appealing description of the dependence structure but presupposes finite second moments. Here we provide an alternative, the Gini autocovariance function (Gini ACV), which captures some key features of the usual ACV while requiring only first moments. For fitting autoregressive, moving average, and ARMA models under just first order assumptions, we derive equations based on the Gini ACV instead of the usual ACV. As another application, we treat a nonlinear autoregressive (Pareto) model allowing heavy tails and obtain via the Gini ACV an explicit correlational analysis in terms of model parameters, whereas the usual ACV even when defined is not available in explicit form. Finally, we formulate a sample Gini ACV that is straightforward to evaluate.

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1 Introduction

In stationary time series modeling, the autocovariance function (ACV) through its associated autocorrelation function (ACF) provides a straightforward and appealing description of the dependence structure, when second moments are finite. Without such an assumption, the ACV is no longer well-defined as a population entity, although sample versions can still be used fruitfully. Here we provide an alternative, the Gini autocovariance function (Gini ACV), which captures some key features of the usual ACV while imposing finiteness only of first moments. While thus serving heavy tailed time series modeling, the Gini ACV complements the usual ACV when standard second order assumptions holds and for some models is even more tractable. For Gaussian models, the Gini autocorrelation function (Gini ACF) and the usual ACF coincide. For fitting autoregressive, moving average, and ARMA models under just first order assumptions, we derive equations based on the Gini ACV instead of the usual ACV. For an important nonlinear autoregressive (Pareto) model defined under a range of moment assumptions including the heavy tailed case, we obtain via the Gini ACV an explicit correlational analysis in terms of model parameters, whereas the usual ACV even when defined is not explicitly available. Finally, we formulate a straightforward sample Gini ACV. (Our treatment extends in part a preliminary paper, Serfling, 2010, also cited in Shelef and Schechtman, 2011, who formulate a Gini partial ACF and develop inference procedures exploiting the two separate Gini autocovariances of each lag.) Our setting for the present paper is that of a strictly stationary time series with continuous marginal distribution $F$.

Just as the usual ACV is based on the usual covariance, the Gini ACV adapts to the time series setting the “Gini covariance” of Schechtman and Yitzhaki (1987), a measure well-defined under just first moment assumptions. Specifically, for $X$ and $Y$ jointly distributed with finite first moments, there are two associated Gini covariances, $\beta(X,Y) = 4\text{Cov}(X,F_Y(Y))$ and $\beta(Y,X) = 4\text{Cov}(Y,F_X(X))$, each involving the usual covariance of one of the variables with the rank of the other. As discussed in Yitzhaki and Olkin (1991), these compromise between Pearson covariance $\text{Cov}(X,Y)$ and the Spearman version $\text{Cov}(F_X(X),F_Y(Y))$. Special applications of Gini covariance are developed in Olkin and Yitzhaki (1992), Xu, Hung, Niranjan, and Shen (2010), and Yitzhaki and Schechtman (2013).

As other relevant background, we mention the connection with L-moments and L-comoments. Hosking (1990) extended the Gini mean difference into a complete series of descriptive measures of all orders, called $L$-moments, which measure spread, skewness, kurtosis, etc., just as do the central moments, but under merely a first moment assumption. The sample L-moments have less sensitivity to extreme data values than the sample central moments. Serfling
and Xiao (2007) extended L-moments to the multivariate case, defining L-
comoments and L-comoment matrices, measuring L-covariance, L-coskewness,
L-cokurtosis, etc., pairwise across the variables, again under first moment
assumptions. The second order L-comoments are, in fact, the Gini covariances.

In Section 2 we formulate the Gini ACV and Gini ACF as \( \gamma^{(G)}(k) = \beta(X_{1+k}, X_1) \) and \( \rho^{(G)}(k) = \gamma^{(G)}(k)/\gamma^{(G)}(0), \) \( k = 0, \pm 1, \pm 2, \ldots \), respectively. Here we also provide relevant technical background on Gini mean difference
and Gini correlation.

Section 3 treats the Gini ACV for linear time series models. Corresponding
Gini cross-correlations are introduced and their key properties developed. For
fitting AR, MA, and ARMA models, systems of equations based on the Gini
ACV are derived. For example, for the zero mean AR(1) process, i.e., \( X_{t+1} = \phi X_t + \xi_t \), we obtain the equation \( \phi = \gamma^{(G)}(+1)/\gamma^{(G)}(0) \), paralleling the equation
based on the usual ACV, but using the first order Gini quantities.

Section 4 applies the Gini ACV to a nonlinear autoregressive time series
model of Pareto type introduced by Yeh, Arnold, and Robertson (1988) and
treated recently by Ferreira (2012). For this model, no closed form expressions
exist for the usual ACF when it is defined under second moment assumptions.
However, closed form expressions for the lag \( \pm 1 \) Gini autocorrelations in terms
of model parameters are obtained, even under first order assumptions. These
facilitate an explicit correlational type representation of dependence structure
for this model and show how that structure changes as a function of model
parameters.

Section 5 formulates a straightforward sample Gini ACV. This supports
useful nonparametric exploratory analysis of a time series without assuming a
particular type of model or assuming second order conditions.

We mention several goals for further work. It is desirable to develop robust
versions of the sample Gini ACV, especially for use with heavy tailed data
arising either as outliers or as innovations. Efficient numerical algorithms
are desired for the nonlinear “Gini” systems of equations for fitting MA(\( q \))
and ARMA(\( p, q \)) models. A standard approach with the usual Yule-Walker
estimates is to recursively fit AR models using the Durbin-Levinson algorithm,
and extension to the Gini-Yule-Walker estimates is desired. It is of interest to
pursue spectral analysis based on the Gini ACV.

Throughout the paper, well-known facts are invoked as needed without
citing particular sources. In such cases, suitable sources are Box and Jenkins
(1976) and Brockwell and Davis (1991).

Finally, for added perspective, we note that several tools already exist for
treating time series under first order (or lower) moment assumptions. For
example, a variant of the usual ACV having a sample version that estimates
it consistently as the sample length increases has a long history (Davis and
1997), although this function lacks a straightforward interpretation and has other limitations. For other approaches, see Peng and Yao (2003), Yang and Zhang (2008), and Chen, Li, and Wu (2012). In Section 3.3.4 below, we also discuss LAD approaches for fitting AR models under minimal moment assumptions. The Gini ACV does not replace any of the diverse existing approaches but rather complements them with an attractive new tool.

2 The Gini autocovariance function

The formulation of our Gini autocovariance function (Gini ACV) draws upon the familiar Gini mean difference and the less familiar Gini covariance. We first introduce these with some relevant details and then proceed to the Gini ACV.

2.1 The Gini mean difference

For a random variable $X$ with distribution $F$, an alternative to the standard deviation as a spread measure was introduced by Gini (1912):

$$
\alpha(X) = E|X_1 - X_2| = E(X_{2:2} - X_{1:2}),
$$

(1)

with $X_{1:2} \leq X_{2:2}$ the ordered values of independent conceptual observations $X_1$ and $X_2$ having distribution $F$. Now known as the Gini mean difference (GMD), $\alpha(X)$ is finite if $F$ has finite mean. An important representation,

$$
\alpha(X) = 2\text{Cov}(X, 2F(X) - 1) = 4\text{Cov}(X, F(X)),
$$

(2)

facilitates an illuminating interpretation: $\alpha(X)$ is 4 times the covariance of $X$ and its “rank” in the distribution $F$, or, more precisely, twice the covariance of $X$ and the classical centered rank function $2F(X) - 1$.

The GMD may also be expressed as an $L$-functional (weighted integral of quantiles),

$$
\alpha(X) = 2\int_0^1 F^{-1}(u) (2u - 1) du,
$$

with $F^{-1}(u) = \inf \{x : F(x) \geq u\}$, $0 < u < 1$, the usual quantile function of $F$. This representation, as well as (1), yields still another useful expression,

$$
\alpha(X) = 2\int x (2F(x) - 1) dF(x),
$$

(3)

which defines an estimator of $\alpha(X)$ by substitution of a sample version of $F$.

For elaboration of the above details, see Hosking (1990) and Serfling and Xiao (2007).
2.2 The Gini covariance

For a bivariate random vector \((X, Y)\) with joint distribution \(F_{X,Y}\) and marginal distributions \(F_X\) and \(F_Y\) having finite means, the so-called “Gini covariance” introduced by Schechtman and Yitzhaki (1987) has two components,

\[
\beta(X, Y) = 2 \text{Cov}(X, 2F_Y(Y) - 1) = 4 \text{Cov}(X, F_Y(Y))
\]

\[
\beta(Y, X) = 2 \text{Cov}(Y, 2F_X(X) - 1) = 4 \text{Cov}(Y, F_X(X)),
\]

the Gini covariances of \(X\) with respect to \(Y\) and of \(Y\) with respect to \(X\), respectively. Note that \(\beta(X, Y)\) is proportional to the covariance of \(X\) and the \(F_Y\)-rank of \(Y\), and \(\beta(Y, X)\) to the covariance of \(Y\) and the \(F_X\)-rank of \(X\). Thus \(\beta(Y, X)\) and \(\beta(X, Y)\) need not be equal, and, when not equal, provide complementary pieces of information about the dependence of \(X\) and \(Y\). The definitions (4) and (5) parallel the representation (2) for \(\alpha(X)\) and reduce to it when \(X = Y\). Also, paralleling (1), we have

\[
\beta(X, Y) = E(X_{[2:2]} - X_{[1:2]}),
\]

where \((X_1, Y_1)\) and \((X_2, Y_2)\) are independent observations on \(F\), and, for \(i = 1, 2\), \(X_{[i:2]}\) denotes the \(X\)-value or “concomitant” matched with \(Y_{i:2}\), with \(Y_{1:2} \leq Y_{2:2}\) the ordered values of \(Y_1\) and \(Y_2\) (see David and Nagaraja, 2003). Further, paralleling (3), we have

\[
\beta(X, Y) = 2 \int \int x (2F_Y(y) - 1) dF(x, y),
\]

facilitating estimation by substitution of appropriate sample distribution functions.

Although \(\beta(X, Y)\) and \(\beta(Y, X)\) are equal for exchangeable \((X, Y)\), this does not hold in general. Such potential asymmetry may at first seem surprising and even unwanted. However, classical higher order extensions of covariance, i.e., the “comoments” introduced by Rubinstein (1973), (including coskewness, cokurtosis, etc.) in the finance setting, are all quite naturally asymmetric. Symmetry of the classical covariance is thus an exception to the general rule. Such asymmetry is also characteristic (even in the second order case) for the recently introduced “L-comoments” of Serfling and Xiao (2007), which parallel the classical central comoments while requiring moment assumptions only of first order instead of increasingly higher order as the order of the comoment increases. The second order L-comoment happens to be the Gini covariance, whose two components thus provide separate pieces of information on dependence. From these one also may craft various symmetric measures if so desired (see Yitzhaki and Olkin, 1991, for further discussion).
A scale-free \textit{Gini correlation} (of \(X\) with respect to \(Y\)) is given by
\[
\rho^{(G)}(X, Y) = \frac{\beta(X, Y)}{\alpha(X)} = \frac{\text{Cov}(X, F_Y(Y))}{\text{Cov}(X, F_X(X))},
\]
and the companion \(\rho^{(G)}(Y, X)\) is defined analogously. These compare with the usual Pearson correlation \(\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}\) defined under second order moment assumptions. In particular, \(\rho^{(G)}(X, Y)\) and \(\rho^{(G)}(Y, X)\) both take values in the interval \([-1, +1]\). Comparatively, \(\rho(X, Y)\) measures the degree of \textit{linearity} in the relationship between \(X\) and \(Y\) and takes values \(\pm 1\) if and only if \(X\) is a linear function of \(Y\), whereas \(\rho^{(G)}(X, Y)\) measures the degree of \textit{monotonicity} and takes values \(\pm 1\) if and only if \(X\) is a monotone function of \(Y\). They coincide under some conditions which are fulfilled for bivariate normal distributions and for certain bivariate Pareto distributions, for example.

For elaboration of the above details, see Schechtman and Yitzhaki (1987) and Serfling and Xiao (2007).

### 2.3 A Gini autocovariance function

Consider a strictly stationary stochastic process \(\{X_t\}\). When the variance is finite, a standard tool is the \textit{autocovariance function}, consisting of the lag \(k\) covariances
\[
\gamma(k) = \text{Cov}(X_{1+k}, X_1), \quad k = 0, \pm 1, \pm 2, \ldots.
\]
Of course, \(\gamma(-k)\) and \(\gamma(+k)\) are equal. Here, however, we assume only first order moments and introduce the \textit{Gini autocovariance function (Gini ACV)}. For each time \(t\) and each lag \(k \geq 1\), there are two Gini covariances of lag \(k\), \(\beta(X_{t+k}, X_t)\) and \(\beta(X_t, X_{t+k})\). By the stationarity assumption, \(\beta(X_{t+k}, X_t) = \beta(X_{1+k}, X_1)\) and \(\beta(X_t, X_{t+k}) = \beta(X_1, X_{1+k})\), for each \(t = 0, \pm 1, \pm 2, \ldots\), and also \(\beta(X_1, X_{1+k}) = \beta(X_{1-k}, X_1)\), each \(k \geq 1\). Consequently, the Gini covariance structure of the time series \(\{X_t\}\) may be characterized succinctly by the \textit{Gini autocovariance function (Gini ACV)}
\[
\gamma^{(G)}(k) = \beta(X_{1+k}, X_1), \quad k = 0, \pm 1, \pm 2, \ldots.
\]
For \(k = 0\), we have \(\gamma^{(G)}(0) = \alpha(X) = \beta(X, X)\), the GMD of \(F\). For lag \(k \neq 0\), \(\gamma^{(G)}(\pm |k|)\) and \(\gamma^{(G)}(\mp |k|)\) provide two measures of dependence which are not necessarily equal, namely, with factors of 4, the covariance between an observation and the rank of the lag \(k\) previous and future observations, respectively. On this basis, directly facilitating practical interpretations, we exhibit the Gini ACV \(\gamma^{(G)}(9)\) in terms of two component functions, each indexed by \(k \geq 0\):
\[
\gamma^{(A)}(k) = \beta(X_{1+k}, X_1), \quad k = 0, 1, 2, \ldots,
\]
and

$$\gamma^{(B)}(k) = \beta(X_1, X_{1+k}), \quad k = 0, 1, 2, \ldots,$$

although for theoretical treatments the function $$\gamma^{(G)}(k)$$ as a whole is more convenient. Corresponding Gini autocorrelation functions (Gini ACFs) are given by $$\rho^{(A)}(k) = \gamma^{(A)}(k)/\gamma^{(G)}(0)$$ and $$\rho^{(B)}(k) = \gamma^{(B)}(k)/\gamma^{(G)}(0)$$.

3 The Gini ACV for Linear Models

Let us consider the linear process $$\{X_t\}$$ generated by a linear filter applied to a series of independent shocks or innovations,

$$X_t = \sum_{i=0}^{\infty} \psi_i \xi_{t-i}, \quad (10)$$

where the $$\xi_t$$ are IID with mean 0, $$\psi_0 = 1$$ and $$\sum_{i=0}^{\infty} |\psi_i| < \infty$$. Under second order assumptions, the $$\xi_t$$ have finite variance $$\sigma^2_{\xi}$$. For the process $$\{X_t\}$$ in our heavy-tailed setting not requiring finite variance, we have the following key result.

**Lemma 1** For any sequence of random variables $$\{\xi_t\}$$ such that $$\sup_t E|\xi_t| < \infty$$, and for any sequence of constants $$\{\psi_i\}$$ such that $$\sum_{i=0}^{\infty} |\psi_i| < \infty$$, the series $$\sum_{i=0}^{\infty} \psi_i \xi_{t-i}$$ converges absolutely with probability 1.

This is just the first statement of Proposition 3.1.1 of Brockwell and Davis (1991). The assumption $$\sup_t E|\xi_t| < \infty$$ is implied by our stationarity and first order assumptions.

Some widely used linear models have only finitely many parameters $$\psi_i$$, and under second order assumptions these parameters may be represented in terms of the usual ACV and thus may be estimated via the sample ACV. Here, requiring only first order assumptions, we obtain alternative methods based on the Gini ACV, thus supporting parameter estimation via the sample Gini ACV (which we introduce in Section 5). In particular, below we obtain Gini equations for model parameters in three important cases: moving average (MA), autoregressive (AR), and ARMA models. For this purpose, we first develop some results on Gini cross-covariances for linear models in general.

3.1 Gini cross-covariances

A useful quantity under second order assumptions is the $$x\xi$$ cross-covariance of lag $$k$$ defined as $$\gamma_{x\xi}(k) = \text{Cov}(X_{t+k}, \xi_t)$$. It is straightforward that

$$\gamma_{x\xi}(k) = \text{Cov}(X_{t+k}, \xi_t) = \text{Cov} \left( \sum_{i=0}^{\infty} \psi_i \xi_{t+k-i}, \xi_t \right) = \begin{cases} 0, & k < 0 \\ \psi_k \sigma^2_{\xi}, & k \geq 0. \end{cases} \quad (11)$$
Reversing the roles of \( \xi \) and \( x \), a \( \xi x \) cross-covariance of lag \( k \) is also defined but, noting that \( \gamma_{\xi x}(k) = \gamma_{x \xi}(-k) \), yields nothing new.

Under first order assumptions, we take \( \gamma_{x \xi}^{(G)}(k) = \beta(X_{t+k}, \xi_t) \) as the Gini \( x \xi \) cross-covariance of lag \( k \) and \( \gamma_{x \xi}^{(G)}(k) = \beta(\xi_{t+k}, X_t) \) as the Gini \( \xi x \) cross-covariance of lag \( k \). The first yields a striking parallel to (11):

\[
\gamma_{x \xi}^{(G)}(k) = \beta(X_{t+k}, \xi_t) = \beta \left( \sum_{i=0}^{\infty} \psi_i \xi_{t+k-i}, \xi_t \right) = \begin{cases} 
0, & k < 0 \\
\psi_k \alpha(\xi), & k \geq 0.
\end{cases} \quad (12)
\]

The second yields a useful expression for the Gini ACV:

\[
\gamma^{(G)}(k) = \beta(X_{1+k}, X_1) = \beta \left( \sum_{i=0}^{\infty} \psi_i \xi_{1+k-i}, X_1 \right) = \sum_{i=\max\{0,k\}}^{\infty} \psi_i \gamma_{x \xi}^{(G)}(k-i) \quad (13)
\]

Let us now suppose further that the linear stationary model is invertible, yielding

\[
\xi_t = \sum_{i=0}^{\infty} \pi_i X_{t-i}, \quad (14)
\]

with \( \pi_0 = 1 \) and \( \sum |\pi_i| < \infty \). The coefficients \( \{\pi_i\} \) and \( \{\psi_i\} \) are related via \( \pi(z) = \psi^{-1}(z) \), where \( \pi(z) = \sum_{i=0}^{\infty} \pi_i z^i \) and \( \psi(z) = \sum_{i=0}^{\infty} \psi_i z^i \), provided that \( \pi(z) \) and \( \psi(z) \) have no common zeros. If follows that the \( \{\pi_i\} \) may be obtained recursively from the \( \{\psi_i\} \) via \( \pi_0 = \psi_0 = 1 \) and

\[
\pi_i = -\left( \psi_i + \pi_1 \psi_{i-1} + \cdots + \pi_{i-1} \psi_1 \right), \quad i \geq 1, \quad (15)
\]

yielding \( \pi_1 = -\psi_1, \pi_2 = \psi_2^2 - \psi_2, \pi_3 = -\psi_3^2 + 2\psi_1 \psi_2 - \psi_3, \) etc. In the second order case, using (14), we obtain for the usual \( \xi x \) cross-covariances convenient expressions in terms of \( \{\pi_i\} \) and the usual ACV:

\[
\gamma_{\xi x}(k) = \gamma_{x \xi}(-k) = \text{Cov}(\xi_{t+k}, X_t) = \begin{cases} 
0, & k > 0 \\
\sum_{i=0}^{\infty} \pi_i \gamma(k-i), & k \leq 0.
\end{cases} \quad (16)
\]

For the first order case, similar steps yield a striking Gini analogue:

\[
\gamma_{\xi x}^{(G)}(k) = \beta(\xi_{t+k}, X_t) = \begin{cases} 
0, & k > 0 \\
\sum_{i=0}^{\infty} \pi_i \gamma^{(G)}(k-i), & k \leq 0.
\end{cases} \quad (17)
\]

### 3.2 Moving Average Processes

We now consider the invertible MA\((q)\) model given by

\[
X_t = \xi_t + \theta_1 \xi_{t-1} + \cdots + \theta_q \xi_{t-q} \quad (18)
\]
for some choice of \( q \geq 1 \) and \( \xi_t \) IID with mean 0. Assuming finite variances, the usual ACV has the representation

\[
\gamma(k) = \begin{cases} 
\sigma^2 \sum_{i=0}^{q-|k|} \theta_i \theta_{i+|k|} & |k| \leq q, \\
0, & |k| > q.
\end{cases}
\]

Under first order assumptions, the above fails but by (13) we obtain a Gini analogue:

\[
\gamma^{(G)}(k) = \begin{cases} 
\sum_{i=\max\{0,k\}}^{q} \theta_i \gamma^{(G)}(k-i) & |k| \leq q, \\
0, & |k| > q.
\end{cases}
\]

Here we have used the facts that \( X_{1+k} \) and \( X_1 \) are independent for \( |k| > q \) and that \( \xi_{1+k} \) and \( \xi_1 \) are independent for \( k > 0 \) and \( k < -q \).

3.2.1 Solving for \( \theta_1, \ldots, \theta_q \) in terms of Gini autocovariances

We start with equations (20), which more explicitly may be written

\[
\begin{align*}
\gamma^{(G)}(0) & \equiv (1) \gamma^{(G)}(0) + \theta_1 \gamma^{(G)}(-1) + \theta_2 \gamma^{(G)}(-2) + \cdots + \theta_q \gamma^{(G)}(-q) \\
\gamma^{(G)}(1) & \equiv (2) \theta_1 \gamma^{(G)}(0) + \theta_2 \gamma^{(G)}(-1) + \cdots + \theta_q \gamma^{(G)}(-q+1) \\
\gamma^{(G)}(2) & \equiv (3) \theta_2 \gamma^{(G)}(0) + \cdots + \theta_q \gamma^{(G)}(-q+2) \\
& \vdots \\
\gamma^{(G)}(q) & \equiv (q+1) \theta_q \gamma^{(G)}(0).
\end{align*}
\]

From the \((q+1)\)th equation, we obtain \( \gamma^{(G)}(0) = \theta_q^{-1} \gamma^{(G)}(q) \) and substitute this into each of the \( q \) preceding equations. Also, in these \( q \) equations, we substitute for the quantities \( \gamma^{(G)}(\cdot) \) using (15) and further reduce using (17).

This leads to a nonlinear system of \( q \) equations for the \( q \) quantities \( \theta_1, \ldots, \theta_q \) in terms of the Gini autocovariances \( \gamma^{(G)}(0), \ldots, \gamma^{(G)}(q) \).

**Illustration for \( q = 1 \).** We have \( \gamma^{(G)}(0) \equiv \gamma^{(G)}(0) + \theta_1 \gamma^{(G)}(-1) \) and \( \gamma^{(G)}(1) \equiv \theta_1 \gamma^{(G)}(0) \). From (2) we obtain \( \gamma^{(G)}(0) = \theta_1^{-1} \gamma^{(G)}(1) \), which substituted into (1) yields \( \gamma^{(G)}(0) = \theta_1^{-1} \gamma^{(G)}(1) + \theta_1 \gamma^{(G)}(-1) \). Now \( \gamma^{(G)}(-1) = \pi_0 \gamma^{(G)}(-1) = \gamma^{(G)}(-1) \), and our equation for \( \theta_1 \) now becomes

\[
\gamma^{(G)}(-1) \theta_1^2 - \gamma^{(G)}(0) \theta_1 + \gamma^{(G)}(1) = 0,
\]

a quadratic equation with solutions

\[
\theta_1 = \frac{\gamma^{(G)}(0) \pm \sqrt{\gamma^{(G)}(0)^2 - 4 \gamma^{(G)}(-1) \gamma^{(G)}(1)}}{2 \gamma^{(G)}(-1)}.
\]
In practice we adopt rules for selecting one of the two solutions, just as is done with equations in terms of the usual ACV in the strictly stationary case under second order assumptions.

**Illustration for \( q = 2 \).** We arrive at the following equations for \( \theta_1 \) and \( \theta_2 \):

\[
\begin{align*}
\gamma^{(G)}(0) &= \theta_2^{-1}\gamma^{(G)}(2) + \theta_1(\gamma^{(G)}(-1) - \theta_1\gamma^{(G)}(-2)) + \theta_2\gamma^{(G)}(-2) \\
\gamma^{(G)}(1) &= \theta_1\theta_2^{-1}\gamma^{(G)}(2) + \theta_2(\gamma^{(G)}(-1) - \theta_1\gamma^{(G)}(-2)).
\end{align*}
\]

### 3.3 AR(\( p \)) Processes

Consider the AR(\( p \)) model given by

\[
X_t = \phi_1X_{t-1} + \cdots + \phi_pX_{t-p} + \xi_t \tag{21}
\]

for some choice of \( p \geq 1 \), with \( \xi_t \) IID with mean 0 and with the *causality* assumption that the process may be represented in the form (10) with \( \sum |\psi_i| < \infty \). We also assume *invertibility*, which yields the recursion

\[
\psi_j = \begin{cases} 
\sum_{0<k\leq j} \phi_k\psi_{j-k}, & 0 \leq j < p, \\
\sum_{0<k\leq p} \phi_k\psi_{j-k}, & j \geq p,
\end{cases}
\]

yielding \( \psi_0 = 1, \psi_1 = \phi_1, \psi_2 = \phi_1^2 + \phi_2 \), etc. To obtain “Yule-Walker” linear systems of \( p \) equations for \( \phi_1, \ldots, \phi_p \), in terms of either the ACV or the Gini ACV, respectively, we apply a general “covariance approach” of possible wider interest.

#### 3.3.1 A general covariance approach

Consider a linear structure

\[
\eta = \sum_{j=1}^{p} \phi_j\alpha_j + \varepsilon, \tag{22}
\]

with \( \varepsilon \) independent of \( \alpha_1, \ldots, \alpha_p \). We seek a linear system

\[
\alpha_i = \sum_{j=1}^{p} b_{ij}\phi_j, \quad i = 1, \ldots, p, \tag{23}
\]

or in matrix form

\[
\alpha = B\phi, \tag{24}
\]
with $a = (a_1, \ldots, a_p)^T$, $\phi = (\phi_1, \ldots, \phi_p)^T$, and $B = (b_{ij})_{p \times p}$ (here $M^T$ is the transpose of matrix $M$). For this we introduce the following simple covariance approach. Note that for any function $Q(\alpha_1, \ldots, \alpha_p)$ of $\alpha_1, \ldots, \alpha_p$ we have

$$\text{Cov}(\eta, Q(\alpha_1, \ldots, \alpha_p)) = \sum_{j=1}^{p} \phi_j \text{Cov}(\alpha_j, Q(\alpha_1, \ldots, \alpha_p)), \quad (25)$$

provided that these covariances are finite. Any choice of the $p$ functions $Q_i(\alpha_1, \ldots, \alpha_p)$, $1 \leq i \leq p$, in (25) yields a linear system of form (24), with

$$a_i = \text{Cov}(\eta, Q_i(\alpha_1, \ldots, \alpha_p)), \quad 1 \leq i \leq p,$$

$$b_{ij} = \text{Cov}(\xi_j, Q_i(\alpha_1, \ldots, \alpha_p)), \quad 1 \leq i, j \leq p.$$ 

If, for some function $g$, we choose these $p$ functions to be of form $Q_i(\alpha_1, \ldots, \alpha_p) = g(\alpha_i)$, $1 \leq i \leq p$, we obtain

$$a_i = \text{Cov}(\eta, g(\alpha_i)), \quad 1 \leq i \leq p, \quad (26)$$

$$b_{ij} = \text{Cov}(\alpha_j, g(\alpha_i)), \quad 1 \leq i, j \leq p. \quad (27)$$

We apply the foregoing device with $(\eta, \alpha_1, \ldots, \alpha_p) = (X_t, X_{t-1}, \ldots, X_{t-p})$, as per the AR($p$) model (21), and obtain both the usual least squares approach under second order assumptions and a Gini approach under first order assumptions.

### 3.3.2 The standard least squares method

Use $g(\alpha_i) = \alpha_i$ and take $Q_i(X_{t-1}, \ldots, X_{t-p}) = X_{t-i}$, $1 \leq i \leq p$, yielding, under second order moment assumptions, $a_i = \text{Cov}(X_t, X_{t-i}) = \gamma(i)$, $1 \leq i \leq p$, and $b_{ij} = \text{Cov}(X_{t-j}, X_{t-i}) = \gamma(|i-j|)$, $1 \leq i, j \leq p$. In this case, (24) gives the usual Yule-Walker equations for $\phi_1, \ldots, \phi_p$. For $p = 1$, this least squares solution is simply $\phi_1 = \gamma(1)/\gamma(0)$.

### 3.3.3 The Gini-Yule-Walker system

With first order assumptions and using $g(\alpha_i) = 2(2F_X(\alpha_i) - 1)$ along with $Q_i(X_{t-1}, \ldots, X_{t-p}) = 2(2F_{X_{t-i}}(X_{t-i}) - 1)$, we obtain $a_i = \beta(X_t, X_{t-i}) = \gamma^G(i)$, $1 \leq i \leq p$, and $b_{ij} = \beta(X_{t-j}, X_{t-i}) = \gamma^G(i-j)$, $1 \leq i, j \leq p$. Then (24) gives a Gini-Yule-Walker system for $\phi_1, \ldots, \phi_p$. For $p = 1$ this Gini solution is simply $\phi_1 = \gamma^G(1)/\gamma^G(0)$. The Gini-Yule-Walker system has the computational structure of the least squares Yule-Walker system but under merely first order moment assumptions.
3.3.4 Comparison of sample versions

As is well known (Maronna, Martin, and Yohai, 2006), the sample least squares estimates of AR parameters actually work well even in the presence of outliers and/or heavy tailed innovations, although there are some pitfalls in fitting AR models with heavy-tailed innovations (Feigin and Resnick, 1999). Also, the least absolute deviations (LAD) method for estimation of AR parameters (e.g., Bloomfield and Steiger, 1983) imposes minimal moment assumptions, but it does not yield closed form expressions nor simply solve a linear system. Convenient review and further LAD approaches are provided by Ling (2005).

On the other hand, the Gini estimators of AR parameters are explicit in terms of an ACV which has a model formulation even under first order moment assumptions. Further, preliminary simulation studies indicate that, in some scenarios of heavy tails and outliers, the Gini estimators perform better than the usual least squares estimators.

3.4 ARMA \((p,q)\) Processes

Including AR and MA as special cases, the ARMA\((p,q)\) model has the form

\[
X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \xi_t + \theta_1 \xi_{t-1} + \cdots + \theta_q \xi_{t-q}
\]

for some choices of \(p,q \geq 1\) and with \(\theta_0 = 1\). Assuming both causality and invertibility, the coefficients \(\{\psi_i\}\) in (10) satisfy the recursion

\[
\psi_j = \begin{cases} 
\theta_j + \sum_{0 < k \leq j} \phi_k \psi_{j-k}, & 0 \leq j < \max\{p, q + 1\}, \\
\sum_{0 < k \leq p} \phi_k \psi_{j-k}, & j \geq \max\{p, q + 1\},
\end{cases}
\]

yielding \(\psi_0 = \theta_0 = 1, \psi_1 = \theta_1 + \phi_1, \) etc.

3.4.1 Solving for \(\phi_1, \ldots, \phi_p\) and \(\theta_1, \ldots, \theta_q\) in terms of the Gini ACV

We illustrate the technique for ARMA\((1,1)\), for which \(\psi_0 = 1\) and \(\psi_j = \theta_1 \phi_1^{j-1} + \phi_1^j, j \geq 1\). Then \(\pi_0 = 1\) and \(\pi_j = (-1)^j (\theta_1 + \phi_1) \phi_1^{j-1}, j \geq 1\), yielding

\[
\gamma_{\xi x}^{(G)}(0) = \sum_{i=0}^{\infty} \pi_i \gamma_{\xi x}^{(G)}(-i) = (\theta_1 + \phi_1) \sum_{i=0}^{\infty} (-1)^i \phi_1^{i-1} \gamma_{\xi x}^{(G)}(-i).
\]

Next we derive the equations \(\gamma_{\xi x}^{(G)}(1) = (\theta_1 \gamma_{\xi x}^{(G)}(0) + \theta_1 \gamma_{\xi x}^{(G)}(0)) = \phi_1 \gamma_{\xi x}^{(G)}(1)\) and \(\gamma_{\xi x}^{(G)}(2) = \phi_1 \gamma_{\xi x}^{(G)}(1)\). The second of these yields the solution for \(\phi_1\):

\[
\phi_1 = \gamma_{\xi x}^{(G)}(2)/\gamma_{\xi x}^{(G)}(1).
\]

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Using (29), the first equation then becomes
\[
\gamma^{(G)}(1) = \phi_1 \gamma^{(G)}(0) + \theta_1(\theta_1 + \phi_1) \sum_{i=0}^{\infty} (-1)^i \phi_1^{i-1} \gamma^{(G)}(-i).
\]
(31)

Since now \(\phi_1\) is given by (30) in terms of \(\gamma^{(G)}(2)\) and \(\gamma^{(G)}(1)\), this represents a quadratic equation for \(\theta_1\) in terms of the Gini autocovariances. Since the terms in the infinite sum in (31) decrease rapidly in magnitude, only a few terms are needed.

4 The Gini ACV for a Nonlinear Process

Here we examine the Gini ACV for a nonlinear type of autoregressive process with possibly infinite variance. Yeh, Arnold, and Robertson (1988) introduce, in different notation, a nonlinear autoregressive Pareto process YARP(III)(1) given by
\[
X_t = \begin{cases} 
p^{-1/\alpha} X_{t-1}, & \text{with probability } p, \\
\min\{p^{-1/\alpha} X_{t-1}, \varepsilon_t\}, & \text{with probability } 1-p,
\end{cases}
\]
where \(0 < p < 1\), with \(\{\varepsilon_t\}\) i.i.d. from the Pareto distribution having survival function
\[
\left[1 + \left( \frac{x-\mu}{\sigma} \right)^{\alpha} \right]^{-1}, \quad x \geq 0,
\]
with the parameters \(\mu, \sigma > 0\), and \(\alpha > 0\) corresponding to location, scale, and tail index, respectively. Henceforth we set \(\mu = 0\) for convenience and denote this distribution by \(G(\sigma, \alpha)\). Only moments of order less than \(\alpha\) are finite.

It is understood that \(\varepsilon_t\) is independent of \(\{X_s, s \leq t - 1\}\). If the series is initiated at time \(t = 0\) with \(X_0\) distributed as \(G(\sigma, \alpha)\), then the series \(\{X_t, t \geq 0\}\) is strictly stationary with marginal distribution \(G(\sigma, \alpha)\). In any case, \(X_t\) converges in distribution to \(G(\sigma, \alpha)\).

The sample paths of YARP(III)(1) exhibit frequent rapid increases to peaks followed by sudden drops to lower levels (Figure 1). As recently discussed in Ferreira (2012), these processes are especially appealing for their tractability in certain respects and their straightforward asymptotically normal estimates of \(p\) and \(\alpha\).

It is intuitively evident from the definition that a YARP(III)(1) process exhibits weak dependence for \(p\) near 0 and increasing dependence as \(p \uparrow 1\). In fact, the probability that the series at time \(t\) starts afresh with a new innovation \(\varepsilon_t\) decreases with \(p\) as follows (proved in the Appendix):
\[
P(X_t = \varepsilon_t) = 1 + \frac{p \log p}{1-p}, \quad 0 < p < 1,
\]
(32)
independently of $\sigma$, $\alpha$, and $t$. In particular, we have

<table>
<thead>
<tr>
<th>$p$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X_t = \varepsilon_t)$</td>
<td>0.74</td>
<td>0.60</td>
<td>0.48</td>
<td>0.40</td>
<td>0.31</td>
<td>0.23</td>
<td>0.17</td>
<td>0.11</td>
<td>0.05</td>
</tr>
</tbody>
</table>

However, a correlation type analysis of YARP(III)(1) poses challenges. Under second order assumptions ($\alpha > 2$), the usual ACV is defined but problematic. Yeh, Arnold, and Robertson (1988) give an explicit formula for $\gamma(0)$ but only an implicit formula involving an incomplete beta integral for $\gamma(1)$ and do not treat $\gamma(k)$, $k \geq 2$. Therefore, alternative approaches have been developed to explore quantitatively the dependence features of YARP(III)(1). Thus Ferreira (2012) establishes that this process has unit upcrossing index (upcrossings of high levels do not cluster) and that its lag $k$ tail dependence coefficient (conditional probability that $X_{t+k}$ is extreme, given that $X_t$ is extreme) is $p^k$, decaying geometrically with $k$.

On the other hand, via the Gini ACV, we obtain for YARP(III)(1) a more explicit correlation type analysis and one that requires only $\alpha > 1$. That is, we give explicit closed-form expressions for $\gamma^{(G)}(0)$, $\gamma^{(G)}(-1)$, and $\gamma^{(G)}(1)$ in terms of the model parameters $\sigma$, $p$, and $\alpha$, as follows. First, as shown in Yeh, Arnold, and Robertson (1988), we have

$$EX_n = \sigma \Gamma \left(1 - \frac{1}{\alpha}\right) \Gamma \left(1 + \frac{1}{\alpha}\right) = \frac{\pi}{\alpha} \csc \left(\frac{\pi}{\alpha}\right).$$  \hspace{2cm} (33)

Since $F$ is continuous, we have $EF(X_1) = 1/2$ and hence our goal becomes evaluation of

$$\gamma^{(G)}(k) = 4 \left[ E(X_{1+k}F(X_1)) - \frac{\sigma}{2} \Gamma \left(1 - \frac{1}{\alpha}\right) \Gamma \left(1 + \frac{1}{\alpha}\right) \right],$$  \hspace{2cm} (34)

reducing the problem to evaluation of $E(X_{1+k}F(X_1))$. Deferring details of proof to the Appendix, here we discuss the solution and its interpretations relative to the parameters $\alpha$ and $p$ of YARP(III)(1). In particular, for $k = 0$ we obtain

$$\gamma^{(G)}(0) = \frac{2\sigma}{\alpha} \Gamma \left(1 - \frac{1}{\alpha}\right) \Gamma \left(1 + \frac{1}{\alpha}\right) = \frac{2\sigma}{\alpha} \frac{\pi}{\alpha} \csc \left(\frac{\pi}{\alpha}\right),$$  \hspace{2cm} (35)

which, of course, is the Gini mean difference of $X_1$ and does not depend upon the parameter $p$. For $k = \pm 1$ we obtain

$$\gamma^{(G)}(-1) = \frac{2\sigma p(1 - p^{1/\alpha})}{1 - p} \frac{\pi}{\alpha} \csc \left(\frac{\pi}{\alpha}\right),$$  \hspace{2cm} (36)

$$\gamma^{(G)}(1) = \frac{2\sigma p(p^{-1/\alpha} - 1)}{1 - p} \frac{\pi}{\alpha} \csc \left(\frac{\pi}{\alpha}\right),$$  \hspace{2cm} (37)
which depend upon all three parameters $\sigma$, $\alpha$, and $p$. Thus the lag $\pm 1$ Gini autocorrelations of YARP(III)(1) are given by

$$
\rho^{(G)}(-1) = \frac{\alpha p(1 - p^{1/\alpha})}{1 - p},
$$
(38)

$$
\rho^{(G)}(1) = \frac{\alpha p^{-1/\alpha} - 1}{1 - p} \left(= p^{-1/\alpha} \rho^{(G)}(-1) \right).
$$
(39)

These explicit functions of $p$ and $\alpha$ provide simple measures showing how the correlation type dependence in YARP(III)(1) varies with these parameters. Such information is not available from the usual ACV and ACF, which lack explicit formulas even when defined ($\alpha > 2$).

For $|k| \geq 2$, explicit formulas for $\gamma^{(G)}(k)$ have not been obtained, but an algorithm for numerical evaluation of $\gamma^{(G)}(k)$ for any $k$ and any desired $p$ and $\alpha$ has been developed, although its computational complexity increases with $k$. The algorithm requires numerical evaluation of 3 integrals for lags $\pm 1$ with a computing time of $< 1$ second, 11 integrals for lags $\pm 2$ with a computing time of about 5 minutes, 49 integrals for lags $\pm 3$ with a computing time of about 9 hours, 261 integrals for lags $\pm 4$, and 1631 integrals for lags $\pm 5$. In principle, Gini autocovariances and autocorrelations for YARP(III)(1) for lags $|k| \leq 3$ are readily available if desired and provide sufficient information on correlation type dependence for many applications.

Views of the Gini autocorrelations $\rho^{(G)}(\pm 1)$ and $\rho^{(G)}(\pm 2)$ of YARP(III)(1) as functions of $p$, $0 < p < 1$, for $\alpha = 1.1$, 1.5, 1.75, 2.0, and 2.5, are provided in Figure 2 and Figure 3, respectively. We note that $\rho^{(G)}(+1)$ changes significantly with $\alpha$ but considerably less with $p$, whereas $\rho^{(G)}(-1)$ changes very significantly with $p$ (approximately as the identity function) but considerably less with $\alpha$. Thus $\rho^{(G)}(-1)$ and $\rho^{(G)}(+1)$ provide complementary pieces of information which, taken together, nicely describe how the correlation type dependence structure of YARP(III)(1) relates to $\alpha$ and $p$. Similar comments apply to the lag $\pm 2$ Gini autocorrelations. For a comparative view of Gini autocorrelations for lags $\pm 1$, $\pm 2$, and $\pm 3$, for $\alpha = 1.5$ and $p = 0.2$, 0.5, and 0.8, see Figure 4. Clearly, the information for lags $\pm 2$ and $\pm 3$ does not greatly enhance the information supplied for the lags $\pm 1$.

Nonparametric estimation of $\gamma^{(G)}(k)$ for $k = 0, \pm 1, \ldots$ can be carried out using the sample versions of Section 6. However, model-based estimates of $\sigma$, $p$, and $\alpha$ are more efficient (straightforward, efficient estimators are provided in Ferreira, 2012). Using these in the explicit expressions or in the numerical algorithms for the Gini autocovariances, one obtains model-based estimates of $\gamma^{(G)}(k)$ conveniently up to lags $|k| \leq 3$. Empirical investigation indicates that these improve upon the nonparametric estimates, as of course they should, being based on the additional information of a parametric model. For higher
order lags, the nonparametric estimators are available and can be computed readily.

5 A Sample Gini ACV

For data \( \{X_1, X_2, \ldots, X_T\} \) from a strictly stationary stochastic process \( \{X_t\} \), we provide a sample Gini ACV based on representation of the Gini ACV in terms of relevant distribution functions and substitution of sample versions of these distribution functions.

5.1 Estimation of the marginal distribution function \( F \)

We start with the usual sample distribution function,

\[
\hat{F}_T(x) = T^{-1} \sum_{t=1}^{T} I\{X_t \leq x\}, \quad -\infty < x < \infty,
\]

with \( I(\cdot) \) the usual “indicator function” defined as \( I(A) = 1 \) or \( 0 \) according as event \( A \) holds or not. This estimator is unbiased for estimation of \( F(x) \), and, assuming that the stationary process \( \{X_i\} \) is ergodic, converges to \( F(x) \) in suitable senses such as in probability or almost surely, as \( T \to \infty \).

5.2 Estimation of \( \gamma^{(G)}(0) \)

We estimate \( \gamma^{(G)}(0) = \alpha(F) \) by the sample analogue estimator \( \alpha(\hat{F}_T) \) based on (3). It is readily checked that this yields

\[
\hat{\gamma}^{(G)}(0) = \alpha(\hat{F}_T) = 2 \int x (2\hat{F}_T(x) - 1) d\hat{F}_T(x) = \frac{2}{T^2} \sum_{t=1}^{T} (2t - T) X_{t:T},
\]

where \( X_{1:T} \leq X_{2:T} \leq \cdots \leq X_{T:T} \) denote the ordered values of the observations.

5.3 Estimation of \( \gamma^{(G)}(k), k \neq 0 \)

For \( k \neq 0 \), we use (7) to write

\[
\gamma^{(G)}(k) = 2 \int \int x (2F_{X_1}(y) - 1) dF_{X_{1+k}, X_1}(x, y).
\]

Then, for \( k \geq +1 \), let \( \hat{F}^* \) be the sample version of \( F_{X_{1+k}, X_1} \) based on the \( T-k \) lag \( k \) bivariate observations

\[
S^* = \{(X_{1+k}, X_1), (X_{2+k}, X_2), \ldots, (X_T, X_{T-k})\}.
\]
Also, for \( k \leq -1 \), let \( \hat{F}^{**} \) be the sample version of \( F_{X_{t+k}, X_t} \) based on the \( T-k \) lag \( k \) bivariate observations

\[
\mathcal{S}^{**} = \{(X_1, X_{1+k}), (X_2, X_{2+k}), \ldots, (X_{T-|k|}, X_T)\},
\]

which are the same pairs as in \( \mathcal{S}^* \) but with the components of each pair reversed in order. These sample bivariate distribution functions, along with \( \hat{F}_T \) again, yield estimators of \( \gamma^{(G)}(k) \) by substitution into (41):

\[
\hat{\gamma}_T^{(G)}(k) = 2 \int \int x(2\hat{F}_T(y) - 1) d\hat{F}_{X_{1+k}, X_1}(x, y),
\]

(42)

where \( \hat{F}_{X_{1+k}, X_1} \) denotes \( \hat{F}^* \) for \( k \geq +1 \) and \( \hat{F}^{**} \) for \( k \leq -1 \). For \( k \geq +1 \), this yields

\[
\hat{\gamma}_T^{(G)}(k) = \frac{2}{T-k} \sum_{t=1}^{T-k} (2\hat{F}_T(X_{t:T-k}) - 1) X_{[t:T-k],1,2}, \quad k \geq 1,
\]

(43)

where \( X_{t:T-k} \) is the \( t \)th ordered second component value, and \( X_{[t:T-k],1,2} \) is the first component value concomitant to the \( t \)th ordered second component value, relative to the bivariate pairs in \( \mathcal{S}^* \). Similarly, for \( k \leq -1 \), we obtain

\[
\hat{\gamma}_T^{(G)}(k) = \frac{2}{T-|k|} \sum_{t=1}^{T-|k|} (2\hat{F}_T(X_{t:T-|k|},2,1) - 1) X_{t:T-|k|}, \quad k \leq -1,
\]

(44)

where \( X_{t:T-|k|} \) is the \( t \)th ordered first component value, and \( X_{[t:T-|k|],2,1} \) is the second component value that is concomitant to the \( t \)th ordered first component value, relative to the bivariate pairs in \( \mathcal{S}^{**} \). However, relative to the set \( \mathcal{S}^* \) of bivariate pairs, this latter equation may be expressed

\[
\hat{\gamma}_T^{(G)}(k) = \frac{2}{T-|k|} \sum_{t=1}^{T-|k|} (2\hat{F}_T(X_{t:T-|k|},1,2) - 1) X_{t:T-|k|}, \quad k \leq -1,
\]

(45)

where \( X_{t:T-|k|} \) denotes the \( t \)th ordered second component value and \( X_{[t:T-|k|],1,2} \) denotes the first component value that is concomitant to the \( t \)th ordered second component value. Thus equations (43) and (44) conveniently express \( \hat{\gamma}_T^{(G)}(k) \), \( k = \pm 1, \pm 2, \ldots, \) in a single notation and relative to the single set of bivariate pairs \( \mathcal{S}^* \).

As discussed earlier, for practical use with data it is helpful to express these as separate Gini autocovariance functions. For this purpose, we put

\[
\hat{\gamma}_T^{(A)}(k) = \frac{2}{T-k} \sum_{t=1}^{T-k} (2\hat{F}_T(X_{t:T-k}) - 1) X_{[t:T-k],1,2}, \quad k \geq 1,
\]

(46)
where $X_{t:T-k}$ is the $t$th ordered second component value, and $X_{[t:T-k],1,2}$ is the first component value concomitant to the $t$th ordered second component value, relative to the bivariate pairs in $S^*$. We view the sample Gini autocovariance function as a descriptive tool rather than an inference procedure. Therefore, asymptotic convergence theory is primarily of technical interest and as such is beyond the scope of the present paper. Moreover, in view of the considerable additional variability in heavy tailed data and modeling, asymptotic distributions would not be applicable except for enormous sample length. Rather, in lieu of asymptotic theory, a bootstrap approach is recommended for practical applications. Also, for a range of fixed sample sizes, and a range of scenarios for heavy tails and for outliers, simulation studies of the sample Gini ACV are being carried out in a separate study.

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References


Appendix

Proof of (32). Put $F(x) = F_0(x/\sigma)$, where $F_0(x) = x^\alpha/(1 + x^\alpha)$, and put $f_0(x) = F'_0(x) = \alpha x^{\alpha-1}/(1 + x^\alpha)^2$. Then it is easily checked that

$$P(X_t = \varepsilon_t) = (1 - p)\alpha \int_0^\infty \frac{x^{2\alpha-1}}{(p + x^\alpha)(1 + x^\alpha)^2} \, dx,$$

which using Mathematica yields (32).
Proofs of (35), (36), and (37). (i) For $k = 0$, we obtain $E(X_1 F(X_1))$ as follows. We have

$$E(X_1 F(X_1)) = \sigma E((X_1/\sigma) F_0(X_1/\sigma)) = \sigma \int_0^\infty x F_0(x) f_0(x) \, dx$$

$$= \sigma \alpha \int_0^\infty \frac{x^\alpha}{1 + x^\alpha} \frac{x^{\alpha-1}}{(1 + x^\alpha)^2} \, dx = \sigma \alpha \int_0^\infty \frac{x^{2\alpha}}{(1 + x^\alpha)^3} \, dx$$

$$= \sigma \alpha \frac{1}{\alpha} \frac{\Gamma(\frac{2\alpha + 1}{\alpha}) \Gamma(3 - \frac{2\alpha + 1}{\alpha})}{\Gamma(3)} = \frac{1}{2} \Gamma \left(2 + \frac{1}{\alpha}\right) \Gamma \left(1 - \frac{1}{\alpha}\right).$$

Here we have used the standard integral formula

$$\int_0^\infty \frac{u^a}{(m + u^b)^c} \, du = \frac{m^{(a + 1 - bc)/b} \Gamma(\frac{a + 1}{b}) \Gamma(c - \frac{a + 1}{b})}{\Gamma(c)}, \quad (47)$$

provided that $a > -1$, $b > 0$, $m > 0$, and $c > \frac{a+1}{b}$. Here we are applying (47) with $a = 2\alpha$, $b = \alpha$, $c = 3$, and $m = 1$, and these substitutions do satisfy the constraints provided that $\alpha > 1$, which we are assuming throughout. (Also, with the substitutions $a = b = \alpha$, $c = 2$, and $m = 1$, which too satisfy the constraints provided that $\alpha > 1$, we obtain (33).) Now returning to (34) for $k = 0$, we readily obtain

$$\gamma^{(G)}(0) = 4 \left[ E(X_1 F(X_1)) - \frac{\sigma}{2} \Gamma \left(1 - \frac{1}{\alpha}\right) \Gamma \left(1 + \frac{1}{\alpha}\right) \right]$$

$$= \frac{2\sigma}{\alpha} \Gamma \left(1 - \frac{1}{\alpha}\right) \Gamma \left(1 + \frac{1}{\alpha}\right) = \frac{2\sigma}{\alpha} \frac{\pi}{\csc \left(\frac{\pi}{\alpha}\right)}.$$

Of course, $\gamma^{(G)}(0)$ is the Gini mean difference of $X_1$.

(ii) Turning now to the case $k = -1$, we evaluate $E(X_0 F(X_1))$, which may be expressed as $\sigma E((X_0/\sigma) F_0(X_1/\sigma)) = \sigma E(X F_0(Y))$, where

$$Y = \begin{cases} p^{-1/\alpha} X, & \text{with probability } p, \\ \min\{p^{-1/\alpha} X, Z\}, & \text{with probability } 1 - p, \end{cases}$$

where $X$ and $Z$ are independent with distribution $F_0$. Letting $W$ be a Bernoulli($p$) random variable independent of $X$ and $Z$, we represent $Y$ as

$$Y = p^{-1/\alpha} X I(W = 1) + \min\{p^{-1/\alpha} X, Z\} I(W = 0)$$

and likewise $X F_0(Y)$ as

$$X F_0(Y) = X F_0(p^{-1/\alpha} X) I(W = 1) + X F_0(\min\{p^{-1/\alpha} X, Z\}) I(W = 0).$$
By the independence assumption regarding $W$,

$$E(XF_0(Y)) = pE(XF_0(p^{-1/\alpha} X)) + (1 - p)E(XF_0(\min\{p^{-1/\alpha} X, Z\})). \quad (48)$$

Note that

$$F_0(p^{-1/\alpha} x) = \frac{p^{-1}x^\alpha}{1 + p^{-1}x^\alpha} = \frac{x^\alpha}{p + x^\alpha}.$$ 

For the first expectation on the righthand side of (48), we have

$$E(XF_0(p^{-1/\alpha} X)) = \int_0^\infty xf_0(p^{-1/\alpha} x)f_0(x)\,dx = \alpha A(p, \alpha), \quad (49)$$

where

$$A(p, \alpha) = \int_0^\infty \frac{x^{2\alpha}}{(p + x^\alpha)(1 + x^\alpha)^2}\,dx.$$ 

For the second expectation on the righthand side of (48), we have

$$E(XF_0(\min\{p^{-1/\alpha} X, Z\})) = \int_0^\infty \left[\int_0^{z^{1/\alpha}} xf_0(p^{-1/\alpha} x)f_0(x)\,dx\right] f_0(z)\,dz$$

$$+ \int_0^\infty F_0(z) \left[\int_{z^{1/\alpha}}^\infty x f_0(x)\,dx\right] f_0(z)\,dz$$

$$= \alpha^2[B(p, \alpha) + C(p, \alpha)], \quad (50)$$

where

$$B(p, \alpha) = \int_0^\infty \left[\int_0^{z^{1/\alpha}} \frac{x^{2\alpha}}{(p + x^\alpha)(1 + x^\alpha)^2}\,dx\right] \frac{z^{\alpha - 1}}{(1 + z^\alpha)^2}\,dz$$

and

$$C(p, \alpha) = \int_0^\infty \left[\int_{z^{1/\alpha}}^\infty \frac{x^\alpha}{(1 + x^\alpha)^2}\,dx\right] \frac{z^{2\alpha - 1}}{(1 + z^\alpha)^3}\,dz.$$ 

We thus have

$$E(XF_0(Y)) = p\alpha A(p, \alpha) + (1 - p)^2\alpha^2(B(p, \alpha) + C(p, \alpha)). \quad (51)$$

(iii) For the case $k = +1$, we evaluate $E(F(X_0)X_1) = \sigma E(F_0(X_0/\sigma)(X_1/\sigma)) = \sigma E(F_0(X)Y)$, with $X$ and $Y$ as above. Similarly to the steps for the case $k = -1$, we obtain

$$E(F_0(X)Y) = pE(F_0(X)p^{-1/\alpha} X) + (1 - p)E(F_0(X)\min\{p^{-1/\alpha} X, Z\}). \quad (52)$$
For the first expectation on the righthand side of (52), we have from previous steps

\[ E(F_0(X)p^{-1/\alpha}X) = p^{-1/\alpha} \int_0^\infty xF_0(x)f_0(x)dx \]

\[ = \frac{p^{-1/\alpha}}{2} \Gamma \left( 2 + \frac{1}{\alpha} \right) \Gamma \left( 1 - \frac{1}{\alpha} \right) \]

\[ = \frac{p^{-1/\alpha}}{2} \left( 1 + \frac{1}{\alpha} \right) \Gamma \left( 1 + \frac{1}{\alpha} \right) \Gamma \left( 1 - \frac{1}{\alpha} \right) \]

\[ = \frac{p^{-1/\alpha}}{2} \left( 1 + \frac{1}{\alpha} \right) \frac{\pi}{\alpha} \csc \left( \frac{\pi}{\alpha} \right). \quad (53) \]

For the second expectation on the righthand side of (52), we have

\[ E(F_0(X)\min\{p^{-1/\alpha}X, Z\}) \]

\[ = \int_0^\infty \left[ \int_0^{zp^{1/\alpha}} xF_0(x)f_0(x)dx \right] f_0(z)dz \]

\[ + \int_0^\infty z \left[ \int_0^{zp^{1/\alpha}} F_0(x)f_0(x)dx \right] f_0(z)dz \]

\[ = \alpha^2 p^{-1/\alpha} D(p, \alpha) + \alpha^2 E(p, \alpha), \quad (54) \]

where

\[ D(p, \alpha) = \int_0^\infty \left[ \int_0^{zp^{1/\alpha}} \frac{x^{2\alpha}}{(1 + x^\alpha)^3} dx \right] \frac{z^{\alpha-1}}{(1 + z^\alpha)^2} dz \]

and

\[ E(p, \alpha) = \int_0^\infty \left[ \int_0^{zp^{1/\alpha}} \frac{x^{2\alpha-1}}{(1 + x^\alpha)^3} dx \right] \frac{z^\alpha}{(1 + z^\alpha)^3} dz. \]

We thus have

\[ E(F_0(X)Y) = \]

\[ \frac{p^{1-1/\alpha}}{2} \left( 1 + \frac{1}{\alpha} \right) \frac{\pi}{\alpha} \csc \left( \frac{\pi}{\alpha} \right) + (1 - p)\alpha^2(p^{-1/\alpha} D(p, \alpha) + E(p, \alpha)). \quad (55) \]

To complete the derivations of \( \gamma^{(G)}(k) \) for \( k = \pm 1 \), we need to evaluate the integrals \( A(p, \alpha), B(p, \alpha), C(p, \alpha), D(p, \alpha), \) and \( E(p, \alpha) \). With the help of
Mathematica, the following formulas are obtained:

\[
A(p, \alpha) = \frac{2\alpha^2(1 - p)[(1 - p) - \alpha p(1 - p^{1/\alpha})]}{2\alpha^3(1 - p)^3} \frac{\pi}{\alpha} \csc \left( \frac{\pi}{\alpha} \right)
\]

\[
B(p, \alpha) = \frac{\alpha[p^{-1/\alpha}(-1 + \alpha + p + \alpha p) - 1 - \alpha + p - \alpha p]}{2\alpha^3(1 - p)^3} \frac{\pi}{\alpha} \csc \left( \frac{\pi}{\alpha} \right)
\]

\[
C(p, \alpha) = \frac{\alpha[(1 - p)(1 + p^{1+1/\alpha}) - 2\alpha p(1 - p^{1/\alpha})]}{2\alpha^3(1 - p)^3} \frac{\pi}{\alpha} \csc \left( \frac{\pi}{\alpha} \right)
\]

\[
D(p, \alpha) = \frac{p[\alpha(1 - p^2) - (1 - p)^2 - 2\alpha^2 p(1 - p^{1/\alpha})]}{2\alpha^3(1 - p)^3} \frac{\pi}{\alpha} \csc \left( \frac{\pi}{\alpha} \right)
\]

\[
E(p, \alpha) = \frac{\alpha p^{-1/\alpha}[2\alpha p^2(1 - p^{1/\alpha}) + p^{1/\alpha}(1 - p)(1 - 2p - p^{1-1/\alpha})]}{2\alpha^3(1 - p)^3} \frac{\pi}{\alpha} \csc \left( \frac{\pi}{\alpha} \right).
\]

Using these in (51) and (55), and combining with (34), we obtain (36) and (37).
Figure 1: Sample paths of an YARP(III)(1) process of time length 200, for $\alpha = 1.5$ and $p = 0.2$ (upper left), $p = 0.5$ (upper right), and $p = 0.8$ (lower). Note that the vertical scales differ.
Figure 2: Lag +1 (upper curve) and Lag −1 (lower curve) Gini autocorrelations of YARP(III)(1) processes, for $\alpha = 1.1$ (upper left), 1.5 (upper right), 1.75 (middle left), 2.0 (middle right), and 2.5 (lower), and $0 < p < 1$. 
Figure 3: Lag +2 (upper curve) and Lag −2 (lower curve) Gini autocorrelations of YARP(III)(1) processes, for $\alpha = 1.1$ (upper left), 1.5 (upper right), 1.75 (middle left), 2.0 (middle right), and 2.5 (lower), and $0 < p < 1$. 
Figure 4: Gini autocorrelations of YARP(III)(1) processes, lags $k = -1, -2, -3$ (clear bars) and $+1, +2, +3$ (shaded bars), for parameters $\alpha = 1.5$ and $p = 0.2$ (top), 0.5 (middle), and 0.8 (bottom).