On Masking and Swamping Robustness of Leading Nonparametric Outlier Identifiers for Univariate Data

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Abstract

In the wide-ranging scope of modern statistical data analysis, a key task is identification of outliers. For any outlier identification procedure, one needs to know its robustness against masking (an “outlier” is undetected as such) and swamping (a “nonoutlier” is classified as an “outlier”). Masking and swamping robustness are interrelated aspects which must be studied together. For such purposes, Serfling and Wang (2014) provide a general framework applicable in any data space. Implementation, however, with particular outlier identifiers in particular types of data space, requires additional theoretical development specialized to the chosen setting. Even the case of univariate data presents nontrivial challenges. Here we apply the framework to study the masking and swamping robustness properties of two leading types of nonparametric outlier identifiers, scaled deviation outlyingness and centered rank outlyingness. The results shed new light on the choice between (Median, MAD) and (trimmed mean, trimmed standard deviation) in using scaled deviation outlyingness. Also, our findings explain how the boxplot, a leading descriptive tool, performs using a hybrid outlyingness function incorporating a quantile-based component to describe the middle half of a data set and a scaled deviation outlyingness component for outlier detection. For both goals, the boxplot greatly favors swamping robustness over masking robustness. We also formulate a variant boxplot offering a more favorable trade-off between these two criteria.

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1 Introduction

Crucial to data analysis is the identification of outliers and anomalies, which may be nuisances to be eliminated or ignored, or possibly targets of special interest. Errors to be avoided in so far as possible are “masking” of an outlier as a nonoutlier and “swamping” of a nonoutlier as an outlier. It is important to evaluate, for any outlier identification procedure, both its masking robustness and its swamping robustness, which are interrelated and trade off against each other.

Quantitative measures for such purposes, the masking breakdown point (MBP) and the swamping breakdown point (SBP), are studied here for two well-established types of univariate nonparametric outlier identifiers, scaled deviation and centered rank. Besides providing better understanding of these two important outlyingness functions, the results clarify how the design of the popular boxplot greatly favors swamping robustness over masking robustness. These insights provide a basis for designing a variant boxplot that balances somewhat more equally across these two performance characteristics.

Notions of MBP and SBP have been developed in a series of papers by Davies and Gather (1993), Becker and Gather (1999), Dang and Serfling (2010), and Serfling and Wang (2014). The latter paper provides the first broad foundational framework for coherent study of MBP and SBP for any outlier identification procedure in any data space, and the treatment includes several key general lemmas aimed at facilitating practical application of the framework. However, application of this general framework is not immediate, but in fact requires innovative further development that is specialized to the particular data setting under consideration. Even the case of univariate data space is nontrivial, and, as the first application of the general framework, is the target of the present paper.

Our setting is nonparametric outlier identification, where the bulk of the data consists of “regular” observations from a distribution $F$ that is unknown and not assumed to belong to a specified parametric family. The central goal is to characterize the outlyingness of points $x$ relative to the distribution $F$, in terms of an outlyingness function $O(x, F)$. Such a function corresponds to a global view of $x$, in comparison with the density function $f(x)$ which quantifies local probability mass at $x$. It yields a “center” (the minimum outlyingness point), a “middle half” region of the 50% least outlying points, and thresholds for selected degrees of outlyingness. The sample version $O(x, X_n)$ analogously structures a data set $X_n$. Being based on a function and thus algorithmic in its formulation, a nonparametric outlier identification procedure does not depend critically on graphical views or other subjective criteria and can be used in online data analysis and statistical learning.

Nonparametric outlier identification differs in orientation and style from
parametric outlier identification, which is oriented to a specified model for the “regular” observations, typically the normal, and has goals such as parametric model-fitting after elimination of outliers, or robust regression modeling in the normal model setting. For example, the “forward search” method (Atkinson and Riani, 2000, and Atkinson, Riani, and Cerioli, 2010) utilizes explicitly the assumed parametric model and carries out diagnostics via graphical displays that are interpreted subjectively.

In our nonparametric treatment, sample “outliers” are points with \( O(x, X_n) \) above some specified threshold \( \lambda \), and these may include both an unknown number of “regular” observations from \( F \) and some “contaminants” arising from other sources than sampling from \( F \) and typically in or toward the tail regions of the data. One goal is to detect the presence of contaminants and sort them out from the regular points. However, such contaminants can seriously disrupt the performance of \( O(x, X_n) \) as a surrogate for \( O(x, F) \), so we need \( O(x, X_n) \) to be robust against both masking and swamping, on the basis of well-defined quantitative criteria.

Our objective measures of masking and swamping robustness, the MBP and SBP, are the minimum fractions of points in \( X_n \) which, if replaced in a suitable way, cause the given procedure to mask outliers or to swarm nonoutliers, respectively. Higher MBP and SBP are better.

More precisely, for each of MBP and SBP, there are two complementary versions, Type A and Type B, making four robustness measures in all. Type A MBP measures the extent to which an extreme outlier of \( F \) can be masked in the sample as a nonoutlier at \( \lambda \) outlyingness level, while Type B MBP measures how deeply (centrally) in the sample a \( \gamma \) level outlier of \( F \) can be masked as a nonoutlier. On the other hand, Type A SBP measures how centrally a nonoutlier of \( F \) can be swamped as a \( \lambda \) level sample outlier, while Type B SBP measures the most extreme sample \( \lambda \) threshold at which a \( \gamma \) level nonoutlier of \( F \) can be swamped as a sample outlier. The Type A measures are based on a given choice of sample threshold \( \lambda \) and are thus paired together, whereas the Type B measures involve a given choice of \( F \) threshold \( \gamma \) and thus are paired together.

Unfortunately, the masking and swamping robustness of outlier identifiers cannot be inferred directly from the “ordinary” robustness properties of the various estimators that may be involved in their formulation. Rather, the notions of MBP and SBP and these four specialized measures are needed. Also, since MBP and SBP trade off against each other, these must be considered in concert.

While the breakdown point (BP) for estimators is a well-established and widely applied concept, notions of MBP and SBP are more problematic and have received only limited treatment prior to the general framework of Serfling and Wang (2014). Davies and Gather (1993) treat certain notions of Type A
MBP and Type B SBP in the univariate parametric setting of the contaminated normal model, Becker and Gather (1999) treat Type A MBP in the setting of the multivariate contaminated normal model, and Dang and Serfling (2010) treat Type A MBP in the general nonparametric multivariate setting.

Here we comprehensively treat MBP and SBP for the sample versions of two important univariate outlyingness functions: scaled deviation outlyingness

\[ \tilde{O}(x, F) = \left| \frac{x - \mu(F)}{\sigma(F)} \right|, \quad -\infty < x < \infty, \]  

(1)

with \( \mu(F) \) and \( \sigma(F) \) location and spread measures, respectively, and centered rank outlyingness

\[ O(x, F) = |2F(x) - 1|, \quad -\infty < x < \infty, \]  

(2)

each increasing as \( x \) moves outward from \( \mu(F) \) or Median(\( F \)), respectively. Scaled deviation outlyingness dates from Mosteller and Tukey (1977), while centered rank outlyingness is rooted in classical inference based on quantiles and ranks. These outlyingness measures are well suited to the nonparametric approach, since they neither require nor put to use an assumption of symmetry as is inherent in the contaminated normal parametric approach.

The results of this paper are as follows. In Section 2 we define the MBP and SBP measures and provide key lemmas instrumental in evaluating them. In Section 3 we develop and discuss MBP and SBP results for scaled deviation outlyingness and centered rank outlyingness. Our results shed new light on the choices (\( \text{Mean, SD} \)), (\( \text{Median, MAD} \)), and (\( \text{trimmed mean, trimmed SD} \)) for (\( \mu(F) \), \( \sigma(F) \)) in scaled deviation outlyingness. In Section 4 we draw upon our MBP and SBP results to explain how the boxplot works as a hybrid of both scaled deviation and centered rank outlyingness. We show that the two chief goals of the boxplot, identification of the middle half and identification of outliers, in common have associated MBP = 0.25 and SBP = 0.75. Also, we formulate a variant boxplot that provides for outlier identification a more appealing trade-off of MBP versus SBP, namely MBP = 0.33 and SBP = 0.67.

Proofs of the results in Section 3 are deferred to the Appendix, but we note that the proof techniques are innovative and of interest in themselves. They provide a needed reference point for extending our results to key multivariate outlyingness functions, the subject of a later investigation. Indeed, our MBP and SBP results not only contribute toward to a fuller understanding of (1) and (2) and their practical differences in the univariate data setting, but also are fundamental to treating the multivariate setting, with “Mahalanobis distance” and “projection” outlyingness arising as two different generalizations of the scaled deviation outlyingness (1) and “spatial” outlyingness as a generalization of the centered rank outlyingness (2).
2 Background definitions and tools

Here we provide the definitions of Type A and Type B MBP and SBP and key lemmas that aid in evaluating these measures. See Serfling and Wang (2014) for elaboration and discussion.

2.1 Robust nonparametric outlier identification

For a probability distribution $F$ on $\mathbb{R}$, an associated outlyingness function $O(x, F)$ measures for each point $x$ its “outlyingness” in the distribution $F$, with higher values for greater “outlyingness”. The point at which $O(x, F)$ is minimized represents a notion of “center” of $F$. For convenience, let $F$ be continuous and let $O(x, F)$ be normalized to satisfy $\inf_x O(x, F) = 0$ and $\sup_x O(x, F) = 1$. Based on a data set $X_n = \{X_1, \ldots, X_n\}$ from $F$, let $O(x, X_n)$ be a sample version estimating $O(x, F)$. For later reference, put $O_n^* = \inf_x O(x, X_n) (\geq 0)$ and $O_{n}^{**} = \sup_x O(x, X_n) (\leq 1)$. Here $O(x, X_n)$ is constructed in a nonparametric fashion, for example by substitution of the usual sample cdf $\hat{F}_n$ for $F$ in $O(x, F)$, rather than by parametric estimation of $F$. The scaled deviation and centered rank outlyingness functions that we consider here are of the nonparametric variety.

One role of outlyingness functions is to rank observations according to outlyingness. Another is to develop contours of equal outlyingness at various levels in order to exhibit the structure of a given data set. Here we focus on their use for “outlier identification” relative to a specified threshold $\lambda$. The “$\lambda$ outliers of $F$” belong to $out(\lambda, F) = \{x : O(x, F) > \lambda\}$, and the “sample $\lambda$ outliers” to $out(\lambda, X_n) = \{x : O(x, X_n) > \lambda\}$.

2.2 Masking and swamping robustness

We formulate four interrelated masking and swamping robustness measures. The notation $\overline{A}$ for the complement of a set $A$ will be used.

2.2.1 Masking robustness measures

Key sets concerning masking are $\mathcal{M}(\lambda, \gamma, X_n, F) = \overline{out(\lambda, X_n)} \cap \overline{out(\gamma, F)}$, for any $\lambda$ and $\gamma$. Masking of some $\gamma$ outliers of $F$ as sample $\lambda$ nonoutliers occurs if

$$\mathcal{M}(\lambda, \gamma, X_n, F) \neq \emptyset,$$

which requires $\lambda > O_n^{**}$. For fixed $\lambda$, masking becomes more severe as $\gamma \uparrow 1$, with increasingly extreme outliers of $F$ becoming masked as sample threshold $\lambda$ nonoutliers. On the other hand, for fixed $\gamma$, masking becomes more severe as $\lambda \downarrow O_n^{**}$, with threshold $\gamma$ outliers of $F$ included within increasingly central
sample nonoutlier regions. The extreme cases correspond to Type A and Type B masking breakdown, respectively.

More precisely, for fixed $k$, consider the modified data sets $X_{n,k}$ obtainable by replacing $k$ observations of $X_n$ by “contaminants”. Corresponding to the “fixed $\lambda$” and “fixed $\gamma$” cases, respectively, two indices that measure in different ways the size of the masking effect are

$$\gamma_M(\lambda, X_n, k) = \text{largest $\gamma$ for which (3) with fixed $\lambda$ holds subject to $k$ replacements}$$

and

$$\lambda_M(\gamma, X_n, k) = \text{smallest $\lambda$ for which (3) with fixed $\gamma$ holds subject to $k$ replacements}.$$ 

The quantity $\gamma_M(\lambda, X_n, k)$ is the largest outlyingness level relative to $F$ that is nonidentifiable at sample outlyingness threshold $\lambda$ due to $k$ replacements. The worst case is $\gamma_M(\lambda, X_n, k) = 1$. With $k_M^{(A)}(\lambda, X_n) = \min\{k : \gamma_M(\lambda, X_n, k) = 1\}$, the Type A Masking Breakdown Point of $O(\cdot, X_n)$ at sample outlyingness threshold $\lambda$ is then $\text{MBP}^{(A)}(\lambda, X_n) = k_M^{(A)}(\lambda, X_n)/n$.

On the other hand, the quantity $\lambda_M(\gamma, X_n, k)$ is the most central level at which a $\gamma$ outlier of $F$ can be masked due to $k$ replacements, with worst case $\lambda_M(\gamma, X_n, k) = O^*_n$. Putting $k_M^{(B)}(\gamma, X_n) = \min\{k : \lambda_M(\gamma, X_n, k) = O^*_n\}$, the Type B Masking Breakdown Point of $O(\cdot, X_n)$ at $F$ outlyingness threshold $\gamma$ is then $\text{MBP}^{(B)}(\gamma, X_n) = k_M^{(B)}(\gamma, X_n)/n$.

2.2.2 Swamping robustness measures

Key sets regarding swamping are $S(\lambda, \gamma, X_n, F) = \text{out}(\lambda, X_n) \cap \overline{\text{out}(\gamma, F)}$, for any $\lambda$ and $\gamma$, and swamping of some $\gamma$ nonoutliers of $F$ as sample threshold $\lambda$ outliers occurs if

$$S(\lambda, \gamma, X_n, F) \neq \emptyset,$$

which requires $\lambda < O^*_n$. For fixed $\lambda$, the swamping becomes more severe as $\gamma \downarrow 0$, with increasingly central nonoutliers of $F$ becoming included in the sample threshold $\lambda$ outlier region (Type A swamping). For fixed $\gamma$, swamping becomes more severe as $\lambda \uparrow O^*_n$, with threshold $\gamma$ nonoutliers of $F$ included within an increasingly extreme sample outlier region (Type B swamping).

Again consider modifications $X_{n,k}$ from $k$ replacements. Corresponding to the “fixed $\lambda$” and “fixed $\gamma$” cases, respectively, two indices related to extreme
instances of swamping are

\[ \gamma_S(\lambda, X_n, k) \]

= smallest \( \gamma \) for which (4) with fixed \( \lambda \) holds subject to \( k \) replacements

= \( \inf \{ \gamma > 0 : \exists \ k \text{ replacements such that } S(\lambda, \gamma, X_{n,k}, F) \neq \emptyset \} \),

and

\[ \lambda_S(\gamma, X_n, k) \]

= largest \( \lambda \) for which (4) with fixed \( \gamma \) holds subject to \( k \) replacements

= \( \sup \{ \lambda < O_n^{**} : \exists \ k \text{ replacements such that } S(\lambda, \gamma, X_{n,k}, F) \neq \emptyset \} \).

The quantity \( \gamma_S(\lambda, X_n, k) \) represents the most central level of nonoutlier of \( F \) that can be swamped at sample outlier threshold \( \lambda \) due to \( k \) replacements. The worst possible case is \( \gamma_S(\lambda, X_n, k) = 0 \). With \( k_S^{(A)}(\lambda, X_n) = \min \{ k : \gamma_S(\lambda, X_n, k) = 0 \} \), the Type A Swamping Breakdown Point of \( O(\cdot, X_n) \) at sample outlyingness threshold \( \lambda \) is \( \text{SBP}^{(A)}(\lambda, X_n) = k_S^{(A)}(\lambda, X_n)/n \).

On the other hand, \( \lambda_S(\gamma, X_n, k) \) is the most extreme sample outlyingness threshold at which a \( \gamma \) nonoutlier of \( F \) can be swamped by \( k \) replacements, with worst possible case \( \lambda_S(\gamma, X_n, k) = O_n^{**} \). Letting \( k_S^{(B)}(\gamma, X_n) = \min \{ k : \lambda_S(\gamma, X_n, k) = O_n^{**} \} \), the Type B Swamping Breakdown Point of \( O(\cdot, X_n) \) at \( F \) outlyingness threshold \( \gamma \) is \( \text{SBP}^{(B)}(\gamma, X_n) = k_S^{(B)}(\gamma, X_n)/n \).

### 2.2.3 The four masking and swamping robustness measures

In exploring a data set \( X_n \) using \( \text{out}(\lambda, X_n) \) as an estimator of \( \text{out}(\lambda, F) \) for a specified outlyingness threshold \( \lambda \), Type A MBP and SBP quite naturally go together as companion robustness measures. They focus on \( \text{out}(\lambda, X_n) \) for some \( \lambda \) and ask what is the largest (smallest) \( \gamma \) such that \( \text{out}(\gamma, F) \) can be masked (swamped) using \( \text{out}(\lambda, X_n) \). On the other hand, one might focus on \( \text{out}(\gamma, F) \) for some \( \gamma \) and ask how centrally (how far away) this outlier region can be masked (swamped) using \( \text{out}(\lambda, X_n) \). For this, the Type B MBP and SBP are companion robustness measures, with roles complementary to the Type A versions. All four of \( \text{MBP}^{(A)}(\lambda, X_n), \text{MBP}^{(B)}(\gamma, X_n), \text{SBP}^{(A)}(\lambda, X_n), \) and \( \text{SBP}^{(B)}(\gamma, X_n) \) should be evaluated in order to understand and quantify the masking and swamping robustness of \( O(\cdot, X_n) \) at the given thresholds.

### 2.3 Four key background lemmas

In Lemmas 1-4 below from Serfling and Wang (2014) we represent the above four complex breakdown points in terms of “ordinary breakdown points” based on Donoho and Huber (1983) and commonly used in practice. This reduces
the problem of evaluation of MBP and SBP to one of evaluation of the more familiar type of breakdown point, although at certain very complex and very nonordinary “inf” and “sup” type statistics.

Let us first recall the formulation of “ordinary breakdown points”. For a real-valued statistic $T(X_n)$ taking values in $[0, 1]$ or $(-\infty, +\infty)$, for example, explosion breakdown of $T(X_n)$ occurs due to replacement of $k$ points of $X_n$ if

$$\sup_{X_{n,k}} |T(X_{n,k})| = \sup_{X_{n,n}} |T(X_{n,n})| =: T^*,$$

with $X_{n,k}$ as previously. Typical values of $T^*$ are 1 or $\infty$. With $k_{\exp}(T(X_n))$ the minimum $k$ such that (5) can occur, the explosion replacement breakdown point of $T(X_n)$ is given by $RBP_{\exp}(T(X_n)) = k_{\exp}(T(X_n))/n$. Likewise, implosion breakdown of $T(X_n)$ occurs with $k$ points of $X_n$ replaced if

$$\inf_{X_{n,k}} |T(X_{n,k})| = \inf_{X_{n,n}} |T(X_{n,n})| =: T^*.$$

A typical value of $T^*$ is 0. With obvious notation, the implosion replacement breakdown point of $T(X_n)$ is given by $RBP_{\imp}(T(X_n)) = k_{\imp}(T(X_n))/n$.

In terms of the above explosion and implosion RBPs, representations for $MBP^{(A)}(\lambda, X_n)$, $MBP^{(B)}(\gamma, X_n)$, $SBP^{(A)}(\lambda, X_n)$, and $SBP^{(B)}(\gamma, X_n)$ are stated as follows.

**Lemma 1** Type A masking breakdown with replacement of $k$ sample values $(\gamma_M(\lambda, X_n, k) = 1)$ holds if and only if $\sup_{X_{n,k}} \sup_{y \in out(\lambda, X_{n,k})} O(y, F) = 1$, and hence $MBP^{(A)}(\lambda, X_n) = RBP_{\exp}\left(\sup_{y \in out(\lambda, X_n)} O(y, F)\right)$.

**Lemma 2** Type B masking breakdown with replacement of $k$ sample values $(\lambda_M(\gamma, X_n, k) = O_n^*)$ holds if and only if $\inf_{X_{n,k}} \inf_{y \in out(\gamma, F)} O(y, X_{n,k}) = O_n^*$, and hence $MBP^{(B)}(\gamma, X_n) = RBP_{\imp}\left(\inf_{y \in out(\gamma, F)} O(y, X_n)\right)$.

**Lemma 3** Type A swamping breakdown with replacement of $k$ sample values $(\gamma_S(\lambda, X_n, k) = 0)$ holds if and only if $\inf_{X_{n,k}} \inf_{y \in out(\lambda, X_{n,k})} O(y, F) = 0$, and hence $SBP^{(A)}(\lambda, X_n) = RBP_{\imp}\left(\inf_{y \in out(\lambda, X_n)} O(y, F)\right)$.

**Lemma 4** Type B swamping breakdown with replacement of $k$ sample values $(\lambda_S(\gamma, X_n, k) = O_n^{**})$ holds if and only if $\sup_{X_{n,k}} \sup_{y \in out(\gamma, F)} O(y, X_{n,k}) = O_n^{**}$, and hence $SBP^{(B)}(\gamma, X_n) = RBP_{\exp}\left(\sup_{y \in out(\gamma, F)} O(y, X_n)\right)$.

### 3 MBP and SBP Results

MBP and SBP results for scaled deviation and centered rank outlyingness are developed in Sections 3.1 and 3.2, respectively. Section 3.3 provides summary and comparison of all the results together.
3.1 MBP and SBP for scaled deviation outlyingness

We treat scaled deviation outlyingness as \( O(x, F) = \tilde{O}(x, F)/(1 + \tilde{O}(x, F)) \), taking values in \([0, 1)\), with \( \tilde{O}(x, F) \) given by (1). The sample version \( O(x, X_n) \) is similarly defined using \( \hat{\mu} = \mu(X_n) \) and \( \hat{\sigma} = \sigma(X_n) \), with \( O_n^* = 0 \) and \( O_n^{**} = 1 \). It is convenient to express Lemmas 1-4 directly in terms of \( \tilde{O}(x, F) \) and \( \tilde{O}(x, X_n) \), obtaining

\[
MBP(A; \lambda, X_n) = RBP_{\exp}\left( \sup_{y \in \text{out}(\lambda, X_n)} \tilde{O}(y, F) \right), \tag{7}
\]

\[
MBP(B; \gamma, X_n) = RBP_{\text{imp}}\left( \inf_{y \in \text{out}(\gamma, F)} \tilde{O}(y, X_n) \right), \tag{8}
\]

\[
SBP(A; \lambda, X_n) = RBP_{\text{imp}}\left( \inf_{y \in \text{out}(\lambda, X_n)} \tilde{O}(y, F) \right), \tag{9}
\]

\[
SBP(B; \gamma, X_n) = RBP_{\exp}\left( \sup_{y \in \text{out}(\gamma, F)} \tilde{O}(y, X_n) \right). \tag{10}
\]

These are treated in turn in Propositions 5-8 below.

We express \( \text{out}(\gamma, F) \) in terms of \( \tilde{O}(\cdot, F) \) and likewise \( \text{out}(\lambda, X_n) \) in terms of \( \tilde{O}(\cdot, X_n) \) via \( \text{out}(\gamma, F) = \{x : O(x, F) > \gamma\} = \{x : \tilde{O}(x, F) > \eta\} \), with \( \eta = \gamma/(1 - \gamma) \), and \( \text{out}(\lambda, X_n) = \{x : O(x, X_n) > \lambda\} = \{x : \tilde{O}(x, X_n) > \beta\} \), with \( \beta = \lambda/(1 - \lambda) \). Here \( \eta \uparrow \infty \) as \( \gamma \uparrow 1 \), \( \beta \uparrow \infty \) as \( \lambda \uparrow 1 \). Accordingly, the above inf and sup expressions involve the regions

\[
\text{out}(\gamma, F) = \left[ \mu(F) - \eta \sigma(F), \mu(F) - \eta \sigma(F) \right]
\]

\[
\text{out}(\lambda, X_n) = \left[ \mu(X_n) - \beta \sigma(X_n), \mu(X_n) - \beta \sigma(X_n) \right]
\]

and their complements.

**Proposition 5** For scaled deviation outlyingness and Type A masking,

\[
MBP^{(A)}(\lambda, X_n) = \min\{RBP_{\exp}(\hat{\mu}), RBP_{\exp}(\hat{\sigma})\}. \tag{11}
\]

**Remarks.** (a) The above result applies to robustness against outliers. (From the proof it is evident that inliers cannot cause Type A masking breakdown for scaled deviation outlyingness.)

(b) Note that \( MBP^{(A)}(\lambda, X_n) \) does not depend upon the threshold \( \lambda \).

(c) In typical cases (including the examples below), we have \( RBP_{\exp}(\hat{\mu}) \leq RBP_{\exp}(\hat{\sigma}) \), in which case simply \( MBP^{(A)}(\lambda, X_n) = RBP_{\exp}(\hat{\mu}) \).
Examples. (i) **Mean and Standard Deviation.** With $\hat{\mu} = \bar{X}$ and $\hat{\sigma} = s$, it is straightforward that $\text{RBP}_{\exp}(\hat{\mu}) = \text{RBP}_{\exp}(\hat{\sigma}) = n^{-1}$, the minimum possible, yielding $\text{MBP}^{(A)}(\lambda, X_n) = n^{-1} \approx 0$.

(ii) **Median and MAD.** With $\hat{\mu} = \text{Med}(X_n)$ and $\hat{\sigma} = \text{MAD}(X_n)$, to obtain $\text{Med} \to \infty$, for $n = 2m + 1$ we require that $m + 1$ observations $\to \infty$ and for $n = 2m$ we require that $m$ observations $\to \infty$. In either case, we have $\text{RBP}_{\exp}(\hat{\mu}) = n^{-1} \left[ \frac{n+1}{2} \right]$. Similarly, to obtain $\text{MAD} \to \infty$, for $n = 2m + 1$ we require that $m + 1$ observations $\to \infty$ (or $-\infty$) and for $n = 2m$ that $m$ observations $\to \infty$ (or $-\infty$). In either case, $\text{RBP}_{\exp}(\hat{\sigma}) = n^{-1} \left[ \frac{n+1}{2} \right]$. This yields $\text{MBP}^{(A)}(\lambda, X_n) = n^{-1} \left[ \frac{n+1}{2} \right] \approx \frac{1}{2}$.

(iii) **$\alpha$-Trimmed Mean and SD.** Let $X_{(n-2|\alpha|)}$ denote the $n - 2|\alpha|$ observations that remain after trimming away the upper $|\alpha|$ observations and the lower $|\alpha|$ observations. Then take $\hat{\mu}$ to be the mean and $\hat{\sigma}$ the standard deviation of the data set $X_{(n-2|\alpha|)}$. It is readily checked that $\text{RBP}_{\exp}(\hat{\mu}) = \text{RBP}_{\exp}(\hat{\sigma}) = (|\alpha| + 1)/n$, yielding $\text{MBP}^{(A)}(\lambda, X_n) = n^{-1}(|\alpha| + 1) \approx \alpha$. This result approaches that in (i) as $\alpha \to 0$ and that in (ii) as $\alpha \to 1/2$. □

**Proposition 6** For scaled deviation outlyingness and Type B masking,

\[
\text{MBP}^{(B)}(\gamma, X_n) = \begin{cases} 
0 & \text{if } \hat{\mu} \notin (\mu - \eta \sigma, \mu + \eta \sigma) \\
\min\{\text{RBP}_{\exp}(\hat{\mu}), \text{RBP}_{\exp}(\hat{\sigma})\} & \text{if } \hat{\mu} \in (\mu - \eta \sigma, \mu + \eta \sigma),
\end{cases}
\]

with $\eta = \gamma/(1 - \gamma)$.

**Remarks.** (a) If $\hat{\mu} \notin (\mu - \eta \sigma, \mu + \eta \sigma)$, then $\text{MBP}^{(B)}(\gamma, X_n) = 0$, meaning that Type B masking breakdown has already occurred for the data $X_n$ without any replacements. That is, outliers of $F$ at threshold $\gamma$ are masked as arbitrarily central sample nonoutliers (i.e., lying in arbitrarily small neighborhoods of $\hat{\mu}$). For a consistent estimator $\hat{\mu}$ of $\mu$, the relevant event $\hat{\mu} \notin (\mu - \eta \sigma, \mu + \eta \sigma)$ has decreasing probability as $n$ increases.

(b) In the typical situation that $\hat{\mu} \in (\mu - \eta \sigma, \mu + \eta \sigma)$, $\text{MBP}^{(B)}(\cdot, X_n)$ is nonzero, does not depend on $\gamma$, and agrees with $\text{MBP}^{(A)}(\cdot, X_n)$.

(c) Thus, overall, $\text{MBP}^{(B)}(\gamma, X_n)$ depends only weakly on $\gamma$.

(d) The above result applies to masking robustness against outliers. From the proof it is seen that inliers also can cause Type B masking breakdown, with associated breakdown points involving the implosion RBP of $\hat{\mu}$. □

**Examples.** When $\hat{\mu}(X_n) \notin (\mu - \eta \sigma, \mu + \eta \sigma)$, we have $\text{MBP}^{(B)}(\cdot, X_n) = 0$. For the case $\hat{\mu}(X_n) \in (\mu - \eta \sigma, \mu + \eta \sigma)$, we obtain the same value as $\text{MBP}^{(A)}(\cdot, X_n)$, and the same values for the examples previously considered. □
**Proposition 7**  For scaled deviation outlyingness and Type A swamping,

\[
SBP(A)(\lambda, X_n) = \begin{cases} 
0 & \text{if } \mu \not\in (\tilde{\mu} - \beta\tilde{\sigma}, \tilde{\mu} + \beta\tilde{\sigma}) \\
\text{RBP}_{\exp \rightarrow +\infty}(|\tilde{\mu}| - \beta\tilde{\sigma}) & \text{if } \mu \in (\tilde{\mu} - \beta\tilde{\sigma}, \tilde{\mu} + \beta\tilde{\sigma}),
\end{cases}
\]

with \( \beta = \lambda/(1 - \lambda) \).

**Remarks.** (a) If \( \mu \not\in (\tilde{\mu} - \beta\tilde{\sigma}, \tilde{\mu} + \beta\tilde{\sigma}) \), then \( SBP(A)(\cdot, X_n) = 0 \), meaning that Type A swamping breakdown has already occurred for the data \( X_n \) without any replacements: arbitrarily central nonoutliers result in (i) as Type A swamping breakdown has already occurred for the data \( X_n \).

(b) In the typical situation that \( \mu \in (\tilde{\mu} - \beta\tilde{\sigma}, \tilde{\mu} + \beta\tilde{\sigma}) \), \( SBP(A)(\cdot, X_n) \) has nonzero value (\( \geq \text{RBP}_{\exp}(\tilde{\mu}) \), note) that depends on \( \lambda \).

(c) The above result applies to swamping robustness against outliers. (As noted in the proof, inliers also can cause Type A swamping breakdown for scaled deviation outlyingness.) \( \Box \)

**Examples.** For the case \( \mu \in (\tilde{\mu} - \beta\tilde{\sigma}, \tilde{\mu} + \beta\tilde{\sigma}) \), we obtain the following results for the examples previously considered.

(i) **Mean and Standard Deviation.** To obtain \( |\tilde{\mu}| - \beta\tilde{\sigma} \rightarrow \infty \), we place \( k \) observations at \( x^* \) and let \( x^* \rightarrow \pm\infty \), resulting in \( \tilde{\mu} \approx \frac{k}{n}x^* \) and \( \tilde{\sigma} \approx \sqrt{n\tilde{\sigma}}(1 - \frac{k}{n})|x^*| \). Then \( |\tilde{\mu}| - \beta\tilde{\sigma} \rightarrow \infty \) if and only if \( k > \frac{\beta^2}{1 + \beta^2}n = \frac{\lambda^2}{\lambda^2 + (1 - \lambda)^2}n \), and thus \( SBP(A)(\cdot, X_n) = n^{-1}\left[\frac{\lambda^2}{\lambda^2 + (1 - \lambda)^2}n\right] \approx \frac{\lambda^2}{\lambda^2 + (1 - \lambda)^2} \).

(ii) **Median and MAD.** Similar steps as in (i) (restricting to \( \lambda < 1/2 \) when \( n \) is even) yield \( SBP(A)(\cdot, X_n) = n^{-1}\left[\frac{n+1}{2}\right] \approx \frac{1}{2} \), which we note is the same as MBP(A)(\cdot, X_n) for the Median and MAD.

(iii) **\( \alpha \)-Trimmed Mean and SD.** Similar steps as in (i) yield \( SBP(A)(\cdot, X_n) = n^{-1}\left[\frac{\lambda^2}{\lambda^2 + (1 - \lambda)^2}n - 2|n\alpha| + |n\alpha|\right] \approx \frac{\lambda^2}{\lambda^2 + (1 - \lambda)^2} (1 - 2\alpha) + \alpha \), approaching the result in (i) as \( \alpha \rightarrow 0 \) and that in (ii) as \( \alpha \rightarrow 1/2 \). \( \Box \)

**Proposition 8**  For scaled deviation outlyingness and Type B swamping,

\[
SBP(B)(\gamma, X_n) = \text{RBP}_{\exp}(\tilde{\mu} | \tilde{\sigma} = o(|\tilde{\mu}|)).
\]

**Remark.** The above result applies to swamping robustness against outliers. (However, inliers also can cause Type B swamping breakdown.) \( \Box \)

**Examples.** (i) **Mean and Standard Deviation.** For \( \tilde{\mu} \rightarrow \infty \) with \( \tilde{\sigma} = o(|\tilde{\mu}|) \), we need that all of the \( n \) data points \( \rightarrow \infty \) in a pattern with their
spread about $\hat{\mu}$ not increasing as fast as $|\hat{\mu}|$. This yields $\text{SBP}^{(B)}(\cdot, X_n) = \text{RBP}_{\exp}^{(\hat{\mu})} |\hat{\sigma} = o(|\hat{\mu}|) = 1$.

(ii) **Median and MAD.** For $n = 2m + 1$, we may take Med and $m$ other observations to $\infty$, resulting in the MAD remaining bounded. For $n = 2m$, we may take the two middle observations and $m - 1$ other observations to $\infty$, again resulting in the MAD remaining bounded. Then, in either case we obtain $\text{SBP}^{(B)}(\cdot, X_n) = \text{RBP}_{\exp}^{(\hat{\mu})} |\hat{\sigma} = o(|\hat{\mu}|) = n^{-1} \left[ \frac{n+1}{2} \right] \approx \frac{1}{2}$.

(iii) **$\alpha$-Trimmed Mean and SD.** For $\hat{\mu} \to \infty$ with $\hat{\sigma} = o(|\hat{\mu}|)$, we need to take $n - \lfloor na \rfloor$ observations to $-\infty$, yielding $\text{SBP}^{(B)}(\cdot, X_n) = \text{RBP}_{\exp}^{(\hat{\mu})} |\hat{\sigma} = o(|\hat{\mu}|) = n^{-1}(n - \lfloor na \rfloor) \approx 1 - \alpha$, which approaches the result in (i) as $\alpha \to 0$ and that in (ii) as $\alpha \to 1/2$. \hfill $\square$

### 3.2 MBP and SBP for centered rank outlyingness

We now consider the **centered rank outlyingness function** given by (2), and for the sample version we employ the usual sample df $\hat{F}_n(x) = n^{-1} \sum_{i=1}^{n} I(X_i \leq x)$ and define $O(x, X_n) = |2\hat{F}_n(x) - 1|$. It is readily checked that $O_n^{**} = 1$, whereas, since $\hat{F}_n(x)$ takes values only of the form $k/n$, $O_n^{*}$ is not strictly 0, but rather

$$O_n^{*} = \left| 2\frac{\lfloor \frac{n+1}{2} \rfloor}{n} - 1 \right| = \begin{cases} 0, & n \text{ even} \\ \frac{1}{n}, & n \text{ odd}. \end{cases}$$

We have

$$\text{out}(\gamma, F) = \{x : |2F(x) - 1| > \gamma\} = \left[ F^{-1}\left(\frac{1-\gamma}{2}\right), F^{-1}\left(\frac{1+\gamma}{2}\right) \right]$$

and

$$\text{out}(\lambda, X_n) = \left[ \hat{F}_n^{-1}\left(\frac{1-\lambda}{2}\right), \hat{F}_n^{-1}\left(\frac{1+\lambda}{2}\right) \right].$$

The following four propositions treat in turn $\text{MBP}^{(A)}(\lambda, X_n)$, $\text{MBP}^{(B)}(\gamma, X_n)$, $\text{SBP}^{(A)}(\lambda, X_n)$, and $\text{SBP}^{(B)}(\gamma, X_n)$.

**Proposition 9** For centered rank outlyingness and Type A masking,

$$\text{MBP}^{(A)}(\lambda, X_n) = n^{-1} \left[ \frac{1-\lambda}{2} \right] \approx \frac{1-\lambda}{2}.$$

**Remarks.**

(a) Note that $\text{MBP}^{(A)}(\lambda, X_n)$ depends upon the threshold $\lambda$ and decreases as $\lambda$ increases.

(b) It is seen from the proof that Type A masking breakdown is attained by replacing observations in such a way that either the $\frac{1-\lambda}{2}$ sample quantile $\to -\infty$ or the $\frac{1+\lambda}{2}$ sample quantile $\to +\infty$, i.e., by explosion breakdown of either of these sample quantiles due to outliers. (Here, inliers cannot cause Type A masking breakdown.) \hfill $\square$
Proposition 10  For centered rank outlyingness and Type B masking, MBP\(^{(B)}(\gamma, X_n)\)
\[
MBP^{(B)}(\gamma, X_n) = \begin{cases} 
O^*_n & \text{if } \hat{F}_{n}^{-1}(\frac{1}{2}) \notin (F^{-1}(\frac{1-\gamma}{2}), F^{-1}(\frac{1+\gamma}{2})) \\
\min\left\{ n^{-1} \left\lceil \frac{n+1}{2} \right\rceil - \hat{F}_{n}^{-1}(F^{-1}(\frac{1-\gamma}{2})), \hat{F}_{n}^{-1}(F^{-1}(\frac{1+\gamma}{2})) - n^{-1} \left\lfloor \frac{n+1}{2} \right\rfloor \right\} & \text{if } \hat{F}_{n}^{-1}(\frac{1}{2}) \in (F^{-1}(\frac{1-\gamma}{2}), F^{-1}(\frac{1+\gamma}{2})) 
\end{cases}
\]
\[
\approx \gamma, \text{ as } n \text{ increases.}
\]

Remarks. (a) If \(\hat{F}_{n}^{-1}(\frac{1}{2}) \notin (F^{-1}(\frac{1-\gamma}{2}), F^{-1}(\frac{1+\gamma}{2}))\), then \(MBP^{(B)}(\gamma, X_n) = 0\), meaning that Type B masking breakdown has occurred for \(X_n\) without any replacements: threshold \(\gamma\) outliers of \(F\) are masked as arbitrarily central sample nonoutliers (i.e., lying in arbitrarily small neighborhoods of \(\hat{F}_{n}^{-1}(\frac{1}{2})\)). Since \(\hat{F}_{n}^{-1}(\frac{1}{2})\) estimates \(F^{-1}(\frac{1}{2})\) consistently, the probability that \(\hat{F}_{n}^{-1}(\frac{1}{2}) \notin (F^{-1}(\frac{1-\gamma}{2}), F^{-1}(\frac{1+\gamma}{2}))\) consistently, the probability that \(\hat{F}_{n}^{-1}(\frac{1}{2}) \notin (F^{-1}(\frac{1-\gamma}{2}), F^{-1}(\frac{1+\gamma}{2}))\) decreases to 0 as \(n \uparrow \infty\).

(b) In the typical situation that \(\hat{F}_{n}^{-1}(\frac{1}{2}) \in (F^{-1}(\frac{1-\gamma}{2}), F^{-1}(\frac{1+\gamma}{2}))\), the value of \(MBP^{(B)}(\gamma, X_n)\) increases as \(\gamma\) increases, in contrast with \(MBP^{(A)}(\lambda, X_n)\) which decreases as \(\lambda\) increases.

(c) From the proof (omitted), it is seen that Type B masking breakdown is produced by replacing observations in such a way that the sample median → either the \(\frac{1-\gamma}{2}\) population quantile or the \(\frac{1+\gamma}{2}\) population quantile. For small \(\gamma\), this represents implosion breakdown of the sample median due to inliers. However, for the typical case of larger \(\gamma\), this represents a moderate form of explosion breakdown of the sample median due to outliers. As \(\gamma \uparrow 1\), this approaches extreme breakdown of the sample median with associated breakdown point 1/2. □

Proposition 11  For centered rank outlyingness and Type A swamping,
(i) If \(F^{-1}(\frac{1}{2}) \notin (\hat{F}_{n}^{-1}(\frac{1-\lambda}{2}), \hat{F}_{n}^{-1}(\frac{1+\lambda}{2}))\), then \(SBP^{(A)}(\lambda, X_n) = 0\).
(ii) Otherwise
\[
SBP^{(A)}(\lambda, X_n) = \min\left\{ 1 - n^{-1} \left\lceil \frac{1-\lambda}{2} \right\rceil - n^{-1} \left\lfloor \frac{1+\lambda}{2} \right\rfloor \right\}
\]
\[
\approx \frac{1+\lambda}{2}, \text{ as } n \text{ increases.}
\]

Remarks. (a) If \(F^{-1}(\frac{1}{2}) \notin (\hat{F}_{n}^{-1}(\frac{1-\lambda}{2}), \hat{F}_{n}^{-1}(\frac{1+\lambda}{2}))\), then \(SBP^{(A)}(\lambda, X_n) = 0\), meaning that Type A swamping breakdown has occurred for the data \(X_n\).
without any replacements: arbitrarily central nonoutliers of \( F \) (i.e., lying in arbitrarily small neighborhoods of \( F^{-1}(\frac{1}{2}) \)) are swamped as sample threshold \( \lambda \) outliers. Since \( \hat{F}_n^{-1}(\frac{1-\lambda}{2}) \) and \( \hat{F}_n^{-1}(\frac{1+\lambda}{2}) \) consistently estimate their population counterparts, the probability of this event decreases to 0 as \( n \) increases.

(b) In the typical situation that \( F^{-1}(\frac{1}{2}) \in (\hat{F}_n^{-1}(\frac{1-\lambda}{2}), \hat{F}_n^{-1}(\frac{1+\lambda}{2})) \), the value of \( \text{SBP}^{(A)}(\lambda, X_n) \) increases with \( \lambda \) up to possible value 1.

(c) From the proof (omitted), it is seen that Type A swamping breakdown due to outliers results from replacing observations in such a way that either the sample \( \frac{1-\lambda}{2} \) quantile moves arbitrarily far above the population median \( F^{-1}(\frac{1}{2}) \) or the sample \( \frac{1+\lambda}{2} \) quantile arbitrarily far below it. This requires replacement of approximately \( \frac{1+\lambda}{2} n \) observations. (On the other hand, if one moves either of these sample quantiles toward the population median \( F^{-1}(\frac{1}{2}) \) in an appropriate way, by replacement of approximately \( \frac{1}{2} n \) observations, one obtains Type A swamping breakdown due to implosion breakdown of an inner sample quantile due to inliers.)

\[ \square \]

**Proposition 12** For centered rank outlyingness and Type B swamping,

(i) If \( (F^{-1}(\frac{1-\gamma}{2}), F^{-1}(\frac{1+\gamma}{2})) \not\subset (X_{1:n}, X_{n:n}) \), then \( \text{SBP}^{(B)}(\gamma, X_n) = 0. \)

(ii) Otherwise

\[
\text{SBP}^{(B)}(\gamma, X_n) = \min \left\{ 1 - \hat{F}_n^{-1} \left( F^{-1} \left( \frac{1-\gamma}{2} \right) \right), \hat{F}_n^{-1} \left( F^{-1} \left( \frac{1+\gamma}{2} \right) \right) \right\} \\
\approx \frac{1 + \gamma}{2}, \text{ as } n \text{ increases.}
\]

Remarks. (a) If \( (F^{-1}(\frac{1-\gamma}{2}), F^{-1}(\frac{1+\gamma}{2})) \not\subset (X_{1:n}, X_{n:n}) \), then Type B swamping breakdown holds already without replacement of observations: threshold \( \gamma \) outliers of \( F \) are swamped as the most extreme sample outliers, either outside the sample cloud or at the boundary. However, this event has probability decreasing to 0 as \( n \to \infty \).

(b) Note that \( \text{SBP}^{(B)}(\gamma, X_n) \) increases with \( \gamma \) up to possible value 1.

(c) From the proof (omitted), it is seen that Type B swamping breakdown due to outliers results from replacements such that either the sample minimum exceeds the \( \frac{1-\gamma}{2} \) population quantile or the sample maximum decedes the \( \frac{1+\gamma}{2} \) population quantile, in either case requiring replacement of approximately \( \frac{1+\gamma}{2} n \) observations, representing explosion breakdown due to outliers. (Type B swamping due to inliers occurs if either the sample minimum exceeds the \( \frac{1-\gamma}{2} \) population quantile or the sample maximum decedes the \( \frac{1+\gamma}{2} \) quantile, through replacement of approximately \( \frac{1-\gamma}{2} n \) observations.)

\[ \square \]
3.3 Summary of MBP and SBP results

Masking and swamping robustness results. For our three versions of scaled deviation outlyingness and for centered rank (CR) outlyingness, the MBP and SBP values are provided below for convenient comparison. Although typically in practice the thresholds of $F$ and sample outlyingness are chosen to be equal, for complete generality here we do not constrain $\lambda$ and $\gamma$. Also, we consider only outlier replacements and for convenience display just the limits as $n \to \infty$.

<table>
<thead>
<tr>
<th></th>
<th>Mean, SD</th>
<th>Med, MAD</th>
<th>$\alpha$-trims, $\alpha &lt; 0.5$</th>
<th>CR</th>
</tr>
</thead>
<tbody>
<tr>
<td>MBP$^{(A)}(\lambda, \mathbb{X}_n)$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\alpha$</td>
<td>$\frac{1-\lambda}{2}$</td>
</tr>
<tr>
<td>MBP$^{(B)}(\gamma, \mathbb{X}_n)$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\alpha$</td>
<td>$\frac{1-\gamma}{2}$</td>
</tr>
<tr>
<td>SBP$^{(A)}(\lambda, \mathbb{X}_n)$</td>
<td>$\frac{\lambda^2}{\lambda^2 + (1-\lambda)^2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\lambda^2}{\lambda^2 + (1-\lambda)^2} (1 - 2\alpha) + \alpha$</td>
<td>$\frac{1+\lambda}{2}$</td>
</tr>
<tr>
<td>SBP$^{(B)}(\gamma, \mathbb{X}_n)$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$1 - \alpha$</td>
<td>$\frac{1+\gamma}{2}$</td>
</tr>
</tbody>
</table>

We interpret these results in the following brief discussions.

Scaled deviation outlyingness. (i) With (Mean, SD), scaled deviation outlyingness achieves for Type B SBP the highest possible value 1 and for Type A SBP a value approaching 1 as $\lambda \uparrow 1$. However, having this optimal SBP imposes too great a cost with respect to masking robustness: MBP = 0. Consequently, this version of scaled deviation outlyingness should be dropped from consideration (as is already well-known).

(ii) With (Median, MAD), scaled deviation outlyingness balances MBP and SBP equally with favorable value MBP = SBP = 0.5.

(iii) With $\alpha$-trims for $\alpha < 0.5$, scaled deviation outlyingness gives SBP higher priority without oversacrificing MBP. For example, with $\alpha = 0.33$ and $\lambda \approx 1$, we have MBP$^{(A)} = $ MBP$^{(B)} = 0.33$ and SBP$^{(B)} = $ SBP$^{(A)} = 0.67$. Likewise, with $\alpha = 0.25$ and $\lambda \approx 1$, we have MBP$^{(A)} = $ MBP$^{(B)} = 0.25$ and SBP$^{(A)} = $ SBP$^{(B)} = 0.75$. As $\alpha \uparrow 0.5$, the imbalance diminishes and MBP and SBP both $\to 0.5$.

Centered rank outlyingness. (i) Here MBP and SBP depend explicitly on the population and sample outlyingness thresholds $\gamma$ and $\lambda$, which thus need to be selected judiciously. One criterion is to enclose a specified proportion $p$ of the distribution. Via the equation $|2F(x) - 1| = \gamma$, the “contour” of equal outlyingness at level $\gamma$ consists of the two points $F^{-1} \left( \frac{1-\gamma}{2} \right)$ and $F^{-1} \left( \frac{1+\gamma}{2} \right)$. Hence, for this contour to enclose proportion $p$, $F(x)$ must equal either $(1-p)/2$ or $(1+p)/2$, in either case yielding $\gamma = p (= \lambda$ also).

(ii) For $\gamma = \lambda = p$, we have MBP $< 0.5 <$ SBP, prioritizing SBP over MBP. For $p = 0.9$, we obtain MBP$^{(A)} = 0.05$, MBP$^{(B)} = 0.45$, and SBP$^{(A)} = $ SBP$^{(B)} = 0.95$, rather overfavoring SBP. Less imbalance is associated with $p = 0.5$, a choice associated with a component of the boxplot, as discussed in Section 4.

Summary. For equal priority on masking and swamping robustness, scaled
deviation outlyingness with \((\text{Med, MAD})\) provides highly favorable \(\text{MPB} = \text{SBP} = 0.5\). For \emph{moderate prioritization of swamping robustness over masking robustness}, one may use scaled deviation outlyingness with \(\alpha\)-trims or centered rank outlyingness. However, at higher levels, centered rank outlyingness has unacceptably low MBP and is not competitive for outlier identification. On the other hand, one can combine the best features of scaled deviation outlyingness and centered rank outlyingness in a hybrid version, the \emph{boxplot}, which we examine next.

4 Robustness Features of the Boxplot

Denote the 1st quartile, median, and 3rd quartile of \(F\) by \(Q_1, M,\) and \(Q_3\) and sample versions by \(\hat{Q}_1, \hat{M},\) and \(\hat{Q}_3,\) respectively. The key aspects of the boxplot are a) the median \(M,\) b) the box representing the “middle half” and corresponding to the interval \((Q_1, Q_3),\) and c) the lower and upper “fences” \(Q_1 - 1.5 \times \text{IQR}\) and \(Q_3 + 1.5 \times \text{IQR}\) demarking “outlyingness” thresholds, where \(\text{IQR} = Q_3 - Q_1,\) the interquartile range.

Regarding robustness features of the sample boxplot, the estimator \(\hat{M}\) for \(M\) has excellent RBP = 0.5, and the estimators \(\hat{Q}_1\) and \(\hat{Q}_3\) for \(Q_1\) and \(Q_3\) each have favorable RBP = 0.25. However, regarding robustness of \emph{identification of the population middle half by the sample middle half} and of \emph{identification of outliers by the sample fences as thresholds}, it is also important to consider and to quantify the relevant \emph{masking and swamping robustness}, via appropriate MBPs and SBPs. Although conceptually different, “middle half identification” and “outlier identification” have in common MBP = 0.25 and SBP = 0.75, as seen in Sections 4.1 and 4.2. A variant boxplot improving MBP to 0.33 at the expense of trading off SBP to 0.67 is formulated in Section 4.3.

4.1 Robustness of middle half identification component

For the sample middle half, based on quantile type thresholds \(\hat{Q}_1\) and \(\hat{Q}_3,\) the relevant MBP and SBP are based on \emph{centered rank outlyingness}. Then Propositions 9-12 with \(\lambda = \gamma = 1/2\) yield MBP = 0.25 and SBP = 0.75.

\textit{Interpretation.} If a sample is modified by over 25% extreme replacement outliers all below \(Q_1\) or all above \(Q_3,\) then either \(\hat{Q}_1\) decreases and becomes below \(Q_1\) (if not already), or \(\hat{Q}_3\) increases and exceeds \(Q_3\) (if not already).

In either case, \emph{masking effects are increased}: additional points outside the \emph{population} middle half become included in the widened \emph{sample} middle half. On the other hand, for extreme outliers in one direction to produce \emph{increased swamping effects}, with additional points inside the population middle half now excluded from the sample middle half, either \(\hat{Q}_1\) must increase and exceed \(Q_3\)
Illustration. Let $F$ be $N(0, 1)$, with middle half $(Q_1, Q_3) = (-0.67, +0.67)$. A sample of size 100 might yield $(\hat{Q}_1, \hat{Q}_3) = (-0.60, 0.79)$, producing “light masking” of the interval $(0.67, 0.79)$ lying outside the population middle half but inside the sample middle half, and also “light swamping” of the interval $(-0.67, -0.60)$ lying within the population middle half but outside the sample middle half. Such mild masking and swamping effects are anticipated for a typical “regular” sample free of contaminating outliers. We now examine the additional effects produced by replacing a fraction of the data by outliers.

(i) 10% contamination. Replace the uppermost 10 sample values by the value 5. The sample middle half remains unchanged, with no additional masking or swamping, consistent with MBP and SBP both $> 0.10$.

(ii) 26% contamination. Replace the uppermost 26 sample values by the value 5, changing the sample middle half to $(-0.60, 5)$. Now substantial additional masking occurs, due to the large interval $(0.67, 5)$ outside the population middle half now included within the sample middle half, but no additional swamping occurs, consistent with SBP $> 0.26 > MBP$.

(iii) 51% contamination. Replace the uppermost 26 sample values by 5 and the next 25 uppermost by 4. In addition to the changes in (ii), $\hat{M}$ now becomes 4. However, the sample middle half and the masking and swamping effects remain the same as for scenario (ii).

(iv) 76% contamination. Replace the uppermost 26 sample values by 5, the next 25 uppermost by 4, and the next 25 uppermost by 3. The sample middle half becomes the interval $(3, 5)$, which is disjoint and even far apart from the population middle half $(-0.67, +0.67)$. Such complete separation of population middle half and sample middle half represents pronounced masking breakdown and pronounced swamping breakdown.

Thus, for identification of the population middle half, with contamination of 25% or less the boxplot exhibits only mild masking and swamping. However, with contamination exceeding 25% but not 75%, a fairly common scenario, the boxplot exhibits substantial additional masking but insubstantial additional swamping: the sample middle half correctly includes the bulk of the population middle half but also excessively includes additional points outside it.

4.2 Robustness of outlier identification component

Outlier identification using the sample boxplot is based on the lower and upper “fences” $\hat{F}_1 = \hat{Q}_1 - 1.5(\hat{Q}_3 - \hat{Q}_1)$ and $\hat{F}_2 = \hat{Q}_3 + 1.5(\hat{Q}_3 - \hat{Q}_1)$, respectively, as
thresholds. Here the multiplier 1.5 of the IQR is, as well-known, such that for the \(N(0,1)\) distribution with fences \(F_1 = Q_1-1.5(Q_3-Q_1) = -0.67-1.5\times1.35 = -2.70\) and \(F_2 = Q_3+1.5(Q_3-Q_1) = +2.70\), respectively, the probabilities of lying below \(F_1\) or above \(F_2\) are 0.0035 (0.35\%) each. This corresponds to a special \(\text{scaled deviation outlyingness function}\), with \(\hat{Q}_1\) and \(\hat{Q}_3\) as (lower and upper) location measures and \(\hat{Q}_3 - \hat{Q}_1\) as spread measure:

\[
\tilde{O}_{box}(x, X_n) = \begin{cases} 
\frac{\hat{Q}_1-x}{\hat{Q}_3-Q_1}, & x < \hat{Q}_1, \\
0, & \hat{Q}_1 \leq x \leq \hat{Q}_3, \\
\frac{x-\hat{Q}_3}{\hat{Q}_3-Q_1}, & x > \hat{Q}_3,
\end{cases}
\]

with threshold \(\beta = 1.5\) and associated outlier region \(\{x : \tilde{O}_{box}(x, X_n) > 1.5\}\), or \(out_{box}(0.6, X_n)\) in terms of \(\lambda = \beta/(1+\beta) = 0.6\). We apply Propositions 5-8.

(a) \(\text{Type A masking breakdown}\) occurs minimally by taking \(\hat{Q}_1\) to \(-\infty\) or \(\hat{Q}_3\) to \(+\infty\) (in either case \(\hat{Q}_3 - \hat{Q}_1 \to +\infty\)), via replacing 25\% of the data by extreme outliers in a single direction, yielding \(\text{MBP}^{(A)} = 0.25\).

(b) \(\text{Type B masking breakdown}\) occurs if \(\hat{Q}_3 \notin (Q_3 - 1.5(Q_3 - Q_1), Q_3 + 1.5(Q_3 - Q_1))\) or if \(\hat{Q}_1 \notin (Q_1 - 1.5(Q_3 - Q_1), Q_1 + 1.5(Q_3 - Q_1))\), with \(\text{MBP}^{(B)} = 0\), or otherwise via \(\hat{Q}_1 \to -\infty\) or \(\hat{Q}_3 \to +\infty\), yielding \(\text{MBP}^{(B)} = 0.25\) (= \(\text{MBP}^{(A)}\)).

(c) \(\text{Type A swamping breakdown}\) occurs if \(Q_3 \notin (\hat{Q}_3 - 1.5(\hat{Q}_3 - \hat{Q}_1), \hat{Q}_3 + 1.5(\hat{Q}_3 - \hat{Q}_1))\), or if \(Q_1 \notin (\hat{Q}_1 - 1.5(\hat{Q}_3 - \hat{Q}_1), \hat{Q}_1 + 1.5(\hat{Q}_3 - \hat{Q}_1))\), with \(\text{SBP}^{(A)} = 0\), or otherwise by taking \(\hat{Q}_1\) and \(\hat{Q}_3\) both to \(+\infty\) or both to \(-\infty\) while maintaining \(\hat{Q}_3 - \hat{Q}_1 \leq |Q_1 - Q_1|/1.5\), via replacing 75\% of the data by extreme outliers in a single direction, with \(\text{SBP}^{(A)} = 0.75\).

(d) \(\text{Type B swamping breakdown}\) is treated similarly to Type A swamping breakdown, obtaining \(\text{SBP}^{(B)} = 0.75\) as well.

The results \(\text{MBP} = 0.25\) and \(\text{SBP} = 0.75\) depend neither on Type A versus Type B, nor on the particular threshold \(\beta = 1.5\).

\textit{Illustration.} Again consider \(F = N(0,1)\) with its lower and upper fences \(F_1 = -2.68\) and \(F_2 = +2.68\), respectively, and again a sample of size 100 with sample quartiles \((\hat{Q}_1, \hat{Q}_3) = (-0.60, 0.79)\) and thus sample fences \(\tilde{F}_1 = -2.69\) and \(\tilde{F}_2 = +2.88\). In this case, there is “light masking” of outliers in the interval \((2.68, 2.88)\) lying above \(F_2\) but within the sample fences, and “very light masking” of outliers in the very short interval \((-2.69, -2.68)\) lying below \(F_1\) but within the sample fences. However, there is no swamping of nonoutliers. We now examine changes caused by the same replacement scenarios as above.
(i) 10% contamination. The sample quartiles and fences remain unchanged, so neither additional masking nor any swamping occurs.

(ii) 26% contamination. With $\hat{Q}_3$ now 5, $\hat{F}_2$ becomes $+13.40$ and there occurs “heavy masking” of the outliers in (2.68, 13.40) but no swamping.

(iii) 51% contamination. No added effects beyond those in (ii).

(iv) 76% contamination. With $\hat{Q}_1 = 3$, $\hat{Q}_3 = 5$, $\hat{F}_1 = +0$, and $\hat{F}_2 = +8$, “heavy masking” of the outliers in (2.68, 8) occurs along with “moderate swamping” of the nonoutliers in ($-2.68, 0$). This breakdown results from the contamination level exceeding both MBP and SBP.

4.3 The MBP versus SBP trade-off for the boxplot

For both middle half identification and outlier identification, the boxplot is only moderately masking robust while overwhelmingly swamping robust, with MBP = 0.25 and SBP = 0.75. The excellent SBP is higher than needed in practice, however. On the other hand, improvement in MBP for the outlier identification component has strong practical appeal and can be achieved by a variant boxplot trading off SBP somewhat to obtain higher MBP, as follows.

To define the fences, use instead of the IQR the intertertile range $\text{ITR} = \hat{T}_2 - \hat{T}_1$ based on the tertiles $\hat{T}_1$ and $\hat{T}_2$ partitioning the sample into thirds. This yields fences $\hat{T}_1 - 2.64(\hat{T}_2 - \hat{T}_1)$ and $\hat{T}_2 + 2.64(\hat{T}_2 - \hat{T}_1)$, the constant 2.64 making tail probabilities outside the fences each 0.0035 for $N(0, 1)$ data, as with IQR-based fences. For this new outlier identification component we obtain MBP = 0.33 and SBP = 0.67. The middle half identification component is unchanged, retaining MBP = 0.25 and SBP = 0.75. This improves the masking and swamping robustness trade-off for the outlier identification component without sacrificing the overall simplicity, appeal, and ease of application of the boxplot.

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Appendix

The following lemma from Serfling and Wang (2014) treats breakdown of a statistic $S(\mathcal{X}_n)$ when the event of breakdown due to $k$ replacements is related to the possible
occurrences of certain events \( E_1, \ldots, E_J \) as a consequence of \( k \) replacements. Let \( k_S \) be the minimal number of replaced data points needed to cause breakdown (either implosion or explosion) of \( S \), and let \( k_1, \ldots, k_J \) be the minimal numbers of replaced data points needed to cause occurrence of the respective events \( E_1, \ldots, E_J \). It is assumed that \( k_S \) and \( k_1, \ldots, k_J \) are well-defined and belong to \( \{1, 2, \ldots, n\} \).

**Lemma 13** (i) If breakdown of \( S \) is implied by occurrence of each one of the events \( E_1, \ldots, E_J \), then \( k_S \leq \min\{k_1, \ldots, k_J\} \).

(ii) If breakdown of \( S \) implies occurrence of at least one of the events \( E_1, \ldots, E_J \), then \( k_S \geq \min\{k_1, \ldots, k_J\} \).

(iii) If breakdown of \( S \) is implied by occurrence of each one of the events \( E_1, \ldots, E_J \) and also implies that at least one of \( E_1, \ldots, E_J \) must occur, then

\[
k_S = \min\{k_1, \ldots, k_J\}.
\]

**Proof of Proposition 5.** Using \( \text{out}(\lambda, X_n) = [\hat{\mu} - \beta \hat{\sigma}, \hat{\mu} + \beta \hat{\sigma}] \) and (7), we have \( \text{MBP}^{(A)}(\lambda, X_n) = \text{RBP}_{\exp}(S(X_n)) \), where

\[
S(X_n) = \sup_{y \in [\mu(X_n) - \beta \sigma(X_n), \mu(X_n) + \beta \sigma(X_n)]} \left| \frac{y - \mu}{\sigma} \right|.
\]

We apply Lemma 13(iii) with this \( S \), some fixed \( k \), and \( E = \{\exists X_{n,k} : S(X_{n,k}) \to \infty\} \), \( E_1 = \{\exists X_{n,k} : |\mu(X_{n,k})| \to \infty\} \), and \( E_2 = \{\exists X_{n,k} : \sigma(X_{n,k}) \to \infty\} \). We need to show that (with \( k \) fixed) \( E \) implies \( E_1 \cup E_2 \) and \( E_1 \) and \( E_2 \) both imply \( E \). The first is immediate. To show that \( E_1 \) implies \( E \), first take the case that \( \mu(X_{n,k}) \to +\infty \). Then \( S(X_{n,k}) \) becomes bounded below by \( (\mu(X_{n,k}) + \beta \sigma(X_{n,k}) - \mu)/\sigma \), which \( \to +\infty \), implying \( E \). The case that \( \mu(X_{n,k}) \to -\infty \) is treated similarly. For the case that \( E_2 \) holds with \( |\mu(X_{n,k})| \to +\infty \), we have that at least one of \( |\mu(X_{n,k}) - \beta \sigma(X_{n,k})| \) and \( |\mu(X_{n,k}) + \beta \sigma(X_{n,k})| \) must \( \to +\infty \), implying \( E \). Also, for the case that \( E_2 \) holds with \( |\mu(X_{n,k})| \) remaining bounded, both of these convergences hold, again yielding \( E \). Since \( E_1 \) and \( E_2 \) can occur only due to replacement of observations by outliers and not by inliers, Type A masking breakdown for scaled deviation outlyingness is associated strictly with outliers. Now applying (13), we obtain \( \text{MBP}^{(A)}(\lambda, X_n) = n^{-1} \min\{k_1, k_2\} \), where \( k_1, k_2 \) are the minimal \( k \) respectively, for occurrence of \( E_1, E_2 \). Of course, \( k_1/n = \text{RBP}_{\exp}(\hat{\mu}) \) and \( k_2/n = \text{RBP}_{\exp}(\hat{\sigma}) \). \( \square \)

**Proof of Proposition 6.** Using \( \text{out}(\gamma, F) = [\mu - \eta \sigma, \mu + \eta \sigma] \) and (8), we have \( \text{MBP}^{(B)}(\gamma, X_n) = \text{RBP}_{\imp}(S(X_n)) \), where

\[
S(X_n) = \inf_{y \in [\mu - \eta \sigma, \mu + \eta \sigma]} \left| \frac{y - \mu(X_n)}{\sigma(X_n)} \right|.
\]

Note that (interpreting 0/0 as 0)

\[
S(X_n) = \begin{cases} 
0 & \text{if } \mu(X_n) \notin (\mu - \eta \sigma, \mu + \eta \sigma) \\
\min \left\{ \frac{\mu(X_n) - \mu - \eta \sigma}{\sigma(X_n)}, \frac{\mu - \eta \sigma - \mu(X_n)}{\sigma(X_n)} \right\} & \text{if } \mu(X_n) \in (\mu - \eta \sigma, \mu + \eta \sigma).
\end{cases}
\]
Thus it is immediate that $\text{MBP}^{(B)}(\gamma, X_n) = 0$ if $\mu(X_n) \notin (\mu - \eta \sigma, \mu + \eta \sigma)$. For the case that $\mu(X_n) \in (\mu - \eta \sigma, \mu + \eta \sigma)$, and for the above $S$, we apply Lemma 13(iii) with some fixed $k$ and the events $E = \{\exists X_{n,k} \in S(X_{n,k}) \to 0\}$ and

\[
E_1 = \{\exists X_{n,k} : \mu(X_{n,k}) \notin (\mu - \eta \sigma, \mu + \eta \sigma)\}
\]
\[
E_2 = \{\exists X_{n,k} : \sigma(X_{n,k}) \to \infty\}
\]
\[
E_3 = \{\exists X_{n,k} : \mu(X_{n,k}) \dashv \mu - \eta \sigma, \text{ with } \sigma(X_{n,k}) \not\to \infty,
\] and $\mu(X_{n,k}) - (\mu - \eta \sigma) = o(\sigma(X_{n,k}))\}
\]
\[
E_4 = \{\exists X_{n,k} : \mu(X_{n,k}) \uparrow \mu + \eta \sigma, \text{ with } \sigma(X_{n,k}) \not\to \infty,
\] and $(\mu + \eta \sigma) - \mu(X_{n,k}) = o(\sigma(X_{n,k}))\}.
\]

Note that (with $k$ fixed) each of $E_1$-$E_4$ implies $E$ and $E$ implies their union (for example, to see that $E_2$ implies $E$, note that if $E_2$ holds with $\mu(X_{n,k}) \in (\mu - \eta \sigma, \mu + \eta \sigma)$, then $E$ trivially follows, and otherwise $E_2$ overlaps $E_1$, which immediately implies $E$). Also, note that $E_1$ and $E_2$ are associated with outliers and explosion, and $E_3$ and $E_4$ with inliers and implosion. Confining attention just to outliers and applying (13), we obtain $\text{MBP}^{(B)}(\gamma, X_n) = n^{-1} \min\{k_1, k_2\}$, with $k_1, k_2$ the minimal values of $k$, respectively, for occurrence of $E_1, E_2$, and, in particular, $k_1/n = \text{RBP}_{\exp}(\mu(X_n))$ and $k_2/n = \text{RBP}_{\exp}(\sigma(X_n))$. □

**Proof of Proposition 7.** Using $\text{out} (\lambda, X_n) = [\mu - \beta \sigma, \mu + \beta \sigma]$ and (9), we have $\text{SBP}^{(A)}(\lambda, X_n) = \text{RBP}_{\text{imp}}(S(X_n))$, where

\[
S(X_n) = \inf_{y \in \mu(X_n) - \beta \sigma(X_n), \mu(X_n) + \beta \sigma(X_n)} \frac{y - \mu}{\sigma}.
\]

Note that (interpreting 0/0 as 0)

\[
S(X_n) = \begin{cases} 0 & \text{if } \mu \notin (\mu(X_n) - \beta \sigma(X_n), \mu(X_n) + \beta \sigma(X_n)) \\
\min \left\{ \frac{\mu - \mu(X_n) + \beta \sigma(X_n)}{\sigma}, \frac{\mu(X_n) + \beta \sigma(X_n) - \mu}{\sigma} \right\} & \text{otherwise.}
\end{cases}
\]

(15)

It is immediate that $\text{SBP}^{(A)}(\lambda, X_n) = 0$ if $\mu \notin (\mu(X_n) - \beta \sigma(X_n), \mu(X_n) + \beta \sigma(X_n))$. For the case $\mu \in (\mu(X_n) - \beta \sigma(X_n), \mu(X_n) + \beta \sigma(X_n))$, and for the above $S$, we apply Lemma 13(iii) with some fixed $k$ and $E = \{\exists X_{n,k} \in S(X_{n,k}) \to 0\}$, $E_1 = \{\exists X_{n,k} : \mu(X_{n,k}) - \beta \sigma(X_{n,k}) \geq \mu\}$, and $E_2 = \{\exists X_{n,k} : \mu(X_{n,k}) + \beta \sigma(X_{n,k}) \leq \mu\}$. Ignoring inliers, and considering only outliers placed in the same direction (toward either $+\infty$ or $-\infty$ but not both simultaneously), we associate $E_1$ with explosion of $\mu(X_{n,k}) - \beta \sigma(X_{n,k})$ to $+\infty$ (which requires $\mu(X_{n,k}) \to +\infty$) and $E_2$ with explosion of $\mu(X_{n,k}) + \beta \sigma(X_{n,k})$ to $-\infty$ (which requires $\mu(X_{n,k}) \to -\infty$). Further, $\mu(X_{n,k}) - \beta \sigma(X_{n,k}) \to +\infty$ is equivalent to $|\mu(X_{n,k})| - \beta \sigma(X_{n,k}) \to +\infty$ with $\mu(X_{n,k}) \to +\infty$, while $\mu(X_{n,k}) + \beta \sigma(X_{n,k}) \to -\infty$ is equivalent to $|\mu(X_{n,k})| - \beta \sigma(X_{n,k}) \to -\infty$ with $\mu(X_{n,k}) \to -\infty$. Thus the event $E$ is associated with $|\mu(X_{n,k})| - \beta \sigma(X_{n,k}) \to +\infty$ (which requires $\mu(X_{n,k}) \to \pm \infty$), and we arrive at
\[ SB^{(A)}(\lambda, X_n) = RBP_{\exp} \to +\infty (|\mu(X_n)| - \beta\sigma(X_n)). \]

**Proof of Proposition 8.** Follows along the lines of the preceding proofs.

**Proof of Proposition 9.** We have \( MB^{(A)}(\lambda, X_n) = RBP_{\exp}(S(X_n)) \), where

\[
S(X_n) = \sup_{y \in OR(\lambda, X_n)} |2F(y) - 1|
\]

\[
= \max \left\{ \left| 2F\left( \hat{F}^{-1}\left( \frac{1 - \lambda}{2} \right) \right) - 1 \right|, \left| 2F\left( \hat{F}^{-1}\left( \frac{1 + \lambda}{2} \right) \right) - 1 \right| \right\}
\]

The first term in the above maximum satisfies \( |2F\left( \hat{F}^{-1}\left( \frac{1 - \lambda}{2} \right) \right) - 1| \to 1 \) if and only if either \( \hat{F}^{-1}\left( \frac{1 - \lambda}{2} \right) \to \infty \) or \( \hat{F}^{-1}\left( \frac{1 - \lambda}{2} \right) \to -\infty \). We apply Lemma 13(iii) with some fixed \( k \) and the events

\[
E = \left\{ \exists\{X_{n,k} : \left| 2F\left( \hat{F}^{-1}_{n,k}\left( \frac{1 - \lambda}{2} \right) \right) - 1 \right| \to 1 \right\}
\]

\[
E_1 = \left\{ \exists\{X_{n,k} : \hat{F}^{-1}_{n,k}\left( \frac{1 - \lambda}{2} \right) \to \infty \right\}
\]

\[
E_2 = \left\{ \exists\{X_{n,k} : \hat{F}^{-1}_{n,k}\left( \frac{1 - \lambda}{2} \right) \to -\infty \right\}
\]

With \( T^{**} = 1 \), (13) yields \( RBP_{\exp}\left( \left| 2F\left( \hat{F}^{-1}\left( \frac{1 - \lambda}{2} \right) \right) - 1 \right| \right) = n^{-1}\min\{k_1, k_2\} \), where \( k_1, k_2 \) are the minimal values of \( k \), respectively, for occurrence of \( E_1, E_2 \).

Specifically, we find \( k_1 = n - \left\lceil \frac{1 - \lambda}{2} n \right\rceil + 1 = \left\lfloor \frac{1 + \lambda}{2} n \right\rfloor + 1 \) and \( k_2 = \left\lfloor \frac{1 - \lambda}{2} n \right\rfloor \leq k_1 \). Thus \( RBP_{\exp}\left( \left| 2F\left( \hat{F}^{-1}\left( \frac{1 - \lambda}{2} \right) \right) - 1 \right| \right) = n^{-1}\left\lceil \frac{1 - \lambda}{2} n \right\rceil \). Similarly we obtain \( RBP_{\exp}\left( \left| 2F\left( \hat{F}^{-1}\left( \frac{1 + \lambda}{2} \right) \right) - 1 \right| \right) = n^{-1}\left(\left\lfloor \frac{1 - \lambda}{2} n \right\rfloor + 1 \right) = n^{-1}\left\lceil \frac{1 - \lambda}{2} n \right\rceil \). \( \square \)

**Proofs of Propositions 10, 11 and 12.** Similar to the foregoing proof.

**References**


