Asymptotic Normality of Shot Noise on Poisson Cluster Processes with Cluster Marks

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Abstract

Asymptotic normality is established for shot noise on Poisson cluster point processes with cluster marks, in the case that the shot noise has finite variance tending to infinity with time. Introducing straightforward conditions on the cluster model and response function, this extends previous work in the literature for the case of no clustering and supports greater flexibility in applying shot noise models. The results are illustrated in detail for power law shot noise and other forms of response function which arise in diverse applications such as semiconductor noise, Cherenkov radiation, biological information communication, financial modeling, insurance, and immigration-death processes.

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1 Introduction and Preliminaries

A shot noise model is given by superposing "shot effects" which initiate at random times and persist through random durations, possibly infinite. Such stochastic processes arise very naturally in diverse fields — see, for example, Parzen (1962), Cox and Isham (1980), Bondesson (1988), Lowen and Teich (1990), Snyder and Miller (1991), and Samorodnitsky (1995, 1998).

In particular, a (possibly random) real-valued shot effect or "impulse response" function $h(t, t', y)$ represents the remaining effect at time t of an impulse initiated at time t' and may depend upon an input y also generated at time t', with $h(t, t', y) \equiv 0$ except when $0 \le t' \le t$. Here we treat shot noise on marked cluster processes with cluster marks, for which a suitable representation is thus

$$X(t) = \sum_j \sum_{k=1}^{K_j} h(t, T_{jk}, V_j, W_{jk}), \ t \ge 0, \tag{1.1}$$

the underlying point process consisting of clusters of points with those of the jth cluster numbering K_j and located at times $T_{jk} = C_j + D_{jk}$, $1 \le k \le K_j$, with C_j a "cluster origin" and the D_{jk}'s "displacements" from the cluster origin. We fix the time origin at 0 (rather than at $-\infty$) and take the D_{jk}'s to be nonnegative, so that all T_{jk}'s *follow* the chosen time origin, yielding a model that includes transient features of $X(\cdot)$ without entailing inconvenient boundary considerations. The case of "no clustering" is specified by $K_j \equiv 1$ and $D_{j1} \equiv 0$, all j.

Associated with the kth point of the jth cluster is a mark (V_j, W_{jk}), with V_j a "cluster mark" shared by all points of that cluster and W_{jk} a "pointwise mark". For example, the marks may play the role of introducing random "amplitudes" and random "durations" into the response function. When the events of a point process are modeled as occurring in clusters, it is desirable to allow the response functions for the points within a given cluster to have some features in common, which may be based on the "cluster mark". In financial modeling, for example, the marks W_{jk} and V_j may represent amplitude components associated with the reaction of the financial market to various events such as interest rate changes, mergers, announcements, expectations, rumors, political events, etc. The inclusion of cluster marks provides considerably enhanced flexibility in applying shot noise models.

Extending previous treatments of shot noise on cluster processes to include a cluster mark, Ramirez-Perez and Serfling (2001) investigated $X(t)$ on arbitrary cluster processes and characterized the exact behavior for fixed t as well as equilibrium behavior and long range dependence properties in the case that the mean and variance have finite limits as $t \to \infty$. In the present paper, confining attention to *Poisson* cluster processes, we treat the complementary case that $X(t)$ is "explosive", having variance tending to infinity and thus not passing to an equilibrium state. Then a desirable property is that $X(t)$, suitably normalized, tend in distribution to a standard normal distribution.

The question of asymptotic normality of $X(t)$ is explored in Section 2. Our main result (Theorem 2.1) establishes straightforward conditions on the response function h under which $\text{Var}\{X(t)\}$ tends to infinity and $X(t)$ is indeed asymptotically normal in distribution, as $t \to \infty$. In effect, this result extends work of Lane (1984) to the case of clustering. A general class of typical response functions is handled by Corollary 2.1.

A range of applications and examples are treated in Section 3, in terms of three special models. Model I encompasses responses formed by a *deterministic shape function* multiplied by a *random amplitude*. Within this framework, and complementing and extending work of Lowen and Teich (1990), Example B treats *power law shot noise*. This structure arises in modeling semiconductor noise, Cherenkov radiation, biological information communication, and financial processes, among other applications. Example C treats insurance claims with delayed or extended payments. Model II encompasses *unit shape functions* with *random amplitudes* and *random durations*, for applications such as the number of individuals in an immigration-death process and the thickness of textile yarn. Example D studies the case of *Pareto law durations*. Model III extends and unifies Models I and II, with Example E treating *power law shot noise combined with Pareto law durations*.

Section 4 provides brief complementary remarks, covering other modes of asymptotics, other types of convergence results, other possible limit laws, and various extensions that might be carried out. The remainder of the present section treats further preliminaries.

A convenient representation for $X(t)$. As in Ramirez-Perez and Serfling (2001), it is useful to represent the shot noise $X(\cdot)$ in (1.1) as shot noise on an ordinary point process *without* clustering, but in terms of a considerably more complicated, *cluster-based* response function g. Defining

$$\gamma = (V, K, D_1, \ldots, D_K, W_1, \ldots, W_K)$$

and

$$g(t, u, \gamma) = \sum_{k=1}^{K} h(t, u + D_k, V, W_k),\ \ t \geq 0, \tag{1.2}$$

we have

$$X(t) = \sum_j g(t, C_j, \gamma_j),\ \ t \geq 0, \tag{1.3}$$

with the γ_j's given by

$$\gamma_j = (V_j, K_j, D_{j1}, \ldots, D_{jK_j}, W_{j1}, \ldots, W_{jK_j}),\ \ j \in \mathbb{N}.$$

Thus $X(t)$ may be viewed simply as shot noise on a marked point process with points $\{C_j\}$ (not necessarily observed) and "marks" $g(\cdot, C_j, \gamma_j)$ constituting the "shot effects". Although well-known in the literature (see Heinrich and Schmidt (1985), p. 716, for discussion), this device has not been exploited previously for the present purposes. In particular, here it enables us to make direct use of existing results of Lane (1984) for Poisson shot noise processes without clustering. □

Assumptions. The following conditions are assumed:

A1 The vectors $\{\gamma_j\}$ are mutually independent and identically distributed.

A2 The sequence of cluster origins $\{C_j\}$ follows a Poisson process with rate function $\lambda(\cdot)$ and is independent of the sequence $\{\gamma_j\}$.

A3 In γ, all components are mutually independent, the D's are identically distributed, and the W's are identically distributed. □

Specialization of Theorems 3.3 and 3.4 of Ramirez-Perez and Serfling (2001) under assumptions A1–A3 yields convenient expressions for the mean and variance of $\{X(t)\}$:

$$\begin{aligned} E\{X(t)\} &= \int_0^t E\{g(t,u,\gamma)\}\,\lambda(u)\,du \\ &= E\{K\}\int_0^t E\{h(t,u+D,V,W)\}\,\lambda(u)\,du \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}\{X(t)\} &= \int_0^t E\{g^2(t,u,\gamma)\}\,\lambda(u)\,du \\ &= E\{K\}\int_0^t E\{h^2(t,u+D,V,W)\}\,\lambda(u)\,du \\ &\quad + E\{K(K-1)\}\int_0^t E_V\{\,[E_{D,W}\,h(t,u+D,V,W)]^2\,\}\,\lambda(u)\,du. \qquad (1.4) \end{aligned}$$

2 General Results

For the shot noise process $X(t)$ given by (1.1) or (1.3), we establish conditions on the impulse function h under which the corresponding normalized process

$$Z(t) = \frac{X(t)-\mu_t}{\sigma_t}$$

converges in distribution to the standard normal distribution $\mathrm{N}(0,1)$ as $t \to \infty$, with $\mu_t = E\{X(t)\}$ and $\sigma_t^2 = \mathrm{Var}\{X(t)\}$ assumed finite, all t. (See Remark 4.3 for discussion of the case that $\mathrm{Var}\{X(t)\}$ is infinite.)

We first establish convergence of $Z(t)$ under conditions on g instead of h. Our starting point is the following lemma, which is obtained by utilizing the representation (1.3) to specialize Theorem 3 of Lane (1984), who treats shot noise on ordinary Poisson processes without clustering. Interpreting $g(t,u,\gamma)$ as the "shot effect" and, for $t \geq 0$ and $y \in \mathbb{R}$, defining the associated integrated tail probability

$$A(t,y) = \int_0^t P\{g(t,u,\gamma) \notin (-y,y]\}\,\lambda(u)\,du,$$

we thus state

Lemma 2.1 *Assume A1 and A2, and suppose that $\sigma_t \to \infty$ as $t \to \infty$. Then the condition*

$$\int_{x\sigma_t}^{\infty} y\,A(t,y)\,dy = o(\sigma_t^2),\ \ t \to \infty,\ \ \textit{each } x > 0, \qquad (2.1)$$

is necessary and sufficient for $Z(t) \xrightarrow{d} \mathrm{N}(0,1)$ as $t \to \infty$.

In applications we prefer, however, to have conditions given directly in terms of the response function h that is used to define the shot noise $X(t)$. For this we establish

Theorem 2.1 *Assume A1–A3 and suppose that h satisfies*

$$\int_0^t E\{h^2(t, u+D, V, W)\}\,\lambda(u)\,du \to \infty,\ \ t \to \infty, \tag{2.2}$$

and, for some $\nu > 2$,

$$\int_0^t E\{|h(t, u+D, V, W)|^\nu\}\,\lambda(u)\,du = o(\sigma_t^\nu),\ \ t \to \infty. \tag{2.3}$$

Then $Z(t) \xrightarrow{d} \mathrm{N}(0,1)$ *as* $t \to \infty$.

Proof. First we note that $E_V\{\,[E_{D,W}\,h(t, u+D, V, W)]^2\,\} \le E\{h^2(t, u+D, V, W)\}$, whence from (1.4) we have $\sigma_t^2 \asymp \int_0^t E\{h^2(t, u+D, V, W)\}\,\lambda(u)\,du$ as $t \to \infty$. Thus, under A1–A3, we have $\sigma_t \to \infty$ if and only if (2.2) holds.

Next note that straightforward application of Markov's inequality yields a convenient sufficient condition for (2.1), as given in Corollary 4 of Lane (1984), namely that

$$\int_0^t E\{|g(t, u, \gamma)|^\nu\}\,\lambda(u)\,du = \mathrm{o}(\sigma_t^\nu),\ \ t \to \infty, \tag{2.4}$$

hold for some $\nu > 2$. Now apply Minkowski's inequality in conjunction with A3 to obtain

$$E\{|g(t, u, \gamma)|^\nu\} \le E\{K^\nu\}\,E\{|h(t, u+D, V, W)|^\nu\},$$

assuming finiteness of the νth moments. In this case (2.4) follows from (2.3). □

A wide class of response functions including many typical cases arising in applications is given by the special structure

$$h(t, u, v, w) = a(v, w^{(1)})\,b(t, u, w^{(2)}), \tag{2.5}$$

with $w = (w^{(1)}, w^{(2)})$ and $b(t, u, w^{(2)}) \equiv 0$ except when $0 \le u \le t$. Relevant assumptions specific to this structure are as follows:

B1 In γ, the component W is of form $(W^{(1)}, W^{(2)})$ with $W^{(1)}$ and $W^{(2)}$ independent.

B2 $E\{|a(V, W^{(1)})|^\nu\} < \infty$ for some $\nu > 2$.

B3 $\int_0^t E\{b^2(t, u+D, W^{(2)})\}\,\lambda(u)\,du \to \infty$ as $t \to \infty$.

B4 For some $\tilde{\nu} > 2$,

$$\int_0^t E\{|b(t, u+D, W^{(2)})|^{\tilde{\nu}}\}\,\lambda(u)\,du = \mathrm{o}\left(\left(\int_0^t E\{b^2(t, u+D, W^{(2)})\}\,\lambda(u)\,du\right)^{\tilde{\nu}/2}\right)$$

as $t \to \infty$ □

Under these assumptions, we have

Corollary 2.1 *Assume A1–A3. Let the response function h be given by (2.5) and satisfy B1–B4. Then $Z(t) \xrightarrow{d} \mathrm{N}(0,1)$ as $t \to \infty$.*

Proof. We apply Theorem 2.1. Using B1–B3, we have

$$\begin{aligned}\int_0^t E\{h^2(t,u+D,V,W)\,\lambda(u)\,du &= \int_0^t E\{a^2(V,W^{(1)})\,b^2(t,u+D,W^{(2)})\}\,\lambda(u)\,du \\ &= E\{a^2(V,W^{(1)})\}\int_0^t E\{b^2(t,u+D,W^{(2)})\}\,\lambda(u)\,du \\ &\to \infty,\ t\to\infty.\end{aligned}$$

Thus follows (2.2) and hence also that $\sigma_t \to \infty$. Then, assuming without loss of generality that ν in B2 and $\tilde{\nu}$ in B4 satisfy $\nu \le \tilde{\nu}$, conditions B2 and B4 yield

$$\begin{aligned}\int_0^t E\{|h(t,u+D,V,W)|^\nu\,\lambda(u)\,du &= E\{|a(V,W^{(1)})|^\nu\}\int_0^t E\{|b(t,u+D,W^{(2)})|^\nu\}\,\lambda(u)\,du \\ &= \mathrm{o}\left(\left(\int_0^t E\{b^2(t,u+D,W^{(2)})\}\,\lambda(u)\,du\right)^{\nu/2}\right),\ t\to\infty, \\ &= \mathrm{o}(\sigma_t^{\tilde{\nu}}),\ t\to\infty.\end{aligned}$$

Thus follows (2.3) with $\nu > 2$. □

Remark 2.1 When B1–B3 hold, a convenient sufficient conditon for B4 is that $|b(\cdot,\cdot,\cdot)|$ is bounded. For, with B_0 denoting the bound, for any $\tilde{\nu} > 2$ we then have

$$\begin{aligned}&\int_0^t E\{|b(t,u+D,W^{(2)})|^{\tilde{\nu}}\}\,\lambda(u)\,du \\ &\qquad \le B_0^{\tilde{\nu}-2}\int_0^t E\{b^2(t,u+D,W^{(2)})\}\,\lambda(u)\,du \\ &\qquad = \mathrm{o}\left(\left(\int_0^t E\{b^2(t,u+D,W^{(2)})\}\,\lambda(u)\,du\right)^{\tilde{\nu}/2}\right),\ t\to\infty.\quad \square\end{aligned}$$

3 Applications

Surprisingly many specific shot noise models arising in applications fall into the framework of (2.5). Here we study some representative cases, for which it is convenient to augment the assumptions A1–A3 with

A4 There exists $\lambda_0 > 0$ such that $\lambda(u) \ge \lambda_0$, all u.

This is satisfied, of course, by a *homogeneous* Poisson process.

3.1 Model I: Arbitrary Shape Functions with Random Amplitudes

Diverse modeling applications are based on response functions of form

$$h(t,u,v,w) = (v+w)\,h_0(t-u), \tag{3.1}$$

with $h_0(\cdot)$ a deterministic shape function satisfying $h_0(x) = 0$ for $x < 0$. For example, the modeling of $1/f^\alpha$ noise in semiconductors and other mechanical, biological, and physical systems is discussed in Gilchrist and Thomas (1975) and Lowen and Teich (1990). Financial modeling applications are discussed in Samorodnitsky (1995) and Ramirez-Perez and Serfling (2001). In insurance applications, the shot noise based on (3.1) can represent the discounted value at time t of all claims submitted between time t and a future time t^*, or the total payment up to time t of claims submitted prior to t and for which payments are extended over time for various possible reasons. For applications such as waterflow modeling, the amplitudes $(v+w)$ in (3.1) represent rainfall amounts delivered by storms prior to time t and $X(t)$ represents the streamflow at time t.

Many typical cases are covered under the following general conditions specific to response functions h of form (3.1):

C1 $E\{|V|^\nu\}$ and $E\{|W|^\nu\}$ are finite for some $\nu > 2$.

C2 Either

(i) h_0 is nonincreasing and satisfies $\int_0^t h_0^2(x)\,dx \to \infty$, $t \to \infty$,

or

(ii) h_0 is nondecreasing.

C3 For some $\tilde{\nu} > 2$,

$$\int_0^t E\{|h_0(t-u-D)|^{\tilde{\nu}}\}\,\lambda(u)\,du = \mathrm{o}\left(\left(\int_0^t E\{h_0^2(t-u-D)\}\,\lambda(u)\,du\right)^{\tilde{\nu}/2}\right)$$

as $t \to \infty$. □

Very usefully, these conditions suffice for asymptotic normality:

Corollary 3.1 *Assume A1–A4. Let the response function h be given by (3.1) and satisfy C1–C3. Then $Z(t) \xrightarrow{d} \mathrm{N}(0,1)$ as $t \to \infty$.*

Proof. We apply Corollary 2.1. Note that (3.1) is of form (2.5) with $w = w^{(1)}$, $a(v,w) = v+w$, and $b(t,u) = h_0(t-u)$, so that B1 is trivially satisfied. Also, immediately, C1 yields B2 and, provided that B3 holds, C3 implies B4. It remains to show that B3 holds, and for this we apply C2 and A4 together. Let c_0 be a number such that $h_0(c_0) > 0$ and $P(D \le c_0) > 0$. Let us first assume C2(i). Then, also using A4, for $t > c_0$ we have

$$\begin{aligned}
&\int_0^t E\{h_0^2(t-u-D)\,\mathbf{1}\{D \le t-u\}\}\,\lambda(u)\,du \\
&\qquad \ge \int_0^t h_0^2(t-u)\,P(D \le t-u)\,\lambda(u)\,du \\
&\qquad \ge \int_0^{t-c_0} h_0^2(t-u)\,P(D \le c_0)\,\lambda(u)\,du \\
&\qquad \ge \lambda_0\,P(D \le c_0)\int_0^{t-c_0} h_0^2(t-u)\,du \\
&\qquad = \lambda_0\,P(D \le c_0)\int_{c_0}^{t} h_0^2(x)\,dx,
\end{aligned}$$

yielding B3. Next we assume C2(ii). Again using A4, for $t > 2c_0$ we now have

$$\begin{aligned}
&\int_0^t E\{h_0^2(t-u-D)\,\mathbf{1}\{D \le t-u\}\}\,\lambda(u)\,du \\
&\qquad \ge \int_0^t E\{h_0^2(t-u-D)\,\mathbf{1}\{D \le (t-u)/2\}\}\,\lambda(u)\,du \\
&\qquad \ge \int_0^t h_0^2(t-u-(t-u)/2)\,P(D \le (t-u)/2)\,\lambda(u)\,du \\
&\qquad \ge \int_0^{t-2c_0} h_0^2((t-u)/2)\,P(D \le c_0)\,\lambda(u)\,du \\
&\qquad \ge \lambda_0\,P(D \le c_0)\,h_0^2(c_0)\,(t-2c_0) \asymp t,\ t \to \infty,
\end{aligned}$$

and again B3 follows. □

Remark 3.1 (i) When C1 and C2 hold, a sufficient conditon for C3 is that $|h_0(\cdot)|$ is bounded. This follows by an argument similar to that used in Remark 2.1.

(ii) Note from the above proof that under C2(ii) and A4 we have

$$t = \mathrm{O}\left(\int_0^t E\{h_0^2(t-u-D)\}\,\lambda(u)\,du\right),\ t \to \infty.$$

This will be useful in establishing C3 for a case of $h_0(x) \to \infty$ in Example C(ii) below. □

We note that shape functions $h_0(x)$ which decrease to 0 at a sufficiently fast rate *do not* satisfy C2. For example, for exponential decay $h_0(x) = e^{-cx}$, $x \ge 0$, the integral in C2 tends to a finite constant and so does $\mathrm{Var}\{X(t)\}$ for typical choices of $\lambda(\cdot)$. In such cases, as seen in Ramirez-Perez and Serfling (2001), $X(t)$ has equilibrium covariance structure and exhibits either short range or long range dependence, depending on whether the convergence of $h_0(x)$ to 0 is very fast or only moderately fast.

When, however, the function $h_0(x)$ decays sufficiently slowly, or even tends to infinity, then C2 is satisfied, $\mathrm{Var}\{X(t)\}$ tends to infinity, and, under conditions C1 and C3, the shot noise is asymptotically normal. We now look at some particular examples and applications, implicitly assuming A1–A4.

Example A *The underlying counting process of a shot noise model.* In the case of *no cluster mark* and *unit response*, i.e., with $V \equiv 0$, $W \equiv 1$, and $h_0(x) \equiv 1$, $x \geq 0$, the shot noise corresponding to (3.1) becomes simply $X(t) = \sum_j K_j$, the *counting process* of the underlying cluster model. Trivially, conditions C1–C3 hold and Corollary 3.1 thus yields asymptotic normality of $X(t)$ as $t \to \infty$ (a well-known result; see Westcott, 1973, or Lane, 1984). □

Example B *Power law decay.* For excellent discussion of power law shot noise in modeling semiconductor noise, in modeling electromagnetic fields produced by Cherenkov radiation arising from a random stream of charged particles, in modeling concentrations of neurotransmitter molecules received at a cell in biological information communication, and in various other physical applications, see Lowen and Teich (1990). For discussion of its role in financial modeling, see Ramirez-Perez and Serfling (2001). As a representative example of power law decay, here we take

$$h_0(x) = \begin{cases} e^{-x}, & 0 < x < 1, \\ e^{-1}\, x^{-\theta}, & x \geq 1, \end{cases} \tag{3.2}$$

with $\theta > 0$.

For $0 < \theta \leq \frac{1}{2}$, it is easily checked that as $t \to \infty$, $\int_0^t h_0^2(x)dx \asymp t^{1-2\theta}$ in the case $\theta < \frac{1}{2}$, and $\asymp \log t$ in the case $\theta = \frac{1}{2}$, whence C2 holds. Assuming C1, the boundedness of h_0 then yields C3 as well, so that Corollary 3.1 yields asymptotic normality of $X(t)$ as $t \to \infty$.

In the complementary case that $\theta > \frac{1}{2}$, $h_0(x)$ decays somewhat faster and C2 fails to hold. Then, as seen in Ramirez-Perez and Serfling (2001), $X(t)$ has equilibrium covariance behavior exhibiting long range dependence in the case $\frac{1}{2} < \theta \leq 1$ and short range dependence in the case $\theta > 1$.

By adopting the time origin $t_0 = 0$, as in the present treatment and in Ramirez-Perez and Serfling (2001), we thus obtain a *complete characterization of the transient behavior of power law shot noise*. Specifically, for the three cases $\theta > 1$, $\frac{1}{2} < \theta \leq 1$, and $0 < \theta \leq \frac{1}{2}$, we have, respectively, stable behavior with short range dependence, stable behavior with long range dependence, and explosive behavior with asymptotic normality.

For comparison, consider the case of a time origin $t_0 = -\infty$, whereby the shot noise $X(t)$ becomes "stationary" for any fixed t. Then, however, the shot noise $X(t)$ based on the above h_0 for any $\theta \in (0, 1)$, or on similar shape functions, can be shown easily to be infinite with probability 1 for each t (see Lowen and Teich, 1990, for a derivation in the homogeneous case with no clustering) and no interesting distributional properties hold. By instead modeling the time t as a finite quantity tending to infinity rather than as a fixed and infinite elapsed time, we adhere more closely to actual physical situations and obtain useful descriptive features of the (actually transient) shot noise. (See Remark 4.1 for further discussion.)

If the power law response function is constrained to become zero after a finite duration that is deterministic, then the shot noise is not explosive but rather tends to equilibrium as $t \to \infty$. If, however, the duration is random with a heavy-tailed distribution, then again explosive behavior results and asymptotic normality holds. Such a variant of h is treated in Example E under Model III below. □

Example C *Insurance claims with delayed or extended payments.* Klüppelberg and Mikosch (1995a,b) discuss modeling of total insurance claims when payments are made as a function of time. In such a setting, $X(t)$ represents the total payout at time t from claims generated by events occurring in $[0,t)$. Here we consider some relevant choices of h_0.

(i) Let $h_0(x)$ be any *bounded* nondecreasing function. Then immediately C2 holds, and, if C1 is also assumed, then C3 follows as well and Corollary 3.1 yields asymptotic normality of $X(t)$ as $t \to \infty$. A typical example is $h_0(x) = 1 - e^{-\theta x}$, $x \geq 0$, with $\theta > 0$, which represents a payment delay that decreases exponentially fast after the claim is filed. For this h_0 in the case of no clustering, asymptotic normality of $X(t)$ has been established by Klüppelberg and Mikosch (1995a).

(ii) In some cases h_0 is strictly increasing over time, for example when payments generated by the claim accumulate over time, as in rehabilitation after accidents. A typical example is $h_0(x) = x^\theta$, $x \geq 0$, with $0 \leq \theta < 1/2$. Immediately C2(ii) holds and we further assume C1. Then, applying Remark 3.1(ii), for $\tilde{\nu} > 2$ we have

$$
\begin{aligned}
&\int_0^t E\{|h_0(t-u-D)|^{\tilde{\nu}}\}\,\lambda(u)\,du \\
&\qquad = \int_0^t E\{|t-u-D|^{\theta\tilde{\nu}}\,\mathbf{1}\{D \leq t-u\}\}\,\lambda(u)\,du \\
&\qquad \leq |t|^{\theta(\tilde{\nu}-2)} \int_0^t E\{(t-u-D)^{2\theta}\,\mathbf{1}\{D \leq t-u\}\}\,\lambda(u)\,du \\
&\qquad = \mathrm{O}\left(\left(\int_0^t E\{h_0^2(t-u-D)\}\,\lambda(u)\,du\right)^{1+\theta(\tilde{\nu}-2)}\right) \quad t \to \infty, \\
&\qquad = \mathrm{o}\left(\left(\int_0^t E\{h_0^2(t-u-D)\}\,\lambda(u)\,du\right)^{\tilde{\nu}/2}\right) \quad t \to \infty,
\end{aligned}
$$

since $\theta < 1/2$ implies that $1 + \theta(\tilde{\nu} - 2) < \tilde{\nu}/2$. Thus C3 follows and Corollary 3.1 yields asymptotic normality of $X(t)$ as $t \to \infty$. In the special case that $D \equiv 0$ and $\lambda(\cdot) \equiv \lambda$, it is easily checked that this h_0 fulfills C3 for any $\theta > 0$. With also $K \equiv 0$ (no clustering), asymptotic normality of $X(t)$ for this h_0 has been established by Klüppelberg and Mikosch (1995a). □

3.2 Model II: Unit Responses with Random Amplitudes and Random Durations

Here we consider the response function

$$h(t, t', v, w) = (v + w^{(1)})\,\mathbf{1}\{0 \leq t - t' \leq w^{(2)}\}, \tag{3.3}$$

where now the pointwise mark is bivariate: $w = (w^{(1)}, w^{(2)})$. Thus the contribution to $X(t)$ from the jth cluster is given by $\sum_{k=1}^{K_j} (V_j + W_{jk}^{(1)})\,\mathbf{1}\{0 \leq t - C_j - D_{jk} \leq W_{jk}^{(2)}\}$.

A special case. In the case of *no cluster mark* and *unit amplitude*, i.e., with $V \equiv 0$ and $W^{(1)} \equiv 1$, the shot noise corresponding to (3.3) becomes simply

$$X(t) = \sum_j \sum_{k=1}^{K_j} \mathbf{1}\{0 \le t - C_j - D_{jk} \le W_{jk}^{(2)}\}. \tag{3.4}$$

For example, consider an immigration-death process with new individuals entering in clusters or groups according to a Poisson cluster process, the kth individual in the jth cluster having random lifetime $W_{jk}^{(2)}$. Then the shot noise $X(t)$ represents the *number of individuals present* at time t, for example the number of persons in the system for a service facility with cluster arrivals and service times $W_{jk}^{(2)}$. As another example, if the underlying point process consists of the points of left ends of fibers forming a textile yarn, with fibers arriving in clusters (having different sources), and $W_{jk}^{(2)}$ represents the random length of the kth fiber in the jth cluster, then $X(t)$ represents the *thickness of the yarn* at the point t. For the case of *no clustering*, i.e., $K \equiv 1$, the model (3.4) is treated by Cox and Isham (1980), §5.6. □

More generally, however, we allow both *cluster marks* and *random amplitudes*, each in itself a desirable extension. Thus, in the immigration-death process example, individuals might arrive in clusters of groups, the kth group in the jth cluster having size $W_{jk}^{(1)}$ and being subject to a common random lifetime $W_{jk}^{(2)}$. (For example, individuals might arrive in clusters of vehicles and depart in the same vehicles.) In this case we might take $V_j \equiv 0$. In the textile yarn example, we now let the fibers have differing thicknesses that may include a cluster-wise component V_j as well as a pointwise component $W_{jk}^{(1)}$.

For the general shot noise model based on (3.3), we introduce the conditions

D1 The components $W^{(1)}$ and $W^{(2)}$ of W are independent.

D2 $E\{|V|^\nu\}$ and $E\{|W^{(1)}|^\nu\}$ are finite for some $\nu > 2$.

D3 $E\{W^{(2)}\} = \infty$. □

We then obtain

Corollary 3.2 *Assume A1–A4. Let the response function h be given by (3.3) and satisfy D1–D3. Then $Z(t) \xrightarrow{d} \mathrm{N}(0,1)$ as $t \to \infty$.*

Proof. We apply Corollary 2.1. Note that (3.3) is of form (2.5) with $a(v,w) = v + w^{(1)}$ and $b(t,u,w^{(2)}) = \mathbf{1}\{0 \le t - u \le w^{(2)}\}$. Condition B1 is just D1, and D2 immediately yields B2. It remains to show that B3 and B4 hold. Letting $\overline{G}$ denote $1 - G$ for any cdf G, and using A4, we have by steps similar to those in proving Corollary 3.1 that

$$\begin{aligned}
&\int_0^t E\{b^2(t, u+D, W^{(2)})\}\,\lambda(u)\,du \\
&\qquad = \int_0^t E_D\{\overline{F}_{W^{(2)}}(t-u-D)\,\mathbf{1}\{0 \le D \le t-u\}\}\,\lambda(u)\,du \\
&\qquad \ge \int_0^t \overline{F}_{W^{(2)}}(t-u)\,P(D \le t-u)\,\lambda(u)\,du
\end{aligned}$$

$$\geq \lambda_0 \, P(D \leq c_0) \int_0^{t-c_0} \overline{F}_{W^{(2)}}(t-u)\, du$$
$$= \lambda_0 \, P(D \leq c_0) \int_{c_0}^{t} \overline{F}_{W^{(2)}}(x)\, dx,$$

from which B3 follows via D3. Finally, noting that $|b(t, u+D, W^{(2)})|^{\alpha} = b(t, u+D, W^{(2)})$ for any $\alpha > 0$, it is easy to obtain B4 from B3. $\square$

Example D *Pareto law durations.* Let $W^{(2)}$ have the Pareto density

$$f(x) = \beta\, \sigma^{\beta}\, (x+\sigma)^{-\beta-1}, \; x > 0,$$

in which case $\overline{F}_{W^{(2)}}(x) = \sigma^{\beta}\, (x+\sigma)^{-\beta}$. For $\beta \leq 1$, condition D3 holds and thus, under D1 and D2, asymptotic normality of $X(t)$ follows. In the complementary case that $\beta > 1$, $W^{(2)}$ has finite mean and as shown in Ramirez-Perez and Serfling (2001) $X(t)$ has equilibrium covariance structure exhibiting long range dependence if $1 < \beta \leq 2$ and short range dependence if $\beta > 2$. $\square$

In general, *equilibrium behavior and long range dependence* for the shot noise based on (3.3) results from *moderate heavy tail behavior* of the random duration $W^{(2)}$, while *explosive behavior and asymptotic normality* results from *strong heavy tail behavior* of $W^{(2)}$. Below we examine this more broadly, for a response function that simultaneously generalizes (3.1) and (3.3).

3.3 Model III: Arbitrary Shape Functions with Random Amplitudes and Random Durations

Here we consider the response function

$$h(t, t', v, w) = (v + w^{(1)})\, \mathbf{1}\{0 \leq t - t' \leq w^{(2)}\}\, h_0(t - t'), \tag{3.5}$$

with $h_0(x) = 0$ for $x > 0$ as in Model I and $w = (w^{(1)}, w^{(2)})$ again bivariate as in Model II. Besides formally unifying Models I and II, the generality of (3.5) has practical appeal in many application contexts, it being reasonable to suppose that a response of form $w\, h_0(t-t')$ or $(v+w)\, h_0(t-t')$ might remain in effect only throughout a finite duration, possibly random. This provides additional flexibility in modeling and greater potential to explain observed behavior of shot noise processes.

With regard to (3.5) we introduce the following general conditions:

E1 The components $W^{(1)}$ and $W^{(2)}$ of W are independent.

E2 $E\{|V|^{\nu}\}$ and $E\{|W^{(1)}|^{\nu}\}$ are finite for some $\nu > 2$.

E3 Either

(i) h_0 is nonincreasing and satisfies $\int_0^t h_0^2(x)\, \overline{F}_{W^{(2)}}(x)\, dx \to \infty$, $t \to \infty$,

or

(ii) h_0 is nondecreasing and $E\{W^{(2)}\} = \infty$.

E4 For some $\tilde{\nu} > 2$,

$$\int_0^t E\{|h_0(t-u-D)|^{\tilde{\nu}}\, \overline{F}_{W^{(2)}}(t-u-D)\, \mathbf{1}\{D \le t-u\}\}\, \lambda(u)\, du$$
$$= \mathrm{o}\left(\left(\int_0^t E\{h_0^2(t-u-D)\, \overline{F}_{W^{(2)}}(t-u-D)\, \mathbf{1}\{D \le t-u\}\}\, \lambda(u)\, du\right)^{\tilde{\nu}/2}\right)$$

as $t \to \infty$. □

Many typical cases are covered by the following result.

Corollary 3.3 *Assume A1–A4. Let the response function h be given by (3.5) and satisfy E1–E4. Then $Z(t) \xrightarrow{d} \mathrm{N}(0,1)$ as $t \to \infty$.*

Proof. We proceed by steps similar to those in the proofs of Corollaries 3.1 and 3.2. Note that (3.1) is of form (2.5) with $w = (w^{(1)}, w^{(2)})$, $a(v,w) = v + w^{(1)}$, and $b(t,u,w^{(2)}) = h_0(t-u)\, \mathbf{1}\{0 \le t-u \le w^{(2)}\}$. Then B1 is just E1, B2 follows immediately from E2, and, provided that B3 holds, B4 follows from E4. It remains to show that B3 holds, for which we apply E3 and A4 together. Let us first assume E3(i). Then, also using A4, for $t > c_0$ we have

$$\begin{aligned}
&\int_0^t E\{b^2(t,u+D,W^{(2)})\}\, \lambda(u)\, du \\
&\quad = \int_0^t E_D\{h_0^2(t-u-D)\, \overline{F}_{W^{(2)}}(t-u-D)\, \mathbf{1}\{0 \le D \le t-u\}\}\, \lambda(u)\, du \\
&\quad \ge \int_0^t h_0^2(t-u)\, \overline{F}_{W^{(2)}}(t-u)\, P(D \le t-u)\, \lambda(u)\, du \\
&\quad \ge \lambda_0\, P(D \le c_0) \int_0^{t-c_0} h_0^2(t-u)\, \overline{F}_{W^{(2)}}(t-u)\, du \\
&\quad = \lambda_0\, P(D \le c_0) \int_{c_0}^t h_0^2(x)\, \overline{F}_{W^{(2)}}(x)\, dx,
\end{aligned}$$

yielding B3. Next we assume E2(ii). Again using A4, for $t > 2c_0$ we now have

$$\begin{aligned}
&\int_0^t E\{b^2(t,u+D,W^{(2)})\}\, \lambda(u)\, du \\
&\quad \ge \int_0^t h_0^2((t-u)/2)\, \overline{F}_{W^{(2)}}(t-u)\, P(D \le (t-u)/2)\, \lambda(u)\, du \\
&\quad \ge \lambda_0\, P(D \le c_0) \int_0^{t-2c_0} h_0^2((t-u)/2)\, \overline{F}_{W^{(2)}}(t-u)\, du \\
&\quad \ge \lambda_0\, P(D \le c_0)\, h_0^2(c_0) \int_0^{t-2c_0} \overline{F}_{W^{(2)}}(t-u)\, du,
\end{aligned}$$

and again B3 follows. □

Remark 3.2 When E2 and E3 hold, a sufficient conditon for E4 is that $|h_0(\cdot)|$ is bounded. This follows by an argument similar to that used in Remark 2.1. □

Example E *Power law response with Pareto law durations.* Let h_0 be of power law form as given by (3.2) in Example B and let $W^{(2)}$ have the Pareto density with parameters σ and β as in Example D. Here we require $\beta + 2\theta \leq 1$. Assume E1 and E2. Since h_0 is bounded, E4 then holds if E3 holds. Now h_0 is nonincreasing and it is easily checked that as $t \to \infty$, $\int_0^t h_0^2(x)\,\overline{F}_{W^{(2)}}(x)\,dx \geq e^{-2}\,\sigma^\beta \int_1^t (x+\sigma)^{-\beta-2\theta}\,dx \asymp t^{1-\beta-2\theta}$ if $\beta + 2\theta < 1$, and $\asymp \log t$ if $\beta + 2\theta = 1$. Thus E3 holds and Corollary 3.3 yields asymptotic normality of $X(t)$. For the equilibrium behavior of $X(t)$ when $\beta + 2\theta > 1$, see Ramirez-Perez and Serfling (2001). □.

Example F Let $h_0(x) = 1 - e^{-\theta x}$, $x \geq 0$, with $\theta > 0$, as in Example C, and again let $W^{(2)}$ have the Pareto density with parameters σ and β as in Example D, with $\beta \leq 1$. Then, assuming E1 and E2, the further conditions of Corollary 3.3 are easily checked and asymptotic normality of $X(t)$ follows. □

4 Complements

Remark 4.1 *Other modes of "asymptotics".* (i) Asymptotics of power law shot noise for fixed time t and time origin $t_0 = -\infty$ are studied by Lowen and Teich (1990) in the case of *no clustering* for a *homogenous* Poisson process with rate λ tending to infinity. In particular, they consider the response function

$$h(t, t') = W\,(t - t')^{-\theta},\ A \leq t - t' < B,$$

with A and B deterministic. For the case $A > 0$, $B < \infty$ and $\theta \leq 1$, for example, they obtain convergence of $X(t)$ (unnormalized) to a normal limit as $\lambda \to \infty$. They obtain no results, however, for the case $B = \infty$ and $\theta \leq 1$, however, since then $X(t)$ is infinite due to the choice of time origin $t_0 = -\infty$. Our results in Examples B and E complement their work in several ways: extending to the clustering case, using time origin $t_0 = 0$, conducting asymptotics as the time t tends to ∞, and establishing asymptotic normality for $X(t)$ when either $B = \infty$ or B is finite but random with a heavy-tailed distribution.

For the homogeneous case, we note that by rescaling the time t we may recast asymptotics for $t \to \infty$ with fixed rate λ into asymptotics for $\lambda \to \infty$ with fixed time t_0. This approach appears less amenable to interpretion, however, in actual physical situations.

(ii) When the shot noise given by (1.1) is defined for points t and $\{T_{jk}\}$ in $\mathbb{R}^d$, it is called "multidimensional shot noise" and represents a form of random field. For the case of an underlying stationary and Brillinger mixing point process in $\mathbb{R}^d$ with intensity λ, Heinrich and Schmidt (1985) develop results on asymptotic normality of $X(t)$ for fixed t as $\lambda \to \infty$. This includes the case of a stationary Poisson process in $\mathbb{R}^d$. Our asymptotic results for $t \to \infty$ in the case $d = 1$ complement their results. □

Remark 4.2 *Other convergence results.* Other types of convergence for explosive Poisson cluster shot noise $X(t)$ as $t \to \infty$ are of interest. For example, laws of large numbers may be derived under straightforward conditions. For the case of no clustering, such results are given in Klüppelberg and Mikosh (1995a), along with a functional central limit theorem. The latter result can be extended to the cluster case under similar conditions as established here for simple asymptotic normality. Likewise, the Berry-Esseen theorem in the no clustering case has been given by Lane (1987) and Klüppelberg and Mikosh (1995a) and can be extended in straightforward fashion to the case of clustering. □

Remark 4.3 *Other limit laws.* Other types of limit law for $Z(t)$ can arise when the variance of $X(t)$ is infinite (in which case the normalizing term in the denominator of $Z(t)$ is defined differently). For the case of no clustering, using the infinite divisibility of shot noise on a Poisson point process and the canonical measure approach of Feller (1971), these results are obtained by Lane (1984) and yield his Theorem 3 that is the basis of our Lemma 2.1. Using the representation (1.3), the shot noise process $X(t)$ given by (1.1) in the clustering case is seen to be infinitely divisible also. Then Theorem 1 of Lane (1984) carries over with some obvious modifications and characterizes the limiting law of $Z(t)$ in terms of the limit of a canonical measure associated with the cdf of the random variable $g(t, u, \gamma)$ given in (1.2). In particular, for the shot noise based on (1.1) with response function (3.3), Example 2.1 of Lane (1984) can be extended to yield the following result: *If the cdf of the amplitude* $V + W^{(1)}$ *belongs to the domain of attraction of a stable law with index* α, $0 < \alpha \leq 2$, *and the extent* $W^{(2)}$ *has infinite mean, then the limiting distribution of* $Z(t)$ *is also stable with index* α. (For the special case $V + W^{(1)} \equiv c$, we have $\alpha = 2$ and the limit law is normal, as already derivable from Corollary 3.2 above.) □

Remark 4.4 *Miscellaneous extensions.* The following extensions are desirable but beyond the scope of the present development. (i) We could allow for "foreshocks" by taking the time interval $[-t, t]$ instead of $[0, t]$. (ii) Accordingly, we could consider impulse functions $h(t, t', y)$ which may be nonzero for $-t \leq t' \leq t$ rather than only for $0 \leq t' \leq t$. (iii) We could permit the rate function $\lambda(\cdot)$ of the Poisson cluster process to be itself a stochastic process, for example a Markov-modulated Poisson process with finitely many environmental states. (iv) We could consider the underlying point process to be $\mathbb{R}^d$-valued and define the response function over suitable regions of $\mathbb{R}^d$, for example compact and convex subsets. □

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