Multivariate Generalized Spatial Signed-Rank Methods

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Abstract

New multivariate generalized signed-rank tests for the one sample location model having favorable efficiency and robustness properties are introduced and studied. Limiting distributions of the tests and related estimates as well as formulae for asymptotic relative efficiencies are found. Relative efficiencies with respect to the classical Hotelling T^2 test (and the mean vector) are evaluated for the multivariate normal, t, and Tukey models. While the tests (estimates) are only rotation invariant (equivariant), versions that are affine invariant (equivariant) are discussed as well.

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1 Introduction

The purpose of this paper is to consider the robustness and efficiency properties of new multivariate generalized signed-rank tests that we introduce here and related multivariate generalized Hodges-Lehmann estimators that were proposed and studied by Chaudhuri (1992). These tests and estimates provide (more or less) robust and efficient competitors of the classical Hotelling T^2 test and the mean vector.

Let $Y = \{\boldsymbol{y}_1, ..., \boldsymbol{y}_n\}$ be a random sample from a k-variate distribution with cdf F (and pdf f) symmetric around $\boldsymbol{\theta}$. We wish to estimate the unknown location centre $\boldsymbol{\theta}$ and test the null hypothesis $H_0 : \ \boldsymbol{\theta} = \mathbf{0}$. Möttönen and Oja (1995) and later Möttönen, Oja, and Tienari (1997) developed and studied multivariate spatial sign and signed-rank tests for this problem. The two test statistics they considered may be written in the form

$$\boldsymbol{W}_{mn} = n^{-m} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \boldsymbol{S}(\boldsymbol{y}_{i_1} + \cdots + \boldsymbol{y}_{i_m}),$$

for $m = 1$ (giving the "spatial sign test statistic") or $m = 2$ (giving the "spatial signed-rank test statistic"). Here the function

$$\boldsymbol{S}(\boldsymbol{y}) = \begin{cases} ||\boldsymbol{y}||^{-1}\boldsymbol{y} & \text{if } \boldsymbol{y} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \boldsymbol{y} = \mathbf{0} \end{cases},$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^k$, denotes the *spatial sign function* of $\boldsymbol{y} \in \mathbb{R}^k$. It is invariant under affine transformations except for those involving heterogeneous scale changes. In this paper, we consider the above statistics $\boldsymbol{W}_{mn}$ for *arbitrary* $m = 1, 2, \ldots$ and call them *generalized signed-rank test statistics.*

Unlike the univariate case with $m = 1$ and 2, the test statistics $\boldsymbol{W}_{mn}$ in general are only conditionally and asymptotically, but not strictly, distribution-free under the null hypothesis. The conditional 'sign-change' tests are then based on the 2^n (under H_0 and symmetry) equiprobable cases $\pm\boldsymbol{y}_1, \ldots, \pm\boldsymbol{y}_n$. These test statistics are not affine invariant, which means that the obtained p-values depend on the chosen coordinate system.

Related location estimators – generalized Hodges-Lehmann estimators – are obtained by a standard approach as follows. First, the observations are centered with respect to a candidate $\boldsymbol{\theta}$ by setting $\boldsymbol{z}_i = \boldsymbol{y}_i - \boldsymbol{\theta}$, $i = 1, ..., n$. Next one calculates the value of the $\boldsymbol{W}_{mn}$ statistic, say $\boldsymbol{W}_{mn}(\boldsymbol{\theta})$, for the centered data. Finally, one obtains the estimator $\boldsymbol{\theta}_{mn}$ via the equation $\boldsymbol{W}_{mn}(\boldsymbol{\theta}_{mn}) = \mathbf{0}$. this yields the class of estimators studied by Chaudhuri (1992). Two special cases are the multivariate spatial median ($m = 1$) and multivariate spatial Hodges-Lehmann estimator ($m = 2$); for these see also Möttönen and Oja (1995). While the estimators $\hat{\boldsymbol{\theta}}_{mn}$ are found to have favorable efficiency and robustness properties, they are not fully affine equivariant, however.

In the paper, the efficiency and robustness properties of generalized spatial signed-rank tests and generalized Hodges-Lehmann estimates are considered in general, so as to include the cases $m \geq 3$. It will be seen, for example, that "efficiency" at normal models increases with m, while "robustness" decreases. It is important, therefore, to establish benchmarks by which m may be selected so as to obtain a suitable trade-off between efficiency and robustness. The desired balance between these criteria depends, of course, upon the given context of application and the particular losses associated with lack of efficiency and lack of robustness.

Numerical asymptotic relative efficiencies are provided for some specific models in Section 4. For $m = 1, 2, \ldots, 5$, and for $k = 1, 2, 3, 4, 6$ and 10, we compare the generalized spatial signed-rank test procedures with the classical Hotelling T^2 test for the k-variate standard normal model and for k-variate t models with 3, 6, and 10 degrees of freedom (see Tables 1-4). For example, for 6-variate standard normal, this ARE increases from 0.920 for $m = 1$ to 0.997 for $m = 5$. On the other hand, for the t distributions, the ARE's are all greater than 1, but do not follow a strict monotonicity pattern. We note, though, that for the 6-variate t distribution with 3 degrees of freedom, the ARE's decrease from 2.344 for $m = 1$ to 1.697 for $m = 5$. Also, for the same range of m and k, we examine both the ARE of the tests and the bias of the estimators, for a family of Tukey models, i.e., contaminated multivariate normal models, with a range of contamination and spread parameters. Favorable robustness properties of our generalized signed-rank methods are indicated by their bounded influence functions and nonzero breakdown points (see Section 3.2). It is of interest that, independently of the dimension, the breakdown point decreases from 0.5 for $m = 1$ to 0.129 for $m = 5$.

The theoretical foundation underlying the above practical results is developed as follows. In Section 2 we treat the asymptotic properties of our generalized signed-rank test procedures. These are obtained via an asymptotic equivalence (Lemma 2) with certain statistics of technically more convenient form expressed in terms of a "multivariate generalized signed rank function", which we define and discuss. In this fashion we obtain chi-square limit distributions for Wald-type generalized signed-rank test statistics, both under the null hypothesis (Theorem 1) and under contiguous alternatives (Theorem 2). The related estimators are discussed in Section 3: asymptotic normality of the estimators is provided by Theorem 3, and natural V-statistic estimators of the limiting covariance matrix are introduced. Proofs are relegated to the Appendix, which contains numerous technical lemmas taken from a more complete source, Möttönen (2002).

In Section 5 we discuss efficiency-robustness trade-offs, we describe the construction of fully affine invariant/equivariant versions, and we briefly make comparisons with selected other procedures. For the latter, an interesting finding is that the comparisons of competing procedures tend to depend upon the dimension.

2 Tests

2.1 Generalized spatial signed-ranks

We first define what we mean by the multivariate generalized spatial signed-rank function:

Definition 1 *Assume that* $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$*,* $m = 1, 2, \ldots$*, is a random sample from a* k*-variate symmetrical distribution with symmetry centre* $\mathbf{0}$ *and cdf* F*. Then the population generalized spatial signed-rank function (of order* m*) is*

$$\boldsymbol{R}_m(\boldsymbol{y}) = E_F\{\boldsymbol{S}(\boldsymbol{y} - \boldsymbol{y}_1 - \ldots - \boldsymbol{y}_{m-1})\}.$$

If $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ *is a random sample from* F*, the corresponding empirical spatial signed-rank function (of order* m*) is*

$$\boldsymbol{R}_{mn}(\boldsymbol{y}) = \text{ave}\{\boldsymbol{S}(\boldsymbol{y} - \boldsymbol{y}_{i_1} - \ldots - \boldsymbol{y}_{i_{m-1}})\},$$

where the average is taken over all possible $(m-1)$*-sets of observations, that is, over all cases* $1 \le i_1 < \ldots < i_{m-1} \le n$*.*

Note that, $\boldsymbol{R}_1(\boldsymbol{y})$ reduces to the spatial sign function and $\boldsymbol{R}_2(\boldsymbol{y})$ is the regular (population) spatial rank function (F symmetric around the origin). Note also that $\boldsymbol{R}_m(\boldsymbol{y})$ is the theoretical rank function for the distribution of $\boldsymbol{y}_1 + \ldots + \boldsymbol{y}_{m-1}$. See Möttönen and Oja (1995). (We could symmetrize $\boldsymbol{R}_{mn}(\boldsymbol{y})$, if desired, by taking the average over cases $\pm\boldsymbol{y}_1, \ldots, \pm\boldsymbol{y}_n$, which would be useful when constructing sign-change tests.) For spherical distributions, the generalized spatial signed-rank function at $\boldsymbol{y}$ ($r = ||\boldsymbol{y}||$ and $\boldsymbol{u} = ||\boldsymbol{y}||^{-1}\boldsymbol{y}$) is

$$\boldsymbol{R}_m(\boldsymbol{y}) = q_m(r)\boldsymbol{u},$$

where $q_m(r) = \varrho_m'(r)$ and $\varrho_m(r) = E_F(||\boldsymbol{y} - \boldsymbol{y}_1 - \ldots - \boldsymbol{y}_{m-1}||)$ (the expectation does not depend on $\boldsymbol{u}$).

As in Möttönen, Oja and Tienari (1997), we can establish

Proposition 1 *Assume that* f *is continuous and bounded. Then*

$$\sup_{\boldsymbol{y}} |\boldsymbol{R}_{mn}(\boldsymbol{y}) - \boldsymbol{R}_m(\boldsymbol{y})| \xrightarrow{P} \mathbf{0}$$

2.2 Test statistics

Definition 2 *The generalized signed-rank test statistic (of order* m*) for testing* $H_0 : \boldsymbol{\theta} = \mathbf{0}$ *is the vector-valued V-statistic*

$$\boldsymbol{W}_{mn} = n^{-m} \sum_{i_1=1}^{n} \ldots \sum_{i_m=1}^{n} \boldsymbol{S}(\boldsymbol{y}_{i_1} + \ldots + \boldsymbol{y}_{i_m}).$$

An asymptotically equivalent version of this test statistic can be constructed using the generalized spatial ranks.

Lemma 1 *Under the null hypothesis, W_{mn} is asymptotically equivalent to*

$$\boldsymbol{T}_{mn} = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{R}_m(\boldsymbol{y}_i),$$

that is,

$$\sqrt{n}(\boldsymbol{W}_{mn} - \boldsymbol{T}_{mn}) \xrightarrow{P} \mathbf{0}.$$

This in fact holds uniformly over all alternatives, since the model is a location model and the statistics are location equivariant.

Under the null hypothesis, the expectation and variance of $\boldsymbol{T}_{mn}$ are

$$E_0(\boldsymbol{T}_{mn}) = \mathbf{0} \;\text{ and }\; \mathrm{Var}_0(\boldsymbol{T}_{mn}) = \frac{1}{n}\, B_m,$$

respectively, where

$$B_m = B_m(F) = E_F[\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{R}_m^T(\boldsymbol{y})]$$

is the *generalized rank covariance matrix.* For $m = 1$ and $m = 2$, the regular spatial sign and rank covariance matrices are obtained. See Visuri *et al.* (2000).

The generalized rank covariance matrix B_m may be consistently estimated by the corresponding sample counterpart (sample rank covariance matrix),

$$B_{mn} = \mathrm{ave}\{\boldsymbol{R}_{mn}(\boldsymbol{y}_i)\boldsymbol{R}_{mn}^T(\boldsymbol{y}_i)\}.$$

Note that related sign-change tests can be constructed, and that asymptotic tests may be constructed (where the p-value is calculated using the limiting distribution). For brevity, we omit the details.

Lemma 2 *Under the null hypothesis,*

$$B_{mn} \xrightarrow{P} B_m,\; n \to \infty.$$

Theorem 1 *Under the null hypothesis, the limiting distribution of*

$$n\boldsymbol{T}_{mn}^T B_{mn}^{-1} \boldsymbol{T}_{mn}$$

is a chi-square distribution with k degrees of freedom.

2.3 Formulae for Pitman efficiency

For limiting efficiencies, we next consider a sequence of contiguous alternatives $H_n : \boldsymbol{\theta} = n^{-1/2}\boldsymbol{\delta}$. Write

$$\boldsymbol{L}(\boldsymbol{y}) = -\nabla \log(f(\boldsymbol{y}))$$

for the optimal location score function and assume that, under the null hypothesis,

$$L_n = \sum_{i=1}^{n}[\log(f(\boldsymbol{y}_i - n^{-1/2}\boldsymbol{\delta}) - \log f(\boldsymbol{y}_i)] = n^{-1/2}\boldsymbol{\delta}^T\boldsymbol{U}_n - \frac{1}{2}\boldsymbol{\delta}^T I_0\boldsymbol{\delta} + o_P(1),$$

where

$$\boldsymbol{U}_n = \sum_{i=1}^{n} \boldsymbol{L}(\boldsymbol{y}_i) \ \text{ and } \ I_0 = E_0(\boldsymbol{L}(\boldsymbol{y})\boldsymbol{L}^T(\boldsymbol{y})).$$

Note that L_n is the log-likelihood test statistic (H_0 vs. H_1), $\boldsymbol{U}_n$ the optimal score statistic, and I_0 the expected Fisher information matrix for a single observation.

Then, under the sequence of contiguous alternatives H_n, the limiting distribution of $n^{1/2}\boldsymbol{T}_{mn}$ is $N_k(A_m\boldsymbol{\delta}, B_m)$ where

$$A_m = E_0(\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{L}^T(\boldsymbol{y})) \ \text{ and } \ B_m = E_0(\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{R}_m^T(\boldsymbol{y})).$$

This can be derived as in Möttönen and Oja (1995). Finally, the limiting distribution of the squared test statistic $n\boldsymbol{T}_{mn}^T B_{mn}^{-1}\boldsymbol{T}_{mn}$ follows.

Theorem 2 *Under the sequence of alternatives H_n, the limiting distribution of*

$$n\boldsymbol{T}_{mn}^T B_{mn}^{-1}\boldsymbol{T}_{mn}$$

is a noncentral chi-square distribution with k degrees of freedom and noncentrality parameter

$$\boldsymbol{\delta}^T A_m^T B_m^{-1} A_m\boldsymbol{\delta}.$$

Asymptotic Pitman efficiencies with respect to the optimal likelihood ratio test test are then simply

$$\frac{\boldsymbol{\delta}^T A_m^T B_m^{-1} A_m\boldsymbol{\delta}}{\boldsymbol{\delta}^T I_0\boldsymbol{\delta}}.$$

This may be used in a standard way, as discussed, for example, in Hettmansperger and McKean (1998).

3 Estimators

3.1 Generalized Hodges-Lehmann estimators

For completeness, we briefly treat the generalized Hodges-Lehmann estimators defined in Section 1.

Lemma 3 *Under general assumptions,*

$$\sqrt{n}\boldsymbol{\theta}_{mn} = \sqrt{n}A_m^{-1}\boldsymbol{W}_{mn} + \boldsymbol{o}_P(1)$$

Theorem 3 *Under general assumptions,*

$$\boldsymbol{\theta}_{mn} \sim AN\left(\boldsymbol{\theta}, \frac{1}{n}A_m^{-1}B_m A_m^{-1}\right).$$

The limiting variance can be estimated by estimating A_m and B_m separately. In Section 2.2 we gave a natural estimate of the generalized rank covariance matrix, namely the empirical rank covariance matrix B_{mn}. To estimate A_m, note that with $\bar{\boldsymbol{y}}_m = (1/m)(\boldsymbol{y}_1 + \ldots + \boldsymbol{y}_m)$ we have

$$A_m = E_F||\bar{\boldsymbol{y}}_m||^{-1}I_k - E\left[\frac{\bar{\boldsymbol{y}}_m\bar{\boldsymbol{y}}_m^T}{||\bar{\boldsymbol{y}}_m||^3}\right].$$

A natural estimate is then the V-statistic

$$A_{mn} = n^{-m}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\left\{||\bar{\boldsymbol{y}}(I)||^{-1}I_k - \left[\frac{\bar{\boldsymbol{y}}(I)\bar{\boldsymbol{y}}(I)^T}{||\bar{\boldsymbol{y}}(I)||^3}\right]\right\},$$

where $I = (i_1, \ldots, i_k)$ and $\bar{\boldsymbol{y}}(I) = (1/m)(\boldsymbol{y}_{i_1} + \ldots + \boldsymbol{y}_{i_m})$.

3.2 Comparison of efficiency and robustness

Comparison of asymptotic covariance matrices $A_m^{-1}B_mA_m^{-1}$ has been carried out in the multivariate normal case by Chaudhuri (1992), in which case it is found that efficiency increases with m.

The influence function of the functional $\boldsymbol{\theta}_m$ is found to be

$$IF(\boldsymbol{y};\boldsymbol{\theta}_m, F) = A_m^{-1}\boldsymbol{R}_m(\boldsymbol{y}),$$

which is bounded since $\boldsymbol{R}_m(\boldsymbol{y})$ is bounded ($||\boldsymbol{R}_m(\boldsymbol{y})|| \leq 1$). The breakdown point of $\hat{\boldsymbol{\theta}}_{mn}$ decreases, although slowly, as m increases, and is independent of the dimension. Specifically,

$$\text{BP}(\hat{\boldsymbol{\theta}}_{mn}) = 1 - (1/2)^{1/m},$$

which takes values 0.5, 0.293, 0.206, 0.159, 0.129 for m = 1, 2, 3, 4, 5.

4 Asymptotic relative efficiency results for some specific models

4.1 Multivariate normal distributions

The multivariate normal case has been considered earlier by Chaudhuri (1992). We assume that $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ is a random sample from the $N(\boldsymbol{0}, I_k)$ distribution. If $||\boldsymbol{y}|| = r$ and $\boldsymbol{u} = ||\boldsymbol{y}||^{-1}\boldsymbol{y}$, then $\boldsymbol{R}_m(\boldsymbol{y}) = q_m(r)\boldsymbol{u}$ where

$$q_m(r) = q\left(\frac{r}{\sqrt{m-1}}\right)$$

and

$$q(r) \;=\; \frac{r}{2^{1/2}} \exp(-\frac{r^2}{2}) \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})} \; {}_1F_1\left(\frac{k+1}{2}; \frac{k+2}{2}; \frac{r^2}{2}\right)$$

is the regular spatial signed-rank function (Möttönen and Oja, 1995).

We then obtain (see the Appendix for details)

$$\begin{aligned}
A_m &= \frac{1}{k}\sqrt{\frac{2}{m}}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} I_k, \\
B_m &= \frac{2\Gamma^2(\frac{k+1}{2})}{mk^2\Gamma^2(\frac{k}{2})} \; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{k+2}{2}; \frac{1}{m^2}\right) I_k, \\
A_m^T B_m^{-1} A_m &= \left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{k+2}{2}; \frac{1}{m^2}\right)\right]^{-1} I_k, \\
ARE_m &= \left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{k+2}{2}; \frac{1}{m^2}\right)\right]^{-1},
\end{aligned}$$

where ARE_m is the Pitman efficiency of the generalized spatial signed-rank test with respect to Hotelling's T^2 test. See Table 1 for values of ARE_m for selected values of k and $m = 1, 2, \ldots, 5$. The notation ${}_2F_1$ denotes the Gauss hypergeometric function.

4.2 Multivariate t distributions

Let us now assume that $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ is a random sample from a k-variate t-distribution with ν degrees of freedom, denoted by $t_{k,\nu}$. See the Appendix. The Pitman asymptotic relative efficiency of the generalized signed-rank test with respect to Hotelling's T^2 is then

$$ARE_m = \frac{\mu(\nu+k)^2}{k(\nu-2)} \left[E\left\{ q_m(r) \frac{r}{\nu + r^2} \right\} \right]^2 \left[E\{q_m^2(r)\} \right]^{-1},$$

Table 1: Asymptotic relative efficiencies of $\boldsymbol{W}_{mn}$ with respect to T^2 in the k-variate standard normal model, for selected values of k and $m = 1, 2, \ldots, 5$.

	m				
k	1	2	3	4	5
1	.637	.955	.981	.989	.993
2	.785	.967	.986	.992	.995
3	.849	.973	.989	.994	.996
4	.884	.978	.991	.995	.997
6	.920	.984	.993	.996	.997
10	.951	.989	.995	.997	.998
∞	1.000	1.000	1.000	1.000	1.000

where r^2/k has a $F(k, \nu)$ distribution. See the Appendix for q_m. Tables 2, 3 and 4 list the Pitman efficiencies of $\boldsymbol{W}_{mn}$ with respect to T^2 for $t_{k,\nu}$ distributions with selected dimensions k and degrees of freedom ν, for $m = 1, 2, \ldots, 5$.

Table 2: Asymptotic relative efficiencies of $\boldsymbol{W}_{mn}$ with respect to T^2 in the case of the k-variate t distribution with $\nu = 3$ degrees of freedom, for selected values of k and $m = 1, 2, \ldots, 5$.

	m				
k	1	2	3	4	5
1	1.621	1.900	1.762	1.660	1.595
2	2.000	1.953	1.789	1.692	1.615
3	2.162	1.994	1.794	1.686	1.618
4	2.250	2.018	1.813	1.699	1.626
6	2.344	2.050	1.830	1.697	1.621
10	2.422	2.093	1.841	1.730	1.639

4.3 Multivariate Tukey distributions

In this section we consider contaminated multivariate normal distributions, called here *multivariate Tukey* distributions. We say that the distribution of $\boldsymbol{y}$ is k-variate Tukey$(\epsilon, \boldsymbol{\mu}, \sigma)$ if its p.d.f. is

$$f(\boldsymbol{y}) = (1 - \epsilon)\phi(\boldsymbol{y}) + \epsilon\sigma^{-k}\phi(\sigma^{-1}(\boldsymbol{y} - \boldsymbol{\mu})),$$

Table 3: Asymptotic relative efficiencies of $\boldsymbol{W}_{mn}$ with respect to T^2 in the case of the k-variate t distribution with $\nu = 6$ degrees of freedom, for selected values of k and $m = 1, 2, \ldots, 5$.

	m				
k	1	2	3	4	5
1	0.879	1.164	1.141	1.121	1.111
2	1.084	1.187	1.159	1.128	1.109
3	1.172	1.200	1.159	1.137	1.116
4	1.220	1.208	1.164	1.133	1.115
6	1.271	1.219	1.170	1.142	1.117
10	1.313	1.229	1.175	1.143	1.121

Table 4: Asymptotic relative efficiencies of $\boldsymbol{W}_{mn}$ with respect to T^2 in the case of the k-variate t distribution with $\nu = 10$ degrees of freedom, for selected values of k and $m = 1, 2, \ldots, 5$.

	m				
k	1	2	3	4	5
1	0.757	1.054	1.056	1.048	1.045
2	0.934	1.071	1.063	1.056	1.048
3	1.009	1.081	1.070	1.058	1.048
4	1.051	1.087	1.073	1.060	1.050
6	1.094	1.095	1.075	1.061	1.052
10	1.131	1.103	1.078	1.061	1.054

where $\phi(\boldsymbol{y})$ is the p.d.f. of $N(\mathbf{0}, I_k)$ and $\epsilon \in [0, 1]$. This model proves useful in considering the asymptotic efficiency and asymptotic bias in the case that part of the data comes from a population of outliers. The problem is then to estimate the location vector of the population associated with the majority of the data.

Now

$$q_m(r) = \sum_{h=0}^{m-1} \binom{m-1}{h} \epsilon^h (1-\epsilon)^{m-1-h} q\left(\frac{r - h||\boldsymbol{\mu}||}{\sqrt{m - 1 + (\sigma^2 - 1)h}} \right),$$

where

$$q(r) = \frac{r}{2^{1/2}} \exp(-\frac{r^2}{2}) \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})} \, {}_1F_1\left(\frac{k+1}{2}; \frac{k+2}{2}; \frac{r^2}{2} \right)$$

is as in Section 4.1. The optimal score function is then

$$\boldsymbol{L}(\boldsymbol{y}) = \frac{(1-\epsilon)\phi(\boldsymbol{y}) + \epsilon\sigma^{-k-2}\phi(\sigma^{-1}\boldsymbol{y})}{(1-\epsilon)\phi(\boldsymbol{y}) + \epsilon\sigma^{-k}\phi(\sigma^{-1}\boldsymbol{y})}\boldsymbol{y}.$$

See the Appendix for details on A_m and B_m. In Figure 1 we exhibit the Pitman efficiencies of $\boldsymbol{W}_{mn}$ with respect to T^2 under k-variate Tukey($\epsilon, \boldsymbol{0}, \sigma$) models with various $\sigma^2 \geq 1$, ϵ, and k, for $m = 1, 2, \ldots, 5$. It is seen that the ARE increases with dimension k and σ and exceeds 1 for $m \geq 2$. Figure 2 exhibits the bias of the functional $\boldsymbol{\theta}_m$, under k-variate Tukey($\epsilon, \boldsymbol{\mu}, \sigma$) models with various $||\boldsymbol{\mu}||$, σ^2, ϵ, and k, for $m = 1, 2, \ldots, 5$. As expected, bias increases with $||\boldsymbol{\mu}||$.

5 Comparisons and further remarks

5.1 Trade-offs between efficiency and robustness

We see that for low degrees of freedom, say $\nu = 3$, the ARE at t distributions becomes decreasing as m increases as soon as the dimension k is greater than 1. Thus the gain in ARE at the normal for higher values of m is paid for by a decrease in ARE at a heavy-tailed distribution such as the multivariate t with 3 degrees of freedom. For degrees of freedom $\nu = 20$, however, the dimension needs to be much higher for the case $m = 1$ to overpower the case $m = 2$.

This approach leads to a broad spectrum of trade-offs between efficiency at the normal and protection against heavier-tailed models. Depending on one's utility for efficiency versus protection, one might choose any one of the cases m = 1, 2, 3, 4, or 5. Tables 1–4 provide some guidance for users to make such selections. For example, W_{mn} for $m = 4$ or 5 proves to be very attractive in comparison with the cases $m = 1$ or 2, in that a significant gain in efficiency at the normal model is achieved while also maintaining good efficiency at heavier-tailed t distributions along with a reasonably high breakdown point.

5.2 Affine invariant/equivariant versions

As mentioned earlier, the generalized signed-rank tests and related estimators are not fully affine invariant/equivariant, due to the use of the spatial sign function and spatial quantile function (see Serfling, 2004, for some discussion). Affine invariant/equivariant tests/estimates, however, may be constructed for the one-sample location problem using the well-known transformation-retransformation technique, as follows. Write

$$C_{mn} = n^{-m} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \boldsymbol{S}(\boldsymbol{y}_{i_1} + \ldots + \boldsymbol{y}_{i_m})\boldsymbol{S}^T(\boldsymbol{y}_{i_1} + \ldots + \boldsymbol{y}_{i_m}).$$

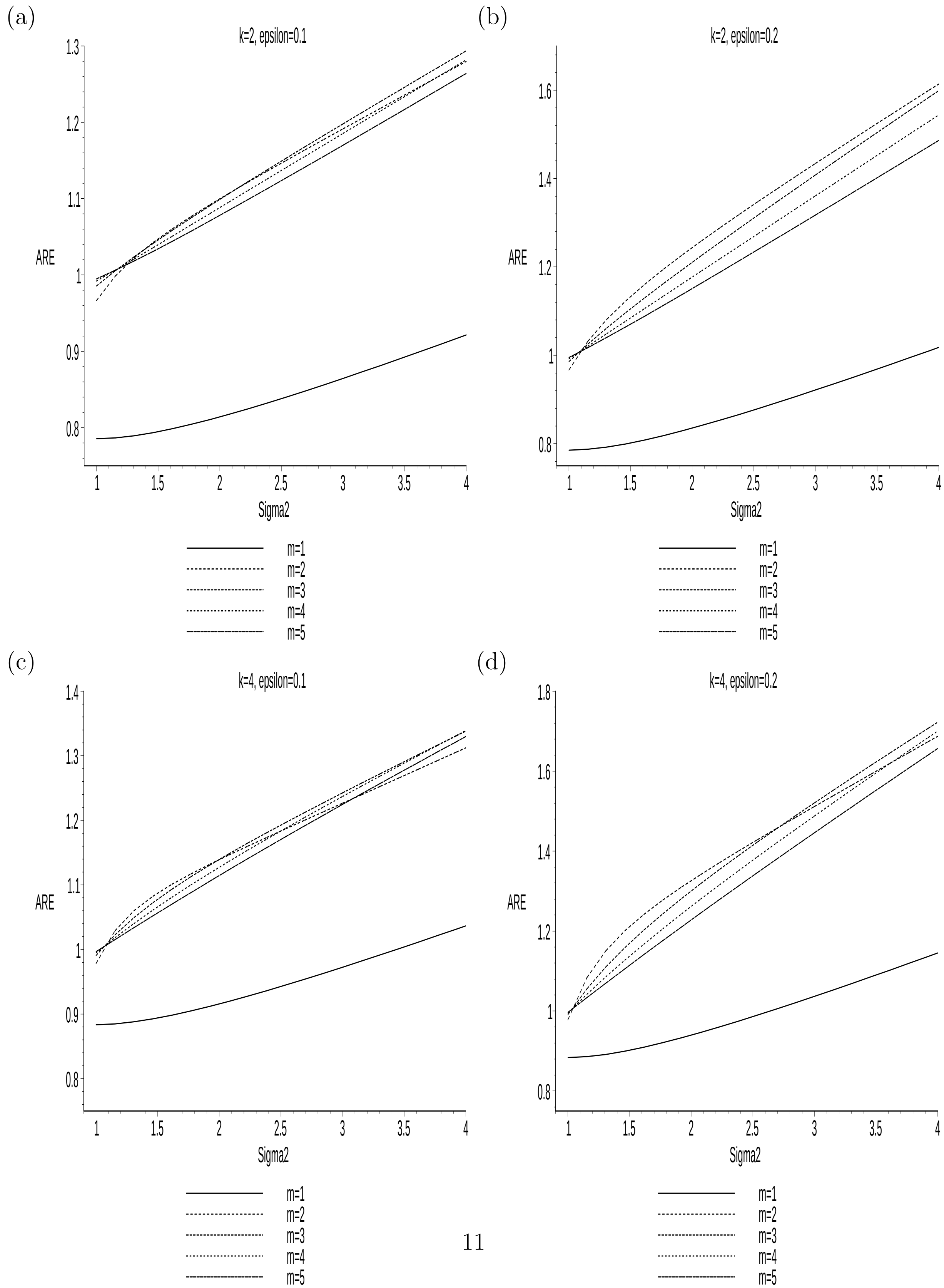

Figure 1: Asymptotic relative efficiencies of $\boldsymbol{W}_{mn}$ with respect to T^2 under k-variate Tukey$(\epsilon, \mathbf{0}, \sigma)$ models with various σ^2, ϵ, and k, for $m = 1, 2, \ldots, 5$.

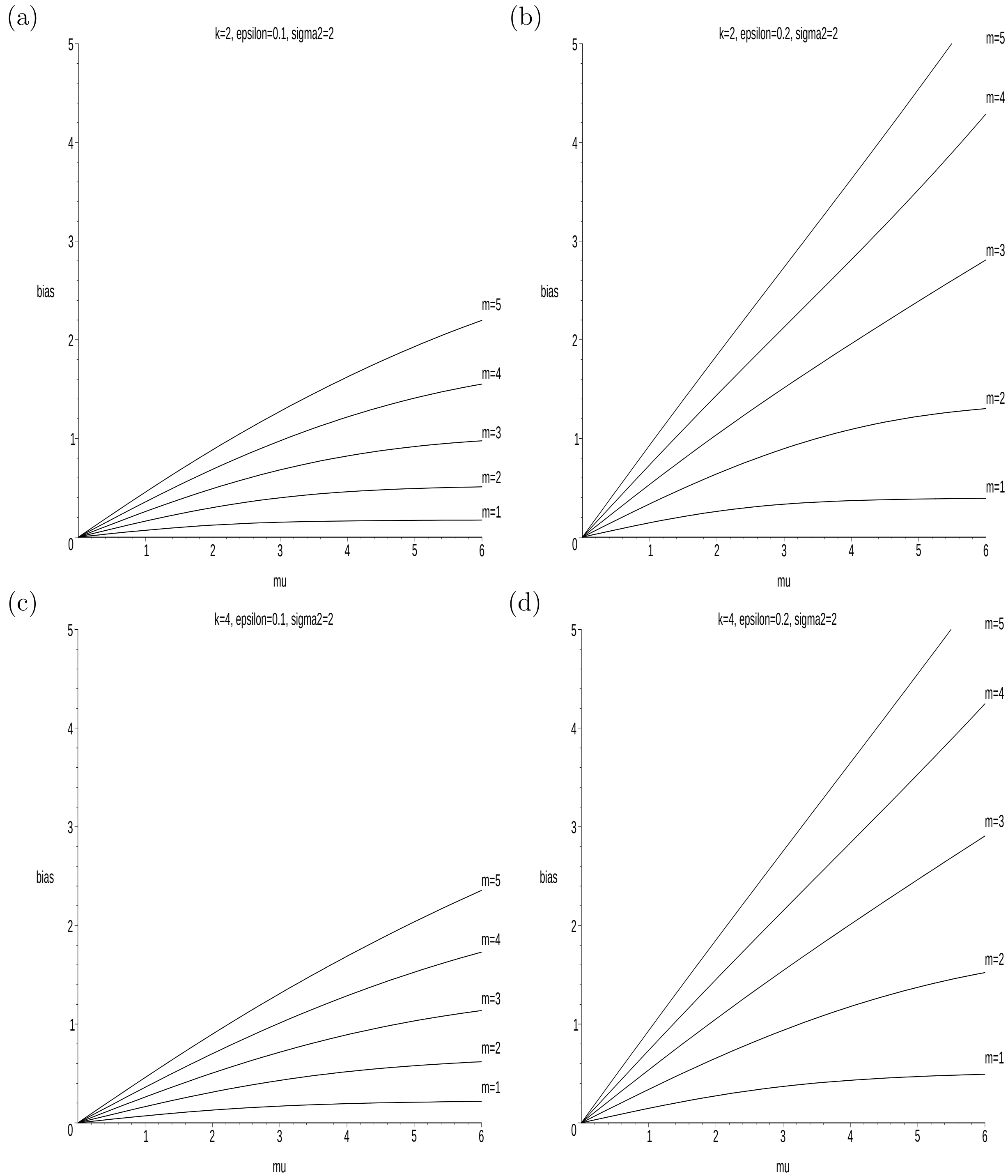

Figure 2: The bias of the functional $\boldsymbol{\theta}_m$ under k-variate Tukey$(\epsilon, \boldsymbol{\mu}, \sigma)$ models with various $||\boldsymbol{\mu}||$, σ^2, ϵ, and k, for $m = 1, 2, \ldots, 5$.

For $m = 1$, the spatial sign covariance matrix is obtained. See Visuri *et al.* (2000). For invariant tests consider a symmetric positive definite $k \times k$-matrix V and standardize the observations by setting $\boldsymbol{z}_i = V^{-1/2}\boldsymbol{y}_i$ ($V^{-1/2}$ also symmetric). Then calculate C_{mn} for the transformed data, and finally solve $C_{mn}(V) = (1/k)I_k$. The estimate $\hat{V}_{mn}$ is then unique up to a multiplicative constant and we can make the choice with $Trace(\hat{V}_{mn}) = k$. Finally, the invariant test statistic for the location problem is $\hat{\boldsymbol{W}}_{mn}$, calculated from the $\hat{V}^{-1/2}\boldsymbol{y}_i$. For $m = 1$, $\hat{V}_{1n}$ is Tyler's scatter matrix estimate (1987), and $\boldsymbol{W}_{1n}$ was introduced by Randles (2000).

Finally, for the affine equivariant location estimate, consider the standardized observations $\boldsymbol{z}_i = V^{-1/2}(\boldsymbol{y}_i - \boldsymbol{\theta})$ with location vector and scatter matrix candidates $\boldsymbol{\theta}$ and V. Calculate the values of the W_{mn} and C_{mn} statistics for the standardized data and solve simultaneous equations

$$\boldsymbol{W}_{m,n}(\boldsymbol{\theta}, V) = \mathbf{0} \quad \text{and} \quad C_{mn}(\boldsymbol{\theta}, V) = \frac{1}{k}I_k.$$

The location estimate is then the generalized transformation-retransformation Hodges-Lehmann estimate. For $m = 1$, the transformation-retransformation spatial median is obtained; Tyler's scatter matrix estimate is used in the transformation. See Chakraborty and Chaudhuri (1998), Chakraborty, Chaudhuri, and Oja (1998), and Hettmansperger and Randles (2002).

5.3 Selected comparisons with other estimators

Restricting to *elliptically symmetric* multivariate distributions, Hallin and Paindaveine (2002) develop linear signed-rank type test statistics based on the interdirections of Randles (1989). Model-based scores yield locally asymptotically maximin tests, which include the following:

V: a new van der Wearden type statistic

S: a sign statistic of Randles (1989)

W: a Wilcoxon statistic of Peters and Randles (1990)

t_3: a statistic optimal for $\boldsymbol{t}$ with 3 degrees of freedom

t_6: a statistic optimal for $\boldsymbol{t}$ with 6 degrees of freedom

t_{15}: a statistic optimal for $\boldsymbol{t}$ with 15 degrees of freedom

Let us – staying within the elliptically symmetric framework – compare V, S, W, t_3, t_6, and t_15 with our generalized spatial signed-rank tests GSSR_1, …, GSSR_5 corresponding to $m = 1, 2, \ldots, 5$, respectively. Let us also include an affine invariant signed-rank test "HMO" of Hettmansperger, Möttönen, and Oja (1997). As a "quick-and-dirty" criterion for comparison, we have compared ARE values for each test under the k-variate normal, $\boldsymbol{t}_3$, and $\boldsymbol{t}_6$ models and eliminate less competitive tests. We omit details and just indicate our findings. Interestingly, the comparisons depend upon the dimension k, as follows.

V, t_{15}, GSSR_2, GSSR_3, and GSSR_4: good for all k

HMO: good for $k = 1$ and $k \geq 6$

GSSR_5: good for $k = 1$

W: good for $k \leq 2$

GSSR_1: good for $k \geq 10$

t_3 and t_6: poor for all k

Of course, many of these statistics – including the GSSR statistics – remain defined and attractive under broader distributional assumptions, such as *central symmetry.* It would be desirable to develop a more extensive comparison study to include other statistics, for example, the "simpler, affine-invariant, multivariate, distribution-free sign test" of Randles (2000), to include other models such as the multivariate Tukey types, and to conclude robustness criteria such as breakdown points and influence functions. This is beyond the scope of the present paper and deferred to future work.

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References

[1] Bose, A. (1998). Bahadur representation of M_m estimates. *Ann. Statist.* **26** 771–777.

[2] Chakraborty, B., and Chaudhuri, P. (1998). On an adaptive transformation–retransformation estimate of multivariate location. *J. Roy. Statist. Soc. B* **60** 145–157.

[3] Chakraborty, B., Chaudhuri, P., and Oja, H. (1998). Operating transformation and retransformation on spatial median and angle test. *Statist. Sinica* **8** 767–784.

[4] Chaudhuri, P. (1992). Multivariate location estimation using extension of R-estimates through U-statistics type approach. *Ann. Statist.* **20** 897–916.

[5] Hallin, M. and Paindaveine, D. (2002). Optimal tests for multivariate location based on interdirections and pseudo-Mahalanobis ranks. *Ann. Statist.* **30** 1103–1133.

[6] Hettmansperger, T. P. and McKean, J. W. (1998). *Robust Nonparametric Statistical Methods.* Arnold, London.

[7] Hettmansperger, T. P., M ott onen, J., and Oja, H. (1997). Affine-invariant multivariate one-sample signed-rank tests *J. Amer. Statist. Assoc.* **92** 1591–1600.

[8] Hettmansperger, T. P., Nyblom, J. and Oja, H. (1994). Affine invariant multivariate one-sample sign tests. *J. R. Statist. Soc. B* **56** 221–234.

[9] Hettmansperger, T. P. and Randles, R. (2002). A practical affine equivariant multivariate median. *Biometrika* **89** 851–860.

[10] Möttönen, J. (2002). Efficiency of the spatial rank test. Unpublished manuscript.

[11] Möttönen, J. and Oja, H. (1995). Multivariate spatial sign and rank methods. *J. Nonpar. Statist.* **5** 201–213.

[12] Möttönen, J., Oja, H. and Tienari, J. (1997). On the efficiency of multivariate spatial sign and rank tests. *Ann. Statist.* **25** 542–552.

[13] Möttönen, J., Hettmansperger, T. P., Oja, H. and Tienari, J. (1998). On the efficiency of affine invariant multivariate rank tests. *J. Mult. Analy.* **66** 118–132.

[14] Peters, D. and Randles, R. H. (1990). A multivariate signed-rank test for the one-sampled location problem. *J. Amer. Statist. Assoc.* **85** 552–557.

[15] Randles, R. H. (1989). A distribution-free multivariate sign test based on interdirections. *J. Amer. Statist. Assoc.* **84** 1045–1050.

[16] Randles, R.(2000). A simpler, affine invariant, multivariate, distribution-free sign test. *J. Amer. Statist. Assoc.* **95** 1263–1268.

[17] Serfling, R. (2004). Nonparametric multivariate descriptive measures based on spatial quantiles. *J. Statist. Plann. and Inf.* **123** 259–278

[18] Visuri, S., Koivunen, V. and Oja, H. (2000). Sign and rank covariance matrices. *J. Statist. Plann. and Inf.* **91** 557–575.

A Asymptotic relative efficiency evaluations for selected models

Here we provide some of the basic lemmas underlying the ARE computations given in the paper. Due to space limitations, many details of proof and some further lemmas are omitted. An expanded version is available on the website `www.utdallas.edu/∼serfling`. Complete details are available in Möttönen (2002).

A.1 Multivariate normal distribution

Lemma 4 *Let $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{m-1}$ be independent $N_k(\mathbf{0}, I_k)$ distributed random vectors and $\boldsymbol{y} = (0, \ldots, 0, r)^T$ a fixed k-variate vector. Then*

$$||\boldsymbol{y} - \boldsymbol{s}||^2 \sim (m-1)\chi^2_k\bigg(\frac{r^2}{m-1}\bigg),$$

where $\boldsymbol{s} = \boldsymbol{y}_1 + \ldots + \boldsymbol{y}_{m-1}$.

Proof. With

$$\boldsymbol{s} \sim N_k(\mathbf{0}, (m-1)I_k),$$

we have

$$\begin{aligned} ||\boldsymbol{y} - \boldsymbol{s}||^2 &= (m-1)\bigg[\frac{s_1^2}{m-1} + \ldots + \frac{s_{k-1}^2}{m-1} + \frac{(r-s_k)^2}{m-1}\bigg] \\ &\sim (m-1)\chi^2_k\bigg(\frac{r^2}{m-1}\bigg). \end{aligned}$$

Lemma 5 *We have*

$$\varrho_m(r) = \sqrt{m-1}\,\varrho\bigg(\frac{r}{\sqrt{m-1}}\bigg),$$

where

$$\varrho(r) = 2^{1/2}\exp(-\frac{r^2}{2})\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\ {}_1F_1\bigg(\frac{k+1}{2};\frac{k}{2};\frac{r^2}{2}\bigg).$$

Proof.

$$\begin{aligned} \varrho_m(r) &= E\Bigg\{\sqrt{(m-1)\chi^2_k\bigg(\frac{r^2}{m-1}\bigg)}\Bigg\} \\ &= \sqrt{m-1}E\Bigg\{\sqrt{\chi^2_k\bigg(\frac{r^2}{m-1}\bigg)}\Bigg\} \\ &= \sqrt{m-1}\,\varrho\bigg(\frac{r}{\sqrt{m-1}}\bigg) \end{aligned}$$

See Lemma 22 of Möttönen (2002).

Lemma 6

$$R_m(r\boldsymbol{u}) = q_m(r)\boldsymbol{u}$$

where

$$q_m(r) = q\left(\frac{r}{\sqrt{m-1}}\right)$$

and

$$q(r) = \frac{r}{2^{1/2}}\exp(-\frac{r^2}{2})\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})}\ {}_1F_1\left(\frac{k+1}{2};\frac{k+2}{2};\frac{r^2}{2}\right)$$

Proof.

$$\begin{aligned} q_m(r) &= \frac{d}{dr}\varrho_m(r) \\ &= \sqrt{m-1}\frac{1}{\sqrt{m-1}}\varrho'\left(\frac{r}{\sqrt{m-1}}\right) \\ &= q\left(\frac{r}{\sqrt{m-1}}\right) \end{aligned}$$

Lemma 7

$$E(q_m(r)r) = \sqrt{2/m}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}$$

Proof.

$$\begin{aligned}
E(q_m(r)r) &= E\left(q\left(\frac{r}{\sqrt{m-1}}\right)r\right) \\
&= E\left(2^{-1/2}\frac{r^2}{\sqrt{m-1}}\exp\left(-\frac{r^2}{2(m-1)}\right)\sum_{i=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k+2}{2}+i)i!}\left(\frac{r^2}{2(m-1)}\right)^i\right) \\
&= 2^{-1/2}\sum_{i=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k+2}{2}+i)2^i i!}\frac{1}{(m-1)^{1/2+i}}E\left((r^2)^{1+i}\exp\left(-\frac{r^2}{2(m-1)}\right)\right) \\
&= 2^{-1/2}\sum_{i=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k+2}{2}+i)2^i i!}\frac{1}{(m-1)^{1/2+i}}\frac{\Gamma(\frac{k}{2}+1+i)2^{1+i}}{\Gamma(\frac{k}{2})(\frac{1}{m-1}+1)^{k/2+1+i}} \\
&= \frac{\sqrt{2}(m-1)^{\frac{k+1}{2}}}{\Gamma(\frac{k}{2})m^{\frac{k+2}{2}}}\sum_{i=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)}{i!}\left(\frac{1}{m}\right)^i \\
&= \frac{\sqrt{2}(m-1)^{\frac{k+1}{2}}}{\Gamma(\frac{k}{2})m^{\frac{k+2}{2}}}\Gamma\left(\frac{k+1}{2}\right)\ {}_1F_0\left(\frac{k+1}{2};\frac{1}{m}\right) \\
&= \frac{\sqrt{2}(m-1)^{\frac{k+1}{2}}}{m^{\frac{k+2}{2}}}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\left(1-\frac{1}{m}\right)^{-\frac{k+1}{2}} \\
&= \sqrt{2/m}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}
\end{aligned}$$

Lemma 8

$$E(q_m^2(r)) = \frac{2}{mk}\frac{\Gamma^2(\frac{k+1}{2})}{\Gamma^2(\frac{k}{2})}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right)$$

Proof.

$$
\begin{aligned}
E(q_m^2(r)) &= E\left(q^2\left(\frac{r}{\sqrt{m-1}}\right)\right) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)\Gamma(\frac{k+1}{2}+j)}{\Gamma(\frac{k+2}{2}+i)\Gamma(\frac{k+2}{2}+j)2^{i+j+1}i!j!}E\left[\left(\frac{r^2}{m-1}\right)^{i+j+1}\exp\left(-\frac{r^2}{m-1}\right)\right] \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)\Gamma(\frac{k+1}{2}+j)}{\Gamma(\frac{k+2}{2}+i)\Gamma(\frac{k+2}{2}+j)2^{i+j+1}i!j!}\left(\frac{1}{m-1}\right)^{i+j+1}\frac{\Gamma(\frac{k}{2}+i+j+1)2^{i+j+1}}{\Gamma(\frac{k}{2})(\frac{2}{m-1}+1)^{k/2+i+j+1}} \\
&= \frac{1}{(m-1)\Gamma(\frac{k}{2})(\frac{2}{m-1}+1)^{k/2+1}} \\
&\quad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{k+2}{2}+i+j)\Gamma(\frac{k+1}{2}+i)\Gamma(\frac{k+1}{2}+j)}{\Gamma(\frac{k+2}{2}+i)\Gamma(\frac{k+2}{2}+j)i!j!}\left(\frac{1}{m+1}\right)^{i}\left(\frac{1}{m+1}\right)^{j} \\
&= \frac{1}{(m-1)\Gamma(\frac{k}{2})(\frac{2}{m-1}+1)^{k/2+1}}\frac{\Gamma(\frac{k+2}{2})\Gamma^2(\frac{k+1}{2})}{\Gamma^2(\frac{k+2}{2})} \\
&\quad F_2\left(\frac{k+2}{2},\frac{k+1}{2},\frac{k+1}{2};\frac{k+2}{2},\frac{k+2}{2};\frac{1}{m+1},\frac{1}{m+1}\right) \\
&= \frac{1}{(m-1)\Gamma(\frac{k}{2})(\frac{2}{m-1}+1)^{k/2+1}}\frac{\Gamma^2(\frac{k+1}{2})}{\frac{k}{2}\Gamma(\frac{k}{2})}\left(1-\frac{1}{m+1}\right)^{-\frac{k+1}{2}}\left(1-\frac{1}{m+1}\right)^{-\frac{k+1}{2}} \\
&\quad {}_2F_1\left(\frac{k+1}{2},\frac{k+1}{2};\frac{k+2}{2};\frac{(\frac{1}{m+1})^2}{(1-\frac{1}{m+1})^2}\right) \\
&= \frac{2\Gamma^2(\frac{k+1}{2})(\frac{m+1}{m})^{k+1}}{(m-1)k\Gamma^2(\frac{k}{2})(\frac{m+1}{m-1})^{k/2+1}}\ {}_2F_1\left(\frac{k+1}{2},\frac{k+1}{2};\frac{k+2}{2};\frac{1}{m^2}\right) \\
&= \frac{2\Gamma^2(\frac{k+1}{2})}{k\Gamma^2(\frac{k}{2})}\frac{(m^2-1)^{k/2}}{m^{k+1}}\left(1-\frac{1}{m^2}\right)^{\frac{k+2}{2}-\frac{k+1}{2}-\frac{k+1}{2}}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right) \\
&= \frac{2}{mk}\frac{\Gamma^2(\frac{k+1}{2})}{\Gamma^2(\frac{k}{2})}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right)
\end{aligned}
$$

Lemma 9

$$
\begin{aligned}
A_m &= \frac{1}{k}\sqrt{\frac{2}{m}}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}I_k \\
B_m &= \frac{2\Gamma^2(\frac{k+1}{2})}{mk^2\Gamma^2(\frac{k}{2})}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right)I_k \\
A_m^T B_m^{-1} A_m &= \left[\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right)\right]^{-1} I_k \\
ARE_m &= \left[\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right)\right]^{-1}
\end{aligned}
$$

Proof.

$$A_m = \frac{1}{k}E(q_m(r)r)I_k = \frac{1}{k}\sqrt{\frac{2}{m}}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}I_k$$

$$B_m = \frac{1}{k}E(q_m^2(r))I_k = \frac{2\Gamma^2(\frac{k+1}{2})}{mk^2\Gamma^2(\frac{k}{2})}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right)I_k$$

$$
\begin{aligned}
A_m^T B_m^{-1} A_m &= \frac{1}{k^2}(\frac{2}{m})\frac{\Gamma^2(\frac{k+1}{2})}{\Gamma^2(\frac{k}{2})}k\frac{m}{2}\frac{k\Gamma^2(\frac{k}{2})}{\Gamma^2(\frac{k+1}{2})}\left[\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right)\right]^{-1} I_k \\
&= \left[\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right)\right]^{-1} I_k
\end{aligned}
$$

$$
\begin{aligned}
ARE &= \frac{\boldsymbol{\delta}^T A_m^T B_m^{-1} A_m \boldsymbol{\delta}}{\boldsymbol{\delta}^T \Sigma^{-1}\boldsymbol{\delta}} \\
&= \frac{\boldsymbol{\delta}^T A_m^T B_m^{-1} A_m \boldsymbol{\delta}}{\boldsymbol{\delta}^T I \boldsymbol{\delta}} \\
&= \frac{\boldsymbol{\delta}^T\boldsymbol{\delta}\left[\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right)\right]^{-1}}{\boldsymbol{\delta}^T\boldsymbol{\delta}} \\
&= \left[\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k+2}{2};\frac{1}{m^2}\right)\right]^{-1}
\end{aligned}
$$

A.2 Multivariate t distribution

Definition 3 *If $x \sim \chi^2_\nu$, $\boldsymbol{z} \sim N_k(\boldsymbol{0}, I_k)$, and x and $\boldsymbol{z}$ are independent, then $\boldsymbol{y} = (x/\nu)^{-1/2}\boldsymbol{z}$ has k-variate t distribution with ν degrees of freedom ($\boldsymbol{y} \sim t_{\nu,k}$).*

Lemma 10 *Let $\boldsymbol{y}_1 = (x_1/\nu)^{-1/2}\boldsymbol{z}_1, \ldots, \boldsymbol{y}_{m-1} = (x_{m-1}/\nu)^{-1/2}\boldsymbol{z}_{m-1}$ be independent $t_{\nu,k}$ distributed random vectors where $x_1, \ldots, x_{m-1}, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{m-1}$ are independent with $x_i \sim \chi^2_\nu$ and $\boldsymbol{z}_i \sim N_k(\boldsymbol{0}, I_k)$. If $\boldsymbol{y} = (0, \ldots, 0, r)^T$ is a fixed k-variate vector and $\boldsymbol{s} = \boldsymbol{y}_1 + \ldots + \boldsymbol{y}_{m-1}$, then*

$$||\boldsymbol{y} - \boldsymbol{s}||^2 \mid x_1, \ldots, x_{m-1} \sim \sigma^2 \chi^2_k\left(\frac{r^2}{\sigma^2}\right)$$

where $\sigma^2 = \sigma^2(\boldsymbol{x}) = \nu/x_1 + \ldots + \nu/x_{m-1}$.

Proof. Suppose that $x_1, \ldots, x_{m-1}$ are given. Then $E(\boldsymbol{s}) = \boldsymbol{0}$ and $\mathrm{Var}(\boldsymbol{s}) = \sigma^2 I_k$ where $\sigma^2 = (x_1/\nu)^{-1} + \ldots + (x_{m-1}/\nu)^{-1}$. Since the components of $\boldsymbol{s}$ are independent and normally distributed we get the result

$$\sigma^{-2}||\boldsymbol{y} - \boldsymbol{s}||^2 = (\frac{s_1}{\sigma})^2 + \ldots + (\frac{s_{k-1}}{\sigma})^2 + (\frac{r - s_k}{\sigma})^2 \sim \chi^2_k\left(\frac{r^2}{\sigma^2}\right)$$

Lemma 11

$$\varrho_m(r) = 2^{1/2}\sum_{i=0}^{\infty} \frac{\Gamma(\frac{k+1}{2} + i)}{\Gamma(\frac{k}{2} + i)i!}\left(\frac{r^2}{2}\right)^i E\left[\exp\left(-\frac{r^2}{2\sigma^2}\right)\left(\frac{1}{\sigma^2}\right)^{i-1/2}\right]$$

Proof.

$$\begin{aligned} E(||\boldsymbol{y} - \boldsymbol{s}|| \mid x_1, \ldots, x_{m-1}) &= E\left(\sigma\sqrt{\chi^2_k\left(\frac{r^2}{\sigma^2}\right)}\right) \\ &= \sum_{i=0}^{\infty} \sigma \frac{\exp(-\frac{r^2}{2\sigma^2})(\frac{r^2}{2\sigma^2})^i \Gamma(\frac{k+1}{2} + i)}{i!\Gamma(\frac{k}{2} + i)} 2^{1/2} \\ &= 2^{1/2}\sum_{i=0}^{\infty} \frac{\Gamma(\frac{k+1}{2} + i)}{\Gamma(\frac{k}{2} + i)i!}\left(\frac{r^2}{2}\right)^i \exp\left(-\frac{r^2}{2\sigma^2}\right)\left(\frac{1}{\sigma^2}\right)^{i-1/2} \end{aligned}$$

$$\varrho_m(r) = E[E(||\boldsymbol{y} - \boldsymbol{s}|| \mid x_1, \ldots, x_{m-1})]$$

Lemma 12 *$\boldsymbol{R}_m(r\boldsymbol{u}) = q_m(r)\boldsymbol{u}$ where $q_m(r) = \frac{d}{dr}\varrho_m(r)$*

A.3 Multivariate Tukey distribution

Definition 4 *We say that the distribution of $\boldsymbol{y}$ is k-variate $Tukey(\epsilon, \boldsymbol{\mu}, \sigma)$ distribution if the p.d.f. of $\boldsymbol{y}$ is*

$$g(\boldsymbol{y}) = (1-\epsilon)f(\boldsymbol{y}) + \epsilon\sigma^{-k} f(\sigma^{-1}(\boldsymbol{y}-\boldsymbol{\mu})),$$

where $f(\boldsymbol{y})$ is the p.d.f. of $N_k(\mathbf{0}, I_k)$-distribution and $\epsilon \in [0,1]$. The c.d.f. of $\boldsymbol{y}$ is then

$$G(\boldsymbol{y}) = (1-\epsilon)F(\boldsymbol{y}) + \epsilon F(\sigma^{-1}(\boldsymbol{y}-\boldsymbol{\mu})).$$

Lemma 13 *Let $v \sim Bin(1,\epsilon)$ and $\boldsymbol{x} \sim N_k(\mathbf{0}, I_k)$ be independent random variables. Then*

$$\boldsymbol{y} = (1-v)\boldsymbol{x} + v(\boldsymbol{\mu} + \sigma\boldsymbol{x}) = v\boldsymbol{\mu} + (1+(\sigma-1)v)\boldsymbol{x}$$

is $Tukey(\epsilon, \boldsymbol{\mu}, \sigma)$ distributed.

Lemma 14 *Let $\boldsymbol{y}_i = v_i\boldsymbol{\mu} + (1+(\sigma-1)v_i)\boldsymbol{x}_i$, $i = 1, \dots, m-1$, be independent $Tukey(\epsilon, \boldsymbol{\mu}, \sigma)$ distributed random vectors. Then the conditional distribution of $\boldsymbol{s} = \sum_{i=1}^{m-1} \boldsymbol{y}_i$ given $v_1, \dots, v_{m-1}$ is $N_k(h\boldsymbol{\mu}, a^2(h)I_k)$ where $h = \sum_{i=1}^{m-1} v_i$ and $a^2(h) = m-1+(\sigma^2-1)h$.*

Proof.

$$\boldsymbol{s} = \left(\sum_{i=1}^{m-1} v_i\right)\boldsymbol{\mu} + \sum_{i=1}^{m-1}(1+(\sigma-1)v_i)\boldsymbol{x}_i$$

$$E(\boldsymbol{s} \mid v_1, \dots, v_{m-1}) = \left(\sum_{i=1}^{m-1} v_i\right)\boldsymbol{\mu} = h\boldsymbol{\mu}$$

$$\begin{aligned}
\mathrm{Var}(\boldsymbol{s} \mid v_1, \dots, v_{m-1}) &= \sum_{i=1}^{m-1}(1+(\sigma-1)v_i)^2 I_k \\
&= \sum_{i=1}^{m-1}(1+2(\sigma-1)v_i + (\sigma-1)^2 v_i^2) I_k \\
&= \sum_{i=1}^{m-1}(1+2(\sigma-1)v_i + (\sigma-1)^2 v_i) I_k \\
&= \sum_{i=1}^{m-1}(1+(\sigma^2-1)v_i) I_k \\
&= (m-1+(\sigma^2-1)h) I_k
\end{aligned}$$

Lemma 15 *Let* $\boldsymbol{y} = (0, \ldots, 0, r)^T$ *and* $\boldsymbol{\mu} = (0, \ldots, 0, \mu_k)^T$ *be fixed k-variate vectors. Then*

$$||\boldsymbol{y} - \boldsymbol{s}|| \mid v_1, \ldots, v_{m-1} \sim \sqrt{a^2(h)\chi_k^2\bigg(\Big(\frac{r - h\mu_k}{a(h)}\Big)^2\bigg)}$$

Proof.

$$\frac{1}{a^2(h)}||\boldsymbol{y} - \boldsymbol{s}||^2 = \frac{s_1^2}{a^2(h)} + \cdots + \frac{s_{k-1}^2}{a^2(h)} + \frac{(r - s_k)^2}{a^2(h)}$$

$$\begin{aligned} E\bigg(\frac{r - s_k}{a(h)}\ \bigg|\ v_1, \ldots, v_{m-1}\bigg) &= \frac{r - h\mu_k}{a(h)} \\ E\bigg(\frac{s_i}{a(h)}\ \bigg|\ v_1, \ldots, v_{m-1}\bigg) &= 0 \quad \text{when } i = 1, \ldots, k-1 \end{aligned}$$

$$\frac{1}{a^2(h)}||\boldsymbol{y} - \boldsymbol{s}||^2 \mid v_1, \ldots, v_{m-1} \sim \chi_k^2\bigg(\Big(\frac{r - h\mu_k}{a(h)}\Big)^2\bigg)$$

Lemma 16

$$E(||\boldsymbol{y} - \boldsymbol{s}|| \mid v_1, \ldots, v_{m-1}) = 2^{1/2}\sum_{i=0}^{\infty}\frac{\Gamma(\frac{k+1}{2} + i)}{\Gamma(\frac{k}{2} + i)i!}\exp\bigg(-\frac{(r - h\mu_k)^2}{2a^2(h)}\bigg)\bigg(\frac{(r - h\mu_k)^2}{2a^2(h)}\bigg)^i a(h)$$

Proof. Using Lemma 4 of Möttönen (2002) we get

$$\begin{aligned} E(||\boldsymbol{y} - \boldsymbol{s}|| \mid v_1, \ldots, v_{m-1}) &= a(h)\sum_{i=0}^{\infty}\frac{\exp(-\frac{(r-h\mu_k)^2}{2a^2(h)})(\frac{(r-h\mu_k)^2}{2a^2(h)})^i}{i!}\frac{\Gamma(\frac{k+1}{2} + i)}{\Gamma(\frac{k}{2} + i)}2^{1/2} \\ &= 2^{1/2}\sum_{i=0}^{\infty}\frac{\Gamma(\frac{k+1}{2} + i)}{\Gamma(\frac{k}{2} + i)i!}\exp\bigg(-\frac{(r - h\mu_k)^2}{2a^2(h)}\bigg)\bigg(\frac{(r - h\mu_k)^2}{2a^2(h)}\bigg)^i a(h) \end{aligned}$$

Lemma 17

$$\begin{aligned} \varrho_m(r) &= E(||\boldsymbol{y} - \boldsymbol{s}||) \\ &= \sum_{h=0}^{m-1}\binom{m-1}{h}\epsilon^h(1 - \epsilon)^{m-1-h}a(h)\varrho\bigg(\frac{r - h\mu_k}{a(h)}\bigg) \end{aligned}$$

where

$$\varrho(r) = 2^{1/2}\exp(-\frac{r^2}{2})\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\ {}_1F_1\bigg(\frac{k+1}{2}; \frac{k}{2}; \frac{r^2}{2}\bigg)$$

Proof. Since $h \sim Bin(m-1, \epsilon)$ we have

$$\begin{aligned}
\varrho_m(r) &= E(||\boldsymbol{y} - \boldsymbol{s}||) \\
&= 2^{1/2} \sum_{i=0}^{\infty} \frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k}{2}+i)i!} E\left(\exp\left(-\frac{(r-h\mu_k)^2}{2a^2(h)}\right)\left(\frac{(r-h\mu_k)^2}{2a^2(h)}\right)^i a(h)\right) \\
&= 2^{1/2} \sum_{i=0}^{\infty} \frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k}{2}+i)i!} \sum_{h=0}^{m-1} \exp\left(-\frac{(r-h\mu_k)^2}{2a^2(h)}\right)\left(\frac{(r-h\mu_k)^2}{2a^2(h)}\right)^i a(h) \\
&\qquad \binom{m-1}{h}\epsilon^h(1-\epsilon)^{m-1-h} \\
&= \sum_{h=0}^{m-1} \binom{m-1}{h}\epsilon^h(1-\epsilon)^{m-1-h} a(h) \sum_{i=0}^{\infty} \frac{\exp(-\frac{(r-h\mu_k)^2}{2a^2(h)})(\frac{(r-h\mu_k)^2}{2a^2(h)})^i}{i!} \frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k}{2}+i)} 2^{1/2} \\
&= \sum_{h=0}^{m-1} \binom{m-1}{h}\epsilon^h(1-\epsilon)^{m-1-h} a(h)\varrho\left(\frac{r-h\mu_k}{a(h)}\right)
\end{aligned}$$

where

$$\begin{aligned}
\varrho(r) &= \sum_{i=0}^{\infty} \frac{\exp(-\frac{r^2}{2})(\frac{r^2}{2})^i}{i!} \frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k}{2}+i)} 2^{1/2} \\
&= 2^{1/2} \exp(-\frac{r^2}{2}) \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} \, {}_1F_1\left(\frac{k+1}{2}; \frac{k}{2}; \frac{r^2}{2}\right)
\end{aligned}$$

See Lemma 22 of Möttönen (2002).

Lemma 18

$$q_m(r) = \sum_{h=0}^{m-1} \binom{m-1}{h}\epsilon^h(1-\epsilon)^{m-1-h} q\left(\frac{r-h\mu_k}{a(h)}\right)$$

where

$$q(r) = \frac{r}{2^{1/2}} \exp(-\frac{r^2}{2}) \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})} \, {}_1F_1\left(\frac{k+1}{2}; \frac{k+2}{2}; \frac{r^2}{2}\right)$$

Proof.

$$\begin{aligned}
q_m(r) &= \frac{d}{dr}\varrho_m(r) \\
&= \sum_{h=0}^{m-1} \binom{m-1}{h}\epsilon^h(1-\epsilon)^{m-1-h} a(h) \frac{d}{dr}\varrho\left(\frac{r-h\mu_k}{a(h)}\right) \\
&= \sum_{h=0}^{m-1} \binom{m-1}{h}\epsilon^h(1-\epsilon)^{m-1-h} q\left(\frac{r-h\mu_k}{a(h)}\right)
\end{aligned}$$

where

$$\begin{aligned} q(r) &= 2^{-1/2} r \exp(-\frac{r^2}{2}) \sum_{i=0}^{\infty} \frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k+2}{2}+i) i!} (\frac{r^2}{2})^i \\ &= \frac{r}{2^{1/2}} \exp(-\frac{r^2}{2}) \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})} \, {}_1F_1\left(\frac{k+1}{2}; \frac{k+2}{2}; \frac{r^2}{2}\right) \end{aligned}$$

See Lemma 23 of Möttönen (2002).

Lemma 19 *Let $\boldsymbol{\mu} = \mathbf{0}$. The optimal score function is then*

$$\boldsymbol{L}(\boldsymbol{y}) = \frac{\alpha(\boldsymbol{y})}{g(\boldsymbol{y})} \boldsymbol{y}$$

where

$$\alpha(\boldsymbol{y}) = (1-\epsilon) f(\boldsymbol{y}) + \epsilon \sigma^{-k-2} f(\sigma^{-1} \boldsymbol{y})$$

Proof.

$$\begin{aligned} \boldsymbol{L}(\boldsymbol{y}) &= -\frac{d}{d\boldsymbol{y}} \log(g(\boldsymbol{y})) \\ &= -\frac{d}{d\boldsymbol{y}} \log((1-\epsilon) f(\boldsymbol{y}) + \epsilon \sigma^{-k} f(\sigma^{-1} \boldsymbol{y})) \\ &= -((1-\epsilon) f(\boldsymbol{y}) + \epsilon \sigma^{-k} f(\sigma^{-1} \boldsymbol{y}))^{-1} [(1-\epsilon) \frac{d}{d\boldsymbol{y}} f(\boldsymbol{y}) + \epsilon \sigma^{-k} \frac{d}{d\boldsymbol{y}} f(\sigma^{-1} \boldsymbol{y})] \\ &= -((1-\epsilon) f(\boldsymbol{y}) + \epsilon \sigma^{-k} f(\sigma^{-1} \boldsymbol{y}))^{-1} [-(1-\epsilon) f(\boldsymbol{y}) \boldsymbol{y} - \epsilon \sigma^{-k-2} f(\sigma^{-1} \boldsymbol{y}) \boldsymbol{y}] \\ &= \frac{(1-\epsilon) f(\boldsymbol{y}) + \epsilon \sigma^{-k-2} f(\sigma^{-1} \boldsymbol{y})}{(1-\epsilon) f(\boldsymbol{y}) + \epsilon \sigma^{-k} f(\sigma^{-1} \boldsymbol{y})} \boldsymbol{y} \\ &= \frac{\alpha(\boldsymbol{y})}{g(\boldsymbol{y})} \boldsymbol{y} \end{aligned}$$

Lemma 20

$$E\left(q\left(\frac{\sigma r}{a(h)}\right) r\right) = \left(\frac{2\sigma^2}{\sigma^2 + a^2(h)}\right)^{1/2} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}$$

Proof.

$$
\begin{aligned}
E\left(q\left(\frac{\sigma r}{a(h)}\right)r\right) &= E\left[r\frac{\sigma r}{\sqrt{2}a(h)}\exp\left(-\frac{\sigma^2 r^2}{2a^2(h)}\right)\sum_{i=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k+2}{2}+i)i!}\left(\frac{\sigma^2 r^2}{2a^2(h)}\right)^i\right] \\
&= \sum_{i=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k+2}{2}+i)i!}\left(\frac{\sigma}{\sqrt{2}a(h)}\right)^{2i+1}E\left[r^{2i+2}\exp\left(-\frac{\sigma^2 r^2}{2a^2(h)}\right)\right] \\
&= \sum_{i=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)}{\Gamma(\frac{k+2}{2}+i)i!}\left(\frac{\sigma}{\sqrt{2}a(h)}\right)^{2i+1}\frac{\Gamma(\frac{k}{2}+i+1)2^{i+1}}{\Gamma(\frac{k}{2})(\frac{\sigma^2}{a^2(h)}+1)^{k/2+i+1}} \\
&= \left(\frac{\sigma}{\sqrt{2}a(h)}\right)\frac{2}{\Gamma(\frac{k}{2})(\frac{\sigma^2}{a^2(h)}+1)^{k/2+1}}\sum_{i=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)}{i!}\left(\frac{\sigma^2}{\sigma^2+a^2(h)}\right)^i \\
&= \left(\frac{\sigma}{\sqrt{2}a(h)}\right)\frac{2\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})(\frac{\sigma^2}{a^2(h)}+1)^{k/2+1}}\sum_{i=0}^{\infty}\frac{(\frac{k+1}{2})_i}{i!}\left(\frac{\sigma^2}{\sigma^2+a^2(h)}\right)^i \\
&= \left(\frac{\sigma}{\sqrt{2}a(h)}\right)\frac{2\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})(\frac{\sigma^2}{a^2(h)}+1)^{k/2+1}}\ {}_1F_0\left(\frac{k+1}{2};\frac{\sigma^2}{\sigma^2+a^2(h)}\right) \\
&= \left(\frac{\sigma}{\sqrt{2}a(h)}\right)\frac{2\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})(\frac{\sigma^2}{a^2(h)}+1)^{k/2+1}}\left(1-\frac{\sigma^2}{\sigma^2+a^2(h)}\right)^{-\frac{k+1}{2}} \\
&= \left(\frac{\sigma}{\sqrt{2}a(h)}\right)\frac{2\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\left(\frac{a^2(h)}{\sigma^2+a^2(h)}\right)^{\frac{k+2}{2}}\left(\frac{a^2(h)}{\sigma^2+a^2(h)}\right)^{-\frac{k+1}{2}} \\
&= \left(\frac{2\sigma^2}{\sigma^2+a^2(h)}\right)^{1/2}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}
\end{aligned}
$$

See Lemma 8 of Möttönen (2002).

Lemma 21 *Let* $\boldsymbol{\mu}=\mathbf{0}$. *Then*

$$
\begin{aligned}
A &= E[\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{L}^T(\boldsymbol{y})] = \frac{\epsilon\sigma^{-1}E(q_m(\sigma r)r)+(1-\epsilon)E(q_m(r)r)}{k}I_k \\
&= \frac{\sqrt{2}}{k}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\sum_{h=0}^{m-1}\binom{m-1}{h}\epsilon^h(1-\epsilon)^{m-1-h}\left\{\frac{\epsilon}{(\sigma^2+a^2(h))^{1/2}}+\frac{1-\epsilon}{(1+a^2(h))^{1/2}}\right\}I_k
\end{aligned}
$$

where $r^2\sim\chi_k^2(0)$.

Proof.

$$
\begin{aligned}
A &= E[\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{L}^T(\boldsymbol{y})] \\
&= \int_{R^k} \frac{\alpha(\boldsymbol{y})}{g(\boldsymbol{y})}\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{y}^T g(\boldsymbol{y})dy_1\dots dy_k \\
&= \int_{R^k} \alpha(\boldsymbol{y})\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{y}^T dy_1\dots dy_k \\
&= (1-\epsilon)\int_{R^k}\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{y}^T f(\boldsymbol{y})dy_1\dots dy_k + \epsilon\sigma^{-2}\int_{R^k}\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{y}^T\sigma^{-k}f(\sigma^{-1}\boldsymbol{y})dy_1\dots dy_k \\
&= (1-\epsilon)\int_{R^k}\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{y}^T f(\boldsymbol{y})dy_1\dots dy_k + \epsilon\sigma^{-2}\int_{R^k}\boldsymbol{R}_m(\sigma\boldsymbol{y})\sigma\boldsymbol{y}^T f(\boldsymbol{y})dy_1\dots dy_k \\
&= (1-\epsilon)E(\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{y}^T) + \epsilon\sigma^{-1}E(\boldsymbol{R}_m(\sigma\boldsymbol{y})\boldsymbol{y}^T) \\
&= (1-\epsilon)E(q_m(r)\boldsymbol{u}r\boldsymbol{u}^T) + \epsilon\sigma^{-1}E(q_m(\sigma r)\boldsymbol{u}r\boldsymbol{u}^T) \\
&= (1-\epsilon)E(q_m(r)r)E(\boldsymbol{u}\boldsymbol{u}^T) + \epsilon\sigma^{-1}E(q_m(\sigma r)r)E(\boldsymbol{u}\boldsymbol{u}^T) \\
&= \frac{1-\epsilon}{k}E(q_m(r)r)I_k + \frac{\epsilon}{k}\sigma^{-1}E(q_m(\sigma r)r)I_k \\
&= \frac{(1-\epsilon)E(q_m(r)r) + \epsilon\sigma^{-1}E(q_m(\sigma r)r)}{k}I_k
\end{aligned}
$$

where $r^2 \sim \chi^2_k(0)$.

$$
\begin{aligned}
\epsilon\sigma^{-1}E(q_m(\sigma r)r) &= \sum_{h=0}^{m-1}\binom{m-1}{h}\epsilon^{h+1}(1-\epsilon)^{m-1-h}E\left(q\left(\frac{\sigma r}{a(h)}\right)r\right)\frac{1}{\sigma} \\
&= \sum_{h=0}^{m-1}\binom{m-1}{h}\epsilon^{h+1}(1-\epsilon)^{m-1-h}\left(\frac{2\sigma^2}{\sigma^2+a^2(h)}\right)^{1/2}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\frac{1}{\sigma} \\
&= \sum_{h=0}^{m-1}\binom{m-1}{h}\epsilon^{h+1}(1-\epsilon)^{m-1-h}\left(\frac{2}{\sigma^2+a^2(h)}\right)^{1/2}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}
\end{aligned}
$$

$$
\begin{aligned}
(1-\epsilon)E(q_m(r)r) &= \sum_{h=0}^{m-1}\binom{m-1}{h}\epsilon^{h}(1-\epsilon)^{m-h}E\left(q\left(\frac{r}{a(h)}\right)r\right) \\
&= \sum_{h=0}^{m-1}\binom{m-1}{h}\epsilon^{h}(1-\epsilon)^{m-h}\left(\frac{2}{1+a^2(h)}\right)^{1/2}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}
\end{aligned}
$$

$$\begin{aligned}
A &= \frac{1}{k}\sum_{h=0}^{m-1}\binom{m-1}{h}\epsilon^h(1-\epsilon)^{m-1-h}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\left\{\epsilon\left(\frac{2}{\sigma^2+a^2(h)}\right)^{1/2}+(1-\epsilon)\left(\frac{2}{1+a^2(h)}\right)^{1/2}\right\}I_k \\
&= \frac{\sqrt{2}}{k}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\sum_{h=0}^{m-1}\binom{m-1}{h}\epsilon^h(1-\epsilon)^{m-1-h}\left\{\frac{\epsilon}{(\sigma^2+a^2(h))^{1/2}}+\frac{1-\epsilon}{(1+a^2(h))^{1/2}}\right\}I_k
\end{aligned}$$

Lemma 22 *The p.d.f of r^2 is*

$$f(t) = (1-\epsilon)g(t) + \epsilon\frac{1}{\sigma^2}g\left(\frac{t}{\sigma^2}\right)$$

where $g(t)$ is the p.d.f. of the $\chi_k^2(0)$ distribution.

Proof.

$$\begin{aligned}
r^2 &= \boldsymbol{y}^T\boldsymbol{y} \\
&= (1+(\sigma-1)u)^2\boldsymbol{x}^T\boldsymbol{x} \\
&= (1+(\sigma^2-1)u)\boldsymbol{x}^T\boldsymbol{x}
\end{aligned}$$

where $u \sim Bin(1,\epsilon)$ and $\boldsymbol{x}^T\boldsymbol{x} \sim \chi_k^2(0)$.

$$\begin{aligned}
F(t) &= P(r^2 \le t) \\
&= P(r^2 \le t \mid u=1)P(u=1) + P(r^2 \le t \mid u=0)P(u=0) \\
&= \epsilon P(\sigma^2\chi_k^2(0) \le t) + (1-\epsilon)P(\chi_k^2(0) \le t) \\
&= \epsilon G\left(\frac{t}{\sigma^2}\right) + (1-\epsilon)G(t) \\
f(t) &= \frac{d}{dt}F(t) \\
&= \epsilon\frac{1}{\sigma^2}g\left(\frac{t}{\sigma^2}\right) + (1-\epsilon)g(t)
\end{aligned}$$

Lemma 23 *Let x be a random variable with the p.d.f.*

$$f(t) = (1-\epsilon)g(t) + \epsilon\frac{1}{\sigma^2}g\left(\frac{t}{\sigma^2}\right)$$

where $g(t)$ is the p.d.f. of the $\chi_k^2(0)$ distribution. Then

$$E(x^i\exp(-dx)) = (1-\epsilon)\frac{\Gamma(\frac{k}{2}+i)2^i}{\Gamma(\frac{k}{2})(2d+1)^{k/2+i}} + \epsilon\sigma^{2i}\frac{\Gamma(\frac{k}{2}+i)2^i}{\Gamma(\frac{k}{2})(2d\sigma^2+1)^{k/2+i}}$$

where $d > -\frac{1}{2\sigma^2}$.

Proof.

$$\begin{aligned}
E(x^i\exp(-dx)) &= \int_0^\infty x^i\exp(-dx)\left((1-\epsilon)g(x)+\epsilon\frac{1}{\sigma^2}g\left(\frac{x}{\sigma^2}\right)\right)dx\\
&= (1-\epsilon)\int_0^\infty x^i\exp(-dx)g(x)dx+\epsilon\frac{1}{\sigma^2}\int_0^\infty x^i\exp(-dx)g\left(\frac{x}{\sigma^2}\right)dx\\
&= (1-\epsilon)\int_0^\infty x^i\exp(-dx)g(x)dx+\epsilon\frac{1}{\sigma^2}\int_0^\infty \sigma^{2i}y^i\exp(-d\sigma^2 y)g(y)\sigma^2dy\\
&= (1-\epsilon)\int_0^\infty x^i\exp(-dx)g(x)dx+\epsilon\sigma^{2i}\int_0^\infty y^i\exp(-d\sigma^2 y)g(y)dy\\
&= (1-\epsilon)\frac{\Gamma(\frac{k}{2}+i)2^i}{\Gamma(\frac{k}{2})(2d+1)^{k/2+i}}+\epsilon\sigma^{2i}\frac{\Gamma(\frac{k}{2}+i)2^i}{\Gamma(\frac{k}{2})(2d\sigma^2+1)^{k/2+i}}
\end{aligned}$$

Lemma 24 *When* $\boldsymbol{\mu}=\mathbf{0}$

$$\begin{aligned}
B &= E[\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{R}_m^T(\boldsymbol{y})]\\
&= \frac{1}{k}E[q_m^2(r)]I_k\\
&= \frac{1}{k}\sum_{h_1=0}^{m-1}\sum_{h_2=0}^{m-1}\binom{m-1}{h_1}\binom{m-1}{h_2}\epsilon^{h_1+h_2}(1-\epsilon)^{2m-2-h_1-h_2}\frac{1}{a(h_1)a(h_2)}\frac{\Gamma^2(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2})}\\
&\qquad\Bigg[\frac{1-\epsilon}{\left(\frac{1}{a^2(h_1)}+\frac{1}{a^2(h_2)}+1\right)^{\frac{k+2}{2}}}(1-x_1)^{-\frac{k+1}{2}}(1-y_1)^{-\frac{k+1}{2}}\\
&\qquad {}_2F_1\left(\frac{k+1}{2},\frac{k+1}{2};\frac{k+2}{2};\frac{x_1y_1}{(1-x_1)(1-y_1)}\right)+\\
&\qquad \frac{\epsilon\sigma^2}{\left(\frac{\sigma^2}{a^2(h_1)}+\frac{\sigma^2}{a^2(h_2)}+1\right)^{\frac{k+2}{2}}}(1-x_2)^{-\frac{k+1}{2}}(1-y_2)^{-\frac{k+1}{2}}\\
&\qquad {}_2F_1\left(\frac{k+1}{2},\frac{k+1}{2};\frac{k+2}{2};\frac{x_2y_2}{(1-x_2)(1-y_2)}\right)\Bigg]I_k
\end{aligned}$$

Proof.

$$\begin{aligned}
B &= E[\boldsymbol{R}_m(\boldsymbol{y})\boldsymbol{R}_m^T(\boldsymbol{y})]\\
&= E[q_m(r)\boldsymbol{u}q_m(r)\boldsymbol{u}^T]\\
&= E[q_m^2(r)]E(\boldsymbol{u}\boldsymbol{u}^T)\\
&= \frac{1}{k}E[q_m^2(r)]I_k
\end{aligned}$$

$$
\begin{aligned}
q_m^2(r) &= \sum_{h_1=0}^{m-1}\sum_{h_2=0}^{m-1}\binom{m-1}{h_1}\binom{m-1}{h_2}\epsilon^{h_1+h_2}(1-\epsilon)^{2m-2-h_1-h_2}q\left(\frac{r}{a(h_1)}\right)q\left(\frac{r}{a(h_2)}\right)\\
&= \sum_{h_1=0}^{m-1}\sum_{h_2=0}^{m-1}\binom{m-1}{h_1}\binom{m-1}{h_2}\epsilon^{h_1+h_2}(1-\epsilon)^{2m-2-h_1-h_2}\\
&\qquad \frac{r^2}{2a(h_1)a(h_2)}\exp\left(-\frac{r^2}{2a^2(h_1)}\right)\exp\left(-\frac{r^2}{2a^2(h_2)}\right)\\
&\qquad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)\Gamma(\frac{k+1}{2}+j)}{\Gamma(\frac{k+2}{2}+i)\Gamma(\frac{k+2}{2}+j)i!j!}\left(\frac{r^2}{2a^2(h_1)}\right)^i\left(\frac{r^2}{2a^2(h_2)}\right)^j\\
&= \sum_{h_1=0}^{m-1}\sum_{h_2=0}^{m-1}\binom{m-1}{h_1}\binom{m-1}{h_2}\epsilon^{h_1+h_2}(1-\epsilon)^{2m-2-h_1-h_2}\frac{1}{2a(h_1)a(h_2)}\\
&\qquad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)\Gamma(\frac{k+1}{2}+j)}{\Gamma(\frac{k+2}{2}+i)\Gamma(\frac{k+2}{2}+j)i!j!}\left(\frac{1}{2a^2(h_1)}\right)^i\left(\frac{1}{2a^2(h_2)}\right)^j\\
&\qquad (r^2)^{i+j+1}\exp\left(-\left(\frac{1}{2a^2(h_1)}+\frac{1}{2a^2(h_2)}\right)r^2\right)
\end{aligned}
$$

$$
\begin{aligned}
E(q_m^2(r)) &= \sum_{h_1=0}^{m-1}\sum_{h_2=0}^{m-1}\binom{m-1}{h_1}\binom{m-1}{h_2}\epsilon^{h_1+h_2}(1-\epsilon)^{2m-2-h_1-h_2}\frac{1}{2a(h_1)a(h_2)}\\
&\qquad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)\Gamma(\frac{k+1}{2}+j)}{\Gamma(\frac{k+2}{2}+i)\Gamma(\frac{k+2}{2}+j)i!j!}\left(\frac{1}{2a^2(h_1)}\right)^i\left(\frac{1}{2a^2(h_2)}\right)^j\\
&\qquad \left[(1-\epsilon)\frac{\Gamma(\frac{k}{2}+i+j+1)2^{i+j+1}}{\Gamma(\frac{k}{2})\left(\frac{1}{a(h_1)^2}+\frac{1}{a(h_2)^2}+1\right)^{k/2+i+j+1}}+\right.\\
&\qquad \left.\epsilon\sigma^{2(i+j+1)}\frac{\Gamma(\frac{k}{2}+i+j+1)2^{i+j+1}}{\Gamma(\frac{k}{2})\left(\frac{\sigma^2}{a(h_1)^2}+\frac{\sigma^2}{a(h_2)^2}+1\right)^{k/2+i+j+1}}\right]
\end{aligned}
$$

$$
\begin{aligned}
E(q_m^2(r)) \;=\; & \sum_{h_1=0}^{m-1}\sum_{h_2=0}^{m-1}\binom{m-1}{h_1}\binom{m-1}{h_2}\epsilon^{h_1+h_2}(1-\epsilon)^{2m-2-h_1-h_2}\frac{1}{a(h_1)a(h_2)}\frac{1}{\Gamma(\frac{k}{2})} \\
& \Bigg[\frac{1-\epsilon}{\left(\frac{1}{a^2(h_1)}+\frac{1}{a^2(h_2)}+1\right)^{\frac{k+2}{2}}} \\
& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)\Gamma(\frac{k+1}{2}+j)\Gamma(\frac{k+2}{2}+i+j)}{\Gamma(\frac{k+2}{2}+i)\Gamma(\frac{k+2}{2}+j)i!j!}x_1^i y_1^j + \\
& \frac{\epsilon\sigma^2}{\left(\frac{\sigma^2}{a^2(h_1)}+\frac{\sigma^2}{a^2(h_2)}+1\right)^{\frac{k+2}{2}}} \\
& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{k+1}{2}+i)\Gamma(\frac{k+1}{2}+j)\Gamma(\frac{k+2}{2}+i+j)}{\Gamma(\frac{k+2}{2}+i)\Gamma(\frac{k+2}{2}+j)i!j!}x_2^i y_2^j\Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
x_1 &= \frac{1}{a^2(h_1)}\left(\frac{1}{a^2(h_1)}+\frac{1}{a(h_2)^2}+1\right)^{-1} \\
y_1 &= \frac{1}{a^2(h_2)}\left(\frac{1}{a^2(h_1)}+\frac{1}{a(h_2)^2}+1\right)^{-1} \\
x_2 &= \frac{1}{a^2(h_1)}\left(\frac{\sigma^2}{a^2(h_1)}+\frac{\sigma^2}{a^2(h_2)}+1\right)^{-1} \\
y_2 &= \frac{1}{a^2(h_2)}\left(\frac{\sigma^2}{a^2(h_1)}+\frac{\sigma^2}{a^2(h_2)}+1\right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
E(q_m^2(r)) \;=\; & \sum_{h_1=0}^{m-1}\sum_{h_2=0}^{m-1}\binom{m-1}{h_1}\binom{m-1}{h_2}\epsilon^{h_1+h_2}(1-\epsilon)^{2m-2-h_1-h_2}\frac{1}{a(h_1)a(h_2)}\frac{1}{\Gamma(\frac{k}{2})} \\
& \Bigg[\frac{1-\epsilon}{\left(\frac{1}{a^2(h_1)}+\frac{1}{a^2(h_2)}+1\right)^{\frac{k+2}{2}}}\frac{\Gamma^2(\frac{k+1}{2})\Gamma(\frac{k+2}{2})}{\Gamma^2(\frac{k+2}{2})} \\
& F_2\left(\frac{k+2}{2},\frac{k+1}{2},\frac{k+1}{2};\frac{k+2}{2},\frac{k+2}{2};x_1;y_1\right)+ \\
& \frac{\epsilon\sigma^2}{\left(\frac{\sigma^2}{a^2(h_1)}+\frac{\sigma^2}{a^2(h_2)}+1\right)^{\frac{k+2}{2}}}\frac{\Gamma^2(\frac{k+1}{2})\Gamma(\frac{k+2}{2})}{\Gamma^2(\frac{k+2}{2})} \\
& F_2\left(\frac{k+2}{2},\frac{k+1}{2},\frac{k+1}{2};\frac{k+2}{2},\frac{k+2}{2};x_2;y_2\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
E(q_m^2(r)) \;=\; & \sum_{h_1=0}^{m-1}\sum_{h_2=0}^{m-1}\binom{m-1}{h_1}\binom{m-1}{h_2}\epsilon^{h_1+h_2}(1-\epsilon)^{2m-2-h_1-h_2}\frac{1}{a(h_1)a(h_2)}\frac{\Gamma^2(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{k+2}{2})} \\
& \Bigg[\frac{1-\epsilon}{\left(\frac{1}{a^2(h_1)}+\frac{1}{a^2(h_2)}+1\right)^{\frac{k+2}{2}}}(1-x_1)^{-\frac{k+1}{2}}(1-y_1)^{-\frac{k+1}{2}} \\
& {}_2F_1\left(\frac{k+1}{2},\frac{k+1}{2};\frac{k+2}{2};\frac{x_1y_1}{(1-x_1)(1-y_1)}\right)+ \\
& \frac{\epsilon\sigma^2}{\left(\frac{\sigma^2}{a^2(h_1)}+\frac{\sigma^2}{a^2(h_2)}+1\right)^{\frac{k+2}{2}}}(1-x_2)^{-\frac{k+1}{2}}(1-y_2)^{-\frac{k+1}{2}} \\
& {}_2F_1\left(\frac{k+1}{2},\frac{k+1}{2};\frac{k+2}{2};\frac{x_2y_2}{(1-x_2)(1-y_2)}\right)\Bigg]
\end{aligned}
$$