Stochastic Inventory Models with Continuous and Poisson Demands and Discounted and Average Costs.

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Abstract

It has been more than ninety years since the classical square-root EOQ formula was given by Harris (1913). Yet there is no continuous-review stochastic inventory model published in the literature with a general enough demand, whose optimal policy would reduce to the square-root formula in the absence of the stochastic components of the underlying demand. Why? In this paper, we surmise the reasons, develop a model with continuous and Poisson demands for the first time, and prove the optimality of an (s, S)-policy. We also verify that the policy reduces to the EOQ formula, as it must, when the intensity of the Poisson process goes to zero. In the process, we develop a new, unified approach of dealing with both the average cost and the discounted cost criteria. We introduce new *average discounted-cost formulas* along with intuitive interpretations. We do not require the surplus cost function to be convex or quasi-convex as has been assumed in the literature. We show that while the optimal ordering level is unique, there may be more than one optimal order-up-to levels.

1 Introduction

In this paper, we bring together two of the most classical results in operations research, namely, the EOQ formula and the optimality of an (s, S)-policy in stochastic inventory models with a fixed ordering cost. The EOQ formula dates back to Ford W. Harris (1913) and the optimality of (s, S)-policy was proved by Herbert Scarf (1960). We should mention, however, that Scarf was not the first to formulate the problem he had solved. That honor belongs to K. J. Arrow, T. Harris and J. Marschak (1951), who formulated the problem and proposed the famous (s, S)-policy. It is also fair to say that Harris (1913) did not provide a rigorous proof that the lot size given by the formula minimized the long-run average cost. Moreover, since an easily accessible reference containing a rigorous proof of the optimality of the EOQ formula could not be found, Beyer and Sethi (1998) supplied a proof involving quasi-variational inequalities (QVI) that arise in the course of dealing with continuous-time optimization problems involving fixed costs.

The marriage of the two classical results is accomplished by formulating a continuous-time stochastic inventory model involving a demand that is the sum of a constant demand rate and a compound Poisson process. In the presence of a fixed cost, we prove that an (s, S)-policy minimizes the long-run average cost. This (s, S)-policy reduces to the square-root EOQ formula, when the intensity of the compound Poisson process becomes zero. And when the constant demand component vanishes, our model reduces to what is referred to in the literature as the continuous-review stochastic inventory model with compound Poisson demand. We should mention that our method of analysis also provides the optimal policy for the discounted cost criterion.

Before we describe the solution methodology we develop and various contributions we make in this paper, we shall briefly review the literature on continuous-review stochastic inventory models involving a fixed cost. In this we limit ourselves to discounted and average cost models allowing for backorders. Even though many of the papers we review consider leadtimes in ordering, we shall not emphasize this issue for expositional purposes. Besides, in most cases, a fixed leadtime can be incorporated without much complications in the standard model by using the *inventory position* rather than the inventory on hand as the system state.

Most of the related early work is devoted to obtaining stationary and limiting distributions of the inventory level under a variety of policies. Also developed are expressions of the expected performance measures associated with the various policies. These tasks were accomplished under a variety of different stochastic demand scenarios. For example, demands may occur continuously in time or they arrive one at a time at random epochs. These epochs may follow a Poisson process, a renewal process, or a counting process. When the demand size at any of these epochs is an iid random variable, then we shall qualify the arrival process by using the word *compound* like in a compound Poisson process. We shall classify these early papers according to the demand process they consider.

Poisson demands are considered by Scarf (1958), Karlin and Scarf (1958), Galliher et al. (1959), and Morse (1958). These early papers are reviewed in Scarf (1963). Poisson demand is generalized by Finch (1961), Rubalskiy (1972a,b) and Sivazlian (1974) to unit demands arriving at epochs following a renewal process.

Compound Poisson demands are treated in Richards (1975), Thompstone and Silver (1975), Archibald and Silver (1978), Feldman (1978) and Federgruen et al. (1983). Feldman allows the intensity of the process to depend on the random state of the environment.

Tijms (1972), Sahin (1979,1983), Federgruen and Schechner (1983), and Zipkin (1986) consider compound renewal demands, whereas Zipkin (1986) consider a compound counting process to model demands. There are others, who obtain steady state distributions of the inventory level with general demand processes. We choose not to review them since they do not obtain cost expressions. Interested readers can refer to

Federgruen and Schechner (1983) and Zipkin (1986) for these references.

Continuously occurring demands are considered by Hadley and Whitin (1963), Bather (1966), Puterman (1975), and Browne and Zipkin (1991). Note that in the case of continuous demands, it is common to consider (Q, r)- policies, where an order in the amount of Q is issued when the inventory level reaches r. It is clear that in the continuous demand case, a (Q, r)-policy is equivalent to an (s, S)-policy with $r = s$ and $Q = S - s$.

After obtaining cost expressions for expected performance measures, many of these papers attempt to obtain the optimal value of the policy parameters s and S that minimize the stationary cost. To our knowledge, none of these papers have taken the next step of showing that the resulting (s, S)-policy is indeed optimal among the class of all non-anticipative or admissible policies as was done by Iglehart (1963) in the discrete-time framework. We should also note that the analysis in Iglehart (1963) is not quit complete. See Beyer and Sethi (1999) for its completion and other details.

This next step in a continuous-review model with Poisson demands is taken by Zheng (1994). Zheng incorporates an additional feature, namely, that discount opportunities arrive according to another Poisson process independent of the demand process, and at these opportunities an order incurring a smaller setup cost may be issued. He proves optimality of a policy known as (s, c, S)-policy in the context of average cost minimization. Such policies are advocated in what is referred to as coordinated inventory replenishment, and readers interested in this literature may consult Zheng (1994) for references. Without the additional discount opportunities, an (s, c, S)-policy reduces to an (s, S)-policy. After Zheng derives the expected cost expression, he minimizes it with respect to the policy parameters. He then shows that the average cost satisfies the optimality equation for the average cost criterion. In order to use, for this purpose, a verification theorem (Theorem 2.1 in Ross (1983)) proved for bounded solutions, Zheng uses a trick of relaxing the constraint that the order sizes must be non-negative. The relaxed model provides a solution of the original problem, because any negative order or disposal will take place only at time zero when the initial inventory level is excessive. But such an action at time zero will not have any effect on the average cost of the policy. We should note, however, that the trick used by Zheng would not work in the discounted cost case.

On the other hand, Song and Zipkin (1993) formulate a continuous-review model with state-dependent Poisson demands. They invoke the standard uniformization procedure (Keilson (1979) and Van Dijk (1990)) to convert their problem to a discrete-time problem, and then use the discrete-time dynamic programming to obtain a state-dependent (s, S)-policy. To be completely rigorous, a verification theorem is required to prove the optimality. This is proved in Beyer, Sethi and Taksar (1998) for Markovian demands and a fairly general surplus cost structure.

One may ask a question as to who was the first to prove the optimality of an (s, S)-policy in the simplest continuous-review model with Poisson demand? In our opinion, this issue is moot since the continuous-time result is a corollary of the discrete-time result in light of the uniformization procedure dating back to Jensen (1953).

We now come to our continuous-review model with a demand that includes both a constant component and a compound Poisson component. We assume the holding cost to be increasing in the inventory level and the shortage cost to be increasing and convex in the amount of backlog. We consider an infinite horizon model with average and discounted costs criteria. We prove the optimality of (s, S)-policies in all cases under consideration. Our paper makes the following important contributions:

i) We develop a unified approach of dealing with both the average cost and the discounted cost criteria.

ii) We do not require surplus cost function to be convex or quasi-convex as has been assumed in the literature.

iii) We prove that the optimal ordering level s is unique. We note that the order-up-to level S may not be unique.

iv) We introduce new *average discounted-cost formulas* required for the development of the unified approach, and we supply their intuitive interpretations.

v) Finally, we allow for a constant demand component. In addition to the interesting fact that this includes EOQ as a special case, there are other reasons why this extension is significant. In what follows we discuss these reasons.

Inclusion of a constant demand rate has never been considered in continuous-time stochastic inventory models. Perhaps there are a number of reasons why this may be so. One is that the presence of the constant demand term means that an (s, S)-policy may order at instants other than the jump epochs of the compound Poisson process. This means that the standard uniformization procedure (Kielson (1979) or Van Dijk (1990)) will not work. In the best case it would need to be modified, and in the worst case it may not be altogether applicable.

On the other hand if one uses a QVI approach in the presence of a constant demand rate, one gets a term involving the first derivative of the value function (resp. potential function) in the discounted (resp. average) cost case. Since one cannot assume the value function to be differentiable a priori, one would require a viscosity solution approach (see, e.g., Fleming and Soner (1992) and Sethi and Zhang (1994)). Even if one could do all this for the discounted cost objective, it would not be still easy to use the standard vanishing discount approach for solving the average cost problem. This is because the value function in the discounted case is K-convex, and not convex, and there is no known result that ensures that the potential function obtained from taking the limit of a continuously differentiable K-convex function, as the discount rate tends to zero, is continuously differentiable. To push the vanishing discount approach through would require a more detailed study of the structure of the value function so that as the discount rate goes to zero, one obtains a continuously differentiable potential function that can be shown to satisfy the average cost QVI.

Finally, the inclusion of a constant demand rate would allow us to take the next step of adding the diffusion term to the demand. The extension would give us a model involving a demand modelled by the sum of a diffusion process and a compound Poisson process. Moreover, the extension would be non-trivial on account of the facts that diffusion can take negative values and that QVI would involve the second derivative of the potential function. Indeed it may be possible to consider a more general demand modelled by a process with stationary independent increments.

In view of all this, we believe that incorporation of a constant demand in stochastic inventory models is a significant contribution

The plan of the remainder of the paper is as follows. In Section 2 we give a rigorous formulation of the problem and formulate the main result concerning the optimality of an (s, S)-policy. In Section 3 we construct a potential function simultaneously for both discounted and average cost problems, and prove some of the required results while relegating the proofs of others of technical nature to the Appendix. In section 4 we complete the proof of the main result. Section 5 concludes the paper.

2 Formulation and Statement of Results

In order to precisely state the problems under consideration, we must specify the probability space, the demand process, the class of admissible ordering policies, the surplus (inventory/backlog) dynamics, surplus and ordering costs, the discount rate, the objective functions, and the assumptions.

We model the demand by a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume the demand to consist of two components, deterministic and stochastic. The deterministic portion of the demand is assumed to be a constant $D \geq 0$ per unit time. In order to define the stochastic portion of the demand, we let $n(t),\ n(0) = 0$, denote a right-continuous process with intensity $\lambda \geq 0$ and $\xi_i \geq 0,\ i = 1, 2, \ldots,$ denote a sequence of iid nonnegative random variables independent of $n(t)$, and having the distribution $G(\cdot)$. The process $n(t)$ provides a sequence of jump times and ξ_i denotes the size of the demand that occurs at the ith jump of $n(t)$. Thus, the cumulative demand in the interval $[0, t]$ is defined to be

$$y(t) = Dt + N(t), \tag{1}$$

where

$$N(t) = \sum_{i \leq n(t)} \xi_i, \ t \geq 0, \tag{2}$$

is a compound Poisson process.

Next we define the class of admissible ordering policies. For this, let $\{\mathcal{F}_t\}$ denote the family of sigma algebras generated by $N(t), \ t \geq 0$. Further, let $\theta_i \geq 0, \ i = 1, 2, \ldots,$ be a strictly increasing sequence of stopping times with respect to the filtration $\{\mathcal{F}_{t+0}\}$ and $u_i > 0$ be a positive random variable adapted to $\mathcal{F}_{\theta_i}, \ i = 1, 2, \ldots$. Simply speaking, θ_i denotes the time of the ith order and u_i denotes the amount of the ith order. Thus,

$$U = (\theta_1, u_1, \theta_2, u_2, \ldots) \tag{3}$$

is referred to as an admissible policy. Let $\mathcal{U}$ denote the set of all admissible policies. Also, the cumulative total order amount $M(t)$ from time 0 to time t can be defined as

$$M(t) = \sum_{\{i : \theta_i < t\}} u_i. \tag{4}$$

We can now easily see that the surplus level $x^U(t)$ at time t under a policy $U \in \mathcal{U}$ is given by the equation

$$x^U(t) = x - Dt - N(t) + M(t), \tag{5}$$

where $x^U(0) = x$ is the initial surplus level at time zero. Note that the surplus x when positive means inventory and when negative means backlog.

Remark 2.1 If at some time t, both the processes $N(t)$ and $M(t)$ jump, then the process $x^U(t)$ is neither continuous from the left nor from the right.

Next we define surplus and ordering costs. We let a nonnegative piecewise continuously differentiable function $f(x)$, $f(0) = 0$, denote the surplus cost. When $x > 0$, the surplus cost refers to the cost of holding inventory and when $x < 0$, it refers to the backlog cost. Some needed properties of the function $f(x)$ will be specified later in Theorem 1 and Remark 2.3.

The cost $c(u)$ of ordering an amount u is given by

$$c(u) = \begin{cases} K + cu, & \text{if } u > 0, \\ 0, & \text{if } u = 0, \end{cases}$$

where $K > 0$ denotes the fixed cost of ordering and c denotes the unit cost of each item ordered.

Remark 2.2 Here we assume $K > 0$ for convenience in exposition. Nevertheless, the results for the special case when $K = 0$, i.e., when there is no fixed cost, follow easily from our analysis. Specifically, when $K = 0$, the optimal policy reduces to a base-stock policy.

We consider both discounted and average cost objective functions in this paper. With $\rho \geq 0$ as the discount rate, the functionals we aim to minimize are stated below:

$$F_\rho(x, U) = \mathbf{E}\left[\int_0^\infty f(x^U(t))e^{-\rho t}dt + \sum_{i=0}^{\infty} c(u_i)e^{-\rho\theta_i}\right] \text{ for } \rho > 0, \tag{6}$$

$$F_0(x, U) = \limsup_{T\to\infty} \frac{1}{T}\mathbf{E}\left[\int_0^T f(x^U(t))dt + \sum_{\{i:\theta_i<T\}} c(u_i)\right] \text{ for } \rho = 0. \tag{7}$$

It is easy to see that when $\rho > 0$ (resp. $\rho = 0$), $F_\rho(x, U)$ represents the total discounted cost (resp. the average cost) over the infinite horizon, when one begins with x as the initial inventory and $U \in \mathcal{U}$ as the policy. In what follows, we mean $\rho \geq 0$, unless we specify $\rho = 0$ or $\rho > 0$.

In order for the cost functionals (6)-(7) to take finite values, we assume that the mean jump size is bounded, i.e., $\mathbf{E}\xi_i = \bar{\xi} < \infty$.

Our goal is to find for any given $\rho \geq 0$,

$$F_\rho(x) = \inf_{U \in \mathcal{U}} F_\rho(x, U), \tag{8}$$

and a policy $U^*_\rho \in \mathcal{U}$ such that $F_\rho(x, U^*_\rho) = F_\rho(x)$.

For $-\infty < s < S < +\infty$, let $U^{s,S}$ denotes the (s, S)-policy given by the following function:

$$U^{s,S}(x) = \begin{cases} 0 & \text{if } x > s, \\ S - x & \text{if } x \leq s. \end{cases} \tag{9}$$

Clearly $U^{s,S} \in \mathcal{U}$. Furthermore, let $\bar{\mathcal{U}} \subset \mathcal{U}$ denote the class of all (s, S)-policies.

Since we expect an (s, S)-policy to be optimal for the problem under consideration, our analysis focuses on selecting a candidate (s, S)-policy in $\bar{\mathcal{U}}$, and then proving that this candidate policy is optimal in the class $\mathcal{U}$ of all admissible policies.

For a given (s, S)-policy, let $Q = S - s$. Let $\tau(Q) = \inf\{t : \ y(t) \geq Q\}$ denote the hitting time of the level Q by the total cumulative demand process $y(t)$ defined in (1). Note that $y(0) = 0$. Define

$$\varphi_\rho(Q) = \mathbf{E}\left[\int_0^{\tau(Q)} e^{-\rho t} dt\right]. \tag{10}$$

It is possible to interpret $\varphi_\rho(Q)$ as the *expected discounted cycle time* under the given (s, S)-policy.

Consider the function

$$\psi_\rho(S, Q) = \mathbf{E}\left[\int_0^{\tau(Q)} f(S - y(t)) e^{-\rho t} dt + (K + cy(\tau(Q))) e^{-\rho\tau(Q)}\right]. \tag{11}$$

The expression $\psi_\rho(S, Q)$ for any given S and Q represents the discounted cost of one cycle, which begins at any time when the surplus process $x(t)$ starts at the level $S > 0$, decreases, and eventually crosses the level $S - Q$, at which instant an order in the amount $y(\tau(Q))$ restores the surplus level back to the level S. That ends the cycle.

Let us define the function $a_\rho(S, Q)$ as the solution of the equation

$$\mathbf{E}\left[\int_0^{\tau(Q)} (f(S - y(t)) - a_\rho(S, Q)) e^{-\rho t} dt + (K + cy(\tau(Q))) e^{-\rho\tau(Q)}\right] = 0. \tag{12}$$

Then from (10)-(12), it is obvious that

$$a_\rho(S,Q) = \frac{\psi_\rho(S,Q)}{\varphi_\rho(Q)}. \tag{13}$$

Note that for $\rho = 0$, the value

$$a_0^{s,S} := a_0(S, S-s) = F_0(x, U^{s,S}) \tag{14}$$

for any x, i.e., it is the long-run average cost associated with the given (s, S)-policy, which in this case coincides with the average cost of the cycle.

For $\rho > 0$, the value

$$a_\rho^{s,S} := a_\rho(S, S-s) \tag{15}$$

will be referred to as *the average discounted cost of a cycle* for the given (s, S)-policy, provided that the initial surplus level $x(0) = S$.

For $\rho > 0$, $x(0) = S$, and a policy $U^{s,S}$, the cost of the first cycle is $\psi_\rho(S, S-s)$ and the length of the first cycle is $\tau(S-s)$. It is immediate from this observation that

$$F_\rho(S, U^{s,S}) = \psi_\rho(S, S-s) + F_\rho(S, U^{s,S})\mathbf{E}\left[e^{-\rho\tau(S-s)}\right].$$

It follows from (10) that $\mathbf{E}\left[e^{-\rho\tau(S-s)}\right] = 1 - \rho\phi_\rho(S-s)$. Using (10) and (13) we obtain

$$F_\rho(S, U^{s,S}) = a_\rho^{s,S}/\rho. \tag{16}$$

While $a_\rho^{s,S}$ has been referred to as the average discounted cost per cycle when $x(0) = S$, its minimization for $\rho > 0$ will not give us the candidate (s, S)-policy we are looking for when $x(0) \neq S$. What we need, therefore, is some way of distributing the cost of the first transient cycle beginning with $x(0) = x \neq S$ and ending with the surplus level S. With this insight, we are ready to find an (s, S)-policy in $\bar{\mathcal{U}}$ as the candidate for the optimal policy.

In the case $\rho > 0$, according to the definition of $U^{s,S}$ and (16), it is clear that for $x \leq s$, we have

$$F_\rho(x, U^{s,S}) = K + c(S-x) + F_\rho(S, U^{s,S}) = K - cx + \frac{1}{\rho}\left(c\rho S + a_\rho(S, S-s)\right). \tag{17}$$

Denote for $\rho \geq 0$,

$$d_\rho(S,Q) = c\rho S + a_\rho(S,Q) \tag{18}$$

and

$$d_\rho = \inf_{S,Q} d_\rho(S,Q). \tag{19}$$

If there exist s_ρ and S_ρ such that $d_\rho(S_\rho, S_\rho - s_\rho) = d_\rho$, then in the case $\rho = 0$ according to (14) for all x, and in the case $\rho > 0$ according to (17) for $x < \min(s, s_\rho)$, we have

$$F_\rho(x, U^{s_\rho, S_\rho}) \leq F_\rho(x, U^{s,S}). \tag{20}$$

It will be shown in Section 4 that the inequality (20) is valid for all x and for any admissible policy $U \in \mathcal{U}$ replacing $U^{s,S}$ in (20). More specifically, the main result of the paper is the following theorem.

Theorem 1. *If there exists a value* σ_ρ *such that the function* $f(x) + c\rho x$ *increases to* ∞ *for* $x > \sigma_\rho$, *and decreases from* ∞ *for* $x < \sigma_\rho$, *then there exist* s_ρ *and* S_ρ, $-\infty < s_\rho < \sigma_\rho < S_\rho < \infty$, *such that*

$$d_\rho = d_\rho(S_\rho, S_\rho - s_\rho) \leq d_\rho(S,Q) \text{ for any } S \text{ and } Q, \tag{21}$$

s_ρ *is unique, and* $F_\rho(x) = F_\rho(x, U^{s_\rho, S_\rho})$, *i.e., the* (s_ρ, S_ρ)*-policy is an optimal policy.*

Remark 2.3 To guarantee the condition on $f(x)+c\rho x$ used in Theorem 1, it is sufficient to assume that $f(0) = 0$, $f(x)$ is decreasing and convex on $(-\infty, 0)$, $f(x)$ is increasing on $(0,\infty)$, $f(x) \to \infty$ as $x \to \infty$, and $\lim_{x\to-\infty} |f'(x)| > c\rho$. It should be noted that for $\rho > 0$, these conditions are sufficient but not necessary for $f(x) + c\rho x$ to satisfy the requirements of Theorem 1. For example, when $\rho > 0$, $f(x)$ does not need to be monotone increasing on $(0,\infty)$. It could even decrease as long as that decrease is less than offset by the linearly increasing part $c\rho x$. Note also that the last limiting condition on $f'(x)$ reduces to the standard condition $p > c\rho$, when $f(x) = -px$ for $x < 0$, with p denoting the unit shortage cost. Otherwise, it would be optimal never to order, which can be accommodated by allowing s to take the value $-\infty$. See also Remark 4.2 in Sethi and Cheng (1997). It is also interesting to point out that σ_ρ plays the role of zero in the discounted case. Finally, we should note that our condition on the surplus cost $f(x)$ generalizes the convexity assumption used by Scarf (1960) and the quasi-convexity assumption used by Veinott (1966).

Note that when $\rho = 0$, we have $d_0(S, S-s) = a_0(S, S-s)$, the long-run average cost. So obviously, *if* an (s, S)-policy is optimal for our problem, then minimizing $d_0(S, S-s)$ over s and S should provide us with an optimal policy. What we have accomplished here is an extension of this idea to the discounted case. This is done by recognizing that there is an initial cost $K + c(S-x)$ of bringing an initial inventory $x < s$ immediately to the level S. A part of this cost, namely $K - cx$, does not depend on S. The remaining part, namely cS, depends on S. Since our purpose is to obtain the best (s, S)-policy for a small x as our candidate, we allocate this cost to the average discounted cycle cost $a_\rho(S, S-s)$, and then minimize this *"modified" average discounted cycle cost.*

First we note that the cost cS at time $t = 0$ is equivalent to a cost rate of $c\rho S$ per unit time. Over the cycle, the expected present value of this cost rate is

$$\mathbf{E}\left[\int_0^{\tau(Q)} c\rho S e^{-\rho t} dt\right] = c\rho S \varphi_\rho(Q). \tag{22}$$

When added to the cycle cost $\psi_\rho(S, Q)$ obtained in (11), we obtain a modified cycle cost as $c\rho S\varphi_\rho(Q) + \psi_\rho(S, Q)$. Then the modified average discounted-cost of the cycle is

$$\frac{c\rho S\varphi_\rho(Q) + \psi_\rho(S, Q)}{\varphi_\rho(Q)} = c\rho S + a_\rho(S, Q), \tag{23}$$

which is precisely $d_\rho(S, Q)$ that we had obtained from (17) as the appropriate quantity to minimize for obtaining the candidate (s, S)-policy.

To summarize, the candidate (s, S)-policy minimizes the modified average discounted cycle cost. When $\rho = 0$, we have $c\rho S = 0$, and so the modified cost is the same as the average cycle cost a_0, and the candidate (s, S)-policy is the best policy in $\bar{\mathcal{U}}$ that minimizes the average cost. No modification of the average cost is required when $\rho = 0$ is consistent with the known intuition that the initial surplus level is of no consequence in the determining the average cost of a policy in $\bar{\mathcal{U}}$. For $\rho > 0$, the candidate policy is the best policy in $\bar{\mathcal{U}}$ for any surplus level "sufficiently" small. Clearly then, if an (s, S)-policy is optimal for our problem for any initial surplus level as we expect, then it must be optimal for a sufficiently small x as well. And, therefore, we obtain our candidate (s, S)-policy by minimizing the modified cycle cost corresponding to small x, and do not concern ourselves with the modification of the average cycle cost by allocating to it, its share of the cost of the first transient cycle when x is large.

To treat the cases with $\rho = 0$ and $\rho > 0$ in a unified manner for $s_\rho > -\infty$, we modify the cost function $F_\rho(x)$ for $\rho > 0$ by subtracting from it an average discounted cost $a_\rho = a_\rho(S_\rho, S_\rho - s_\rho)$ corresponding to the (s_ρ, S_ρ)-policy. Note that for $\rho > 0$. the dynamic programming equation in the integral form can be written as

$$F_\rho(x) = \inf_{U \in \mathcal{U}} \mathbf{E} \left[\int_0^T f(x^U(t)) e^{-\rho t} dt + \sum_{\{i:\theta_i < T\}} c(u_i) e^{-\rho \theta_i} + F_\rho(x^U(T)) e^{-\rho T} \right] \tag{24}$$

for any $T > 0$. By subtracting a_ρ from both sides and recognizing that $a_\rho = \int_0^T a_\rho \rho e^{-\rho t} dt + a_\rho e^{-\rho T}$, we can rewrite (24) as

$$W_\rho(x) = \inf_{U \in \mathcal{U}} \mathbf{E} \left[\int_0^T (f(x^U(t)) - a_\rho) e^{-\rho t} dt + \sum_{\{i:\theta_i < T\}} c(u_i) e^{-\rho \theta_i} + W_\rho(x^U(T)) e^{-\rho T} \right], \tag{25}$$

where

$$W_\rho(x) = F_\rho(x) - \frac{a_\rho}{\rho} \quad \text{for } \rho > 0. \tag{26}$$

In the case when $\rho = 0$, the function satisfying (25) is called a *potential function*, and together with a number a_0 and a policy U_0^* which provides the infimum in (25), they are called a *canonical triplet* written as $\{a_0, W_0(x), U_0^*\}$; see Dynkin and Yushkevich (1978). Furthermore, a_0 turns out to be the minimum average cost, i.e.,

$$F_0(x) = a_0 \quad \text{for any} \quad x. \tag{27}$$

Our analysis and the derivation of (25) has permitted the extension of the notions of the potential function and the canonical triplet to the case $\rho > 0$. Accordingly, in Section 3 we will construct for $\rho \geq 0$, a canonical triplet $\{a_\rho, W_\rho(x), U_\rho^*\}$ that is continuous in $\rho \geq 0$. In other words, we construct a number a_ρ, a potential function $W_\rho(x)$ satisfying (25), and a policy U_ρ^* which provides the infimum in (25). As will be seen later, it is the (s_ρ, S_ρ)-policy mentioned earlier that would give us the policy U_ρ^*.

Remark 2.4 Note that the relation (26) when $\rho > 0$ has the interpretation that the potential function is the difference between the optimal cost (or the value function) and the capitalized value of the a cost stream, equal to the weighted average cost a_ρ, discounted at the rate ρ.

Theorem 1 will be proved in Section 4. There we will show using a verification theorem that Theorem 1 follows from (25) and the properties of the constructed $W_\rho(x)$ derived in the next section.

3 Construction of the Potential Function

Using functions $\psi_\rho(S, Q)$ and $\varphi_\rho(Q)$, we construct in four steps a potential function $W_\rho(x)$, and show that it satisfies relation (25).

In Step 1, we begin with a policy $U^{s,S}$ and construct a function $P_\rho^{s,S}(x)$, which satisfies the following relation analogous to (25), i.e.,

$$\begin{aligned} P_\rho^{s,S}(x) \; &= \mathbf{E}\left[\int_0^T (f(x^{s,S}(t)) - a_\rho^{s,S})e^{-\rho t}dt \right. \\ &\left. + \sum_{\{i:\theta_i^{s,S}<T\}} \Big(K + c(S - x^{s,S}(\theta_i^{s,S}))\Big)e^{-\rho\theta_i^{s,S}} + P_\rho^{s,S}(x^{s,S}(T))e^{-\rho T}\right], \end{aligned} \tag{28}$$

where $\theta_i^{s,S}$ is the time of the ith crossing of the surplus level s by the process $x^{s,S}(t) := x^{U^{s,S}}(t)$, $i = 1, 2, \ldots$. Note that if the initial surplus x is less or equals to s, then $\theta_1^{s,S} = 0$, and it is, by definition, the time of the first crossing of the level s. Just as in (26) and (27), we have

$$P_\rho^{s,S}(x) = F_\rho(x, U^{s,S}) - \frac{a_\rho^{s,S}}{\rho} \quad \text{for } \rho > 0, \tag{29}$$

and

$$a_0^{s,S} = F_0(x, U^{s,S}) \quad \text{for any} \quad x. \tag{30}$$

In Step 2, we give necessary conditions for finite s_ρ and S_ρ to satisfy (21).

In Step 3, we show that the constructed function

$$P_\rho(x) = P_\rho^{s_\rho, S_\rho}(x) \tag{31}$$

satisfies the quasi-variational inequalities (QVI) for our problem. These inequalities are specified later as (47) and (49). In Remark 3.5, we rewrite the QVI for the discounted case in the form of a dynamic programming equation.

In Step 4, we show that from the fact that $P_\rho(x)$ satisfies the QVI, it follows that $P_\rho(x)$, a_ρ, and the (s_ρ, S_ρ)-policy solve (25). Setting $W_\rho(x) = P_\rho(x)$ and $U_\rho^* = U^{s_\rho, S_\rho}$ completes our construction of the potential function and the canonical triplet $\{a_\rho, W_\rho(x), U_\rho^*\}$.

In what follows we carry out the details of these four steps.

STEP 1. Let

$$\begin{aligned} m_\rho(S,Q) : &= \psi_\rho(S,Q) + c\rho S \varphi_\rho(Q) \\ &= \mathbf{E}\left[\int_0^{\tau(Q)} (f(S - y(t)) + c\rho S) e^{-\rho t} dt + (K + cy(\tau(Q))) e^{-\rho\tau(Q)}\right]. \end{aligned} \tag{32}$$

Then according to (18) and (13),

$$d_\rho(S,Q) = \frac{m_\rho(S,Q)}{\varphi_\rho(Q)}. \tag{33}$$

We begin with the following lemma.

Lemma 1. *The function $P_\rho^{s,S}(x)$ defined in (29) for $\rho > 0$ can be specified as*

$$P_\rho^{s,S}(x) = \begin{cases} K + c(S - x) & \text{if } x \le s, \\ m_\rho(x, x - s) - d_\rho(S, S - s)\varphi_\rho(x - s) + c(S - x) & \text{if } x > s. \end{cases} \tag{34}$$

Furthermore, $P_\rho^{s,S}(x)$ *for* $\rho \geq 0$ *satisfies* (28), *where* $P_0^{s,S}(x)$ *is obtained by setting* $\rho = 0$ *in* (34).

Proof. First we consider the case $\rho > 0$. If $x \leq s$, then (34) follows from (29) and (17). If $x > s$, then according to the (s,S)-policy we do not order until $y(t) \geq x - s$, at which time we order the amount $y(\tau(x-s)) + (S - x)$ and jump to the inventory level S. These observations, the definition (32) of $m_\rho(x, x-s)$, and the expression (16) for $F_\rho(S, U^{s,S})$ imply

$$\begin{aligned} F_\rho(x, U^{s,S}) &= \mathbf{E}\Bigg[\int_0^{\tau(x-s)} f(x - y(t))e^{-\rho t}dt \\ &\quad + \Big(K + cy(\tau(x-s)) + (S-x)c + F_\rho(S, U^{s,S})\Big)e^{-\rho\tau(x-s)}\Bigg] \\ &= m_\rho(x, x-s) - c\rho x\varphi_\rho(x-s) + \mathbf{E}\left[\left(c(S-x) + \frac{a_\rho^{s,S}}{\rho}\right)e^{-\rho\tau(x-s)}\right]. \end{aligned} \tag{35}$$

From the fact that $\mathbf{E}\left[\left(1 - e^{-\rho\tau(x-s)}\right)\right] = \rho\varphi_\rho(x-s)$ and the relation between $a_\rho^{s,S}$ and $d_\rho(S, S-s)$ (see (13), (15), and (18)), equation (35) reduces to

$$\begin{aligned} F_\rho(x, U^{s,S}) &= m_\rho(x, x-s) - c\rho x\varphi_\rho(x-s) + c(S-x) + \frac{a_\rho^{s,S}}{\rho} \\ &\quad + \left[c(S-x) + \frac{a_\rho^{s,S}}{\rho}\right]\rho\varphi_\rho(x-s) \\ &= m_\rho(x, x-s) - d_\rho(S, S-s)\varphi_\rho(x-s) + c(S-x) + \frac{a_\rho^{s,S}}{\rho}. \end{aligned} \tag{36}$$

This proves (34) for $x > s$ and completes the proof of (29).

It follows from (29) and (6) that

$$\begin{aligned} P_\rho^{s,S}(x) &= \mathbf{E}\left[\int_0^\infty (f(x^{U^{s,S}}(t)) - a_\rho^{s,S})e^{-\rho t}dt + \sum_{i=1}^\infty (K + cu_i)e^{-\rho\theta_i^{s,S}}\right] \\ &= \mathbf{E}\left[\int_0^T (f(x^{U^{s,S}}(t)) - a_\rho^{s,S})e^{-\rho t}dt + \sum_{\{i:\theta_i^{s,S} < T\}} (K + cu_i)e^{-\rho\theta_i^{s,S}}\right] \\ &\quad + \mathbf{E}\left[\int_T^\infty (f(x^{U^{s,S}}(t)) - a_\rho^{s,S})e^{-\rho t}dt + \sum_{\{i:\theta_i^{s,S} \geq T\}} (K + cu_i)e^{-\rho\theta_i^{s,S}}\right]. \end{aligned} \tag{37}$$

This relation proves (28) for $\rho > 0$, since the second expectation in the right-hand side of (37) equals $e^{-\rho T}\mathbf{E}\left[P_\rho^{s,S}(x^{U^{s,S}}(T))\right]$.

Note that the function $P_\rho^{s,S}(x)$ is defined also for $\rho = 0$. Taking the limit as $\rho \to 0$ in (28), we prove (28) for $\rho = 0$. Dividing (28) by T for $\rho = 0$ and taking the limit as $T \to \infty$, we obtain (30).

STEP 2. For obtaining the necessary conditions for (21) to hold, we derive the following results in the next two lemmas proved in Appendix.

Lemma 2. *The function $\varphi_\rho(Q)$ has the following derivative with respect to Q:*

$$\varphi'_\rho(Q) = \frac{1}{D}\mathbf{E}\left[e^{-\rho\tau(Q)} I_{[y(\tau(Q))=Q]}\right]. \tag{38}$$

where the indicator function $I_A(\omega) = 1$ if $\omega \in A$, and $I_A(\omega) = 0$ otherwise.

Lemma 3. *The function $m_\rho(S, Q)$ has the following partial derivative with respect to Q:*

$$\frac{\partial}{\partial Q} m_\rho(S,Q) = \varphi'_\rho(Q)\left[f(S-Q) - \rho K + c(D + \bar{\xi}\lambda + \rho(S-Q))\right], \tag{39}$$

where $\bar{\xi} = \int_0^\infty z dG(z)$, the expected value of the Poisson demand.

From (33) and (39), it follows that

$$\frac{\partial}{\partial Q} d_\rho(S,Q) = \frac{\varphi'_\rho(Q)}{\varphi^2_\rho(Q)} \varepsilon_\rho(S,Q), \tag{40}$$

where

$$\begin{aligned} \varepsilon_\rho(S,Q) &= \frac{\varphi_\rho(Q)}{\varphi'_\rho(Q)} \frac{\partial}{\partial Q} m_\rho(S,Q) - m_\rho(S,Q) \\ &= \varphi_\rho(Q)[f(S-Q) - \rho K + c(D + \bar{\xi}\lambda + \rho(S-Q))] - m_\rho(S,Q). \end{aligned} \tag{41}$$

Denote the partial derivative of $m_\rho(S,Q)$ with respect to S as

$$\tilde{m}_\rho(S,Q) := \frac{\partial}{\partial S} m_\rho(S,Q) = \mathbf{E}\left[\int_0^{\tau(Q)} (f'(S - y(t)) + c\rho) e^{-\rho t} dt\right]. \tag{42}$$

The necessary conditions for s_ρ and S_ρ, $-\infty < s_\rho < S_\rho < +\infty$, to satisfy (21) are

$$\varepsilon_\rho(S_\rho, S_\rho - s_\rho) = 0, \quad \tilde{m}_\rho(S_\rho, S_\rho - s_\rho) = 0. \tag{43}$$

According to (33), these conditions can be rewritten in the form

$$d_\rho = f(s_\rho) - \rho K + c(D + \bar{\xi}\lambda + \rho s_\rho), \tag{44}$$

$$\mathbf{E}\left[\int_0^{\tau(S_\rho - s_\rho)} (f'(S_\rho - y(t)) + c\rho)e^{-\rho t}dt\right] = 0 \tag{45}$$

Remark 3.1 Requirements of Theorem 1 on the function $f(x) + c\rho x$ and (45) imply that if s_ρ and S_ρ exist, then $s_\rho < \sigma_\rho < S_\rho$.

Remark 3.2 It follows from (44) and the conditions on $f(x) + c\rho x$ that if s_ρ exists, then it is unique. It proves the statement of Theorem 1 regarding the uniqueness of s_ρ.

Remark 3.3 In Section 4 we show that there exists an $\bar{S}_\rho \leq +\infty$ such that for any given S, $\sigma_\rho < S < \bar{S}_\rho$, there exists a unique $Q(S)$ such that $\tilde{m}_\rho(S, Q(S)) = 0$. However, it is possible that for a given s there may exist more than one value of S for which $\tilde{m}_\rho(S, S - s) = 0$. So, we do not know if S_ρ is unique.

Remark 3.4 It is straightforward to check that the two conditions in (43) are equivalent to the following two conditions: (a) the function $P_\rho(x) - (S_\rho - x)c$ has a minimum at $x = S_\rho$ and (b) the function $P_\rho(x) := P_\rho^{s_\rho, S_\rho}(x)$ is smooth (continuously differentiable) at the point $x = s_\rho$. The condition (b) is commonly referred to as a *smooth pasting* condition in the literature.

STEP 3. Now we will prove the following lemma.

Lemma 4. *The potential function $P_\rho(x)$ satisfies the following relations:*

$$\rho P_\rho(x) = f(x) - a_\rho + \lambda \mathbf{E}\left[P_\rho(x - \xi) - P_\rho(x)\right] - DP'_\rho(x) \text{ for } x > s_\rho. \tag{46}$$

$$\rho P_\rho(x) \leq f(x) - a_\rho + \lambda \mathbf{E}\left[P_\rho(x - \xi) - P_\rho(x)\right] - DP'_\rho(x) \text{ for } -\infty < x < \infty, \tag{47}$$

$$P_\rho(x) = K + c(S_\rho - x) + P_\rho(x + (S_\rho - x)) \text{ for } x \leq s_\rho, \tag{48}$$

$$P_\rho(x) \leq K + cu + P_\rho(x + u) \text{ for all } -\infty < x < \infty \text{ and } u > 0, \tag{49}$$

Remark 3.5 Inequalities (47) and (49) are called quasi-variational inequalities (QVI) in the literature devoted to the control of continuous processes with jumps. They can be rewritten for the value function $F_\rho(x)$, $\rho > 0$, as the following dynamic programming equation:

$$\begin{aligned} 0 = & \min \begin{cases} \min_{u>0}[K + cu + F_\rho(x+u) - F_\rho(x)] & \text{(order)}, \\ f(x) + \lambda\mathbf{E}[F_\rho(x-\xi) - F_\rho(x)] - DF'_\rho(x) - \rho F_\rho(x) & \text{(no order)}, \end{cases} \\ = & \min_{u\geq 0}[K(1-\delta(u)) + cu + F_\rho(x+u) - F_\rho(x)] \\ & +\delta(u)\Big(f(x) + \lambda\mathbf{E}[F_\rho(x-\xi) - F_\rho(x)] - DF'_\rho(x) - \rho F_\rho(x)\Big), \end{aligned} \tag{50}$$

where $\delta(u) = 1$ if $u = 0$, and $\delta(u) = 0$ if $u \neq 0$. When $\rho = 0$, the dynamic programming equation is replaced by an ergodic equation in terms of the average cost a_0 and the potential function $W_0(x)$. This equation is

$$\begin{aligned} 0 = \; & \min_{u\geq 0}[K(1-\delta(u)) + cu + W_0(x+u) - W_0(x)] \\ & +\delta(u)\Big(f(x) - a_0 + \lambda\mathbf{E}[W_0(x-\xi) - W_0(x)] - DW'_0(x)\Big). \end{aligned} \tag{51}$$

Proof of Lemma 4. For $\rho > 0$, (46) follows from the definition (6) of $F_\rho(x, U)$, the equality $P_\rho(x) = F_\rho(x, U^{s_\rho,S_\rho}) - a_\rho/\rho$, and the fact that $x^{s_\rho,S_\rho}(t)$ corresponding to U^{s_ρ,S_ρ} is a Markov process. The case $\rho = 0$ follows by continuity in ρ.

For proving (47), consider

$$A_\rho(x) = \rho P_\rho(x) + DP'_\rho(x) - \lambda\mathbf{E}\left[P_\rho(x-\xi) - P_\rho(x)\right]. \tag{52}$$

Relation (47) is equivalent to

$$A_\rho(x) \leq f(x) - a_\rho \quad \text{for } -\infty < x < +\infty. \tag{53}$$

For $x > s_\rho$, according to (46) we have the equality in (53). For $x < s_\rho$, according to (31) and (34), $P_\rho(x) = K + c(S_\rho - x)$ and $\mathbf{E}\left[P_\rho(x-\xi)\right] = K + c(S_\rho - x) + c\bar{\xi}$. Using these relations, the equality $\rho K - Dc - \lambda c\bar{\xi} = f(s_\rho) + c\rho s_\rho - d_\rho$ (see (44)), and the relation

(18) between d_ρ and a_ρ, we get for $x < s_\rho$,

$$\begin{aligned} A_\rho(x) &= \rho K + c\rho(S_\rho - x) - Dc - \lambda c\bar{\xi} = f(s_\rho) + c\rho s_\rho - d_\rho + c\rho S_\rho - c\rho x \\ &= f(x) - a_\rho + f(s_\rho) + c\rho s_\rho - (f(x) + c\rho x). \end{aligned} \tag{54}$$

It follows from Remark 3.1 that $s_\rho < \sigma_\rho$. According to the assumptions of Theorem 1, the function $f(u)+c\rho u$ decreases on (x, s_ρ). Consequently, $f(s_\rho)+c\rho s_\rho-(f(x)+c\rho x) < 0$ for $x < s_\rho$. This completes the proof of (53) and, in turn, of (47).

From (34) for $x = S_\rho$, it follows that $P_\rho(S_\rho) = 0$. So (48) follows from (34).

To complete the proof of Lemma 4, it remains to prove (49) for $x > s_\rho$. To this end, denote $f^\rho(x) := f(x + s_\rho)$,

$$m^\rho(x) := m_\rho(x+s_\rho, x) = \mathbf{E}\left[\int_0^{\tau(x)} (f^\rho(x - y(t)) + c\rho(x + s_\rho))e^{-\rho t}dt + (K + cy(\tau(x)))e^{-\rho\tau(x)}\right], \tag{55}$$

and

$$\hat{P}_\rho(x) := \begin{cases} K \quad \text{if } x \leq 0, \\ m^\rho(x) - d_\rho\varphi_\rho(x) \text{ if } x > 0. \end{cases} \tag{56}$$

It follows from the definition (34) of $P_\rho(x)$ and (56) that $\hat{P}_\rho(x) = P_\rho(x+s_\rho) - c(S_\rho - s_\rho - x)$. So for completing the proof of Lemma 4, it suffices to prove that

$$\hat{P}_\rho(x) \leq K + \hat{P}_\rho(x + u) \text{ for all } x > 0 \text{ and } u > 0, \tag{57}$$

i.e., $\hat{P}_\rho(x)$ is K-nondecreasing.

From the definition (56) of $P_\rho(x)$, the definition (55) of $m_\rho(x)$, and the definition of s_ρ and S_ρ, it follows that

$$\hat{P}_\rho(x) \geq 0 \quad \text{for} \quad -\infty < x < +\infty, \quad \hat{P}_\rho(S_\rho - s_\rho) = 0. \tag{58}$$

It follows from the definition (56) of $\hat{P}_\rho(x)$, the definition (55) of $m_\rho(x)$, expressions (39) and (42) for partial derivatives of $m_\rho(S, Q)$, and the expression for d_ρ from the necessary condition (44) that

$$\begin{aligned}\frac{d}{dx}\hat{P}_\rho(x) = & \frac{\partial}{\partial S}m_\rho(S,Q)\Big|_{\substack{S=x+s_\rho,\\ Q=x}} + \frac{\partial}{\partial Q}m_\rho(S,Q)\Big|_{\substack{S=x+s_\rho,\\ Q=x}} - d_\rho\frac{d}{dQ}\varphi_\rho(Q)\Big|_{Q=x} \\ = & \frac{\partial}{\partial S}m_\rho(S,Q)\Big|_{\substack{S=x+s_\rho,\\ Q=x}} = \mathbf{E}\left[\int_0^{\tau(x)}\Big(f'(s_\rho+x-y(t))+c\rho\Big)e^{-\rho t}dt\right].\end{aligned} \tag{59}$$

Since $\hat{P}_\rho(0)=K$ (see (56)), $\hat{P}_\rho(\sigma_\rho-s_\rho)=0$ (see (58)), and $f(x)+c\rho x$ decreases for $x<\sigma_\rho$, we have from (59) that $\hat{P}_\rho(x)$ decreases on $(0,\sigma_\rho-s_\rho)$ from K to 0. Consequently,

$$\hat{P}_\rho(x)\le K \quad \text{for} \quad 0<x<\sigma_\rho-s_\rho. \tag{60}$$

Using (60) and the fact that $\hat{P}_\rho(x+u)\ge 0$, we get

$$\hat{P}_\rho(x+u)-\hat{P}_\rho(x)\ge 0-K=-K \quad \text{for} \quad 0<x<\sigma_\rho-s_\rho,$$

which proves (57) for $0<x<\sigma_\rho-s_\rho$. To consider the case $x>\sigma_\rho-s_\rho$, we need the following preliminary lemma which is also of independent interest.

Lemma 5. *For any* $0<v\le x$ *and* $u>0$,

$$\begin{aligned}\hat{P}_\rho(x+u)-\hat{P}_\rho(x) &= \mathbf{E}\left[\int_0^{\tau(v)}\Big(f^\rho(x+u-y(t))-f^\rho(x-y(t))+c\rho u\Big)e^{-\rho t}dt\right] \\ &+\mathbf{E}\left[e^{-\rho\tau(v)}\Big(\hat{P}_\rho(x+u-y(\tau(v)))-\hat{P}_\rho(x-y(\tau(v)))\Big)\right].\end{aligned} \tag{61}$$

Proof. Let $x>0$. If $y(\tau(x))\ge x+u$, then $\tau(x+u)=\tau(x)$ and $y(\tau(x+u))=y(\tau(x))$. If $y(\tau(x))<x+u$, then for a fixed $y(\tau(x))$, the difference $\tau(x+u)-\tau(x)$ has the same distribution as that of $\tau(x+u-y(\tau(x)))$. It follows from here that

$$\begin{aligned}\varphi_\rho(x+u)-\varphi_\rho(x) &= \mathbf{E}\left[e^{-\rho\tau(x)}\int_0^{\tau(x+u)-\tau(x)}e^{-\rho t}dt\right] \\ &= \mathbf{E}\left[e^{-\rho\tau(x)}\varphi_\rho(x+u-y((\tau(x)))I_{[y(\tau(x))<x+u]}\right],\end{aligned} \tag{62}$$

$$m^\rho(x+u)-m^\rho(x)=\mathbf{E}\left[\int_0^{\tau(x)}\Delta_u^\rho(x-y(t))e^{-\rho t}dt\right]+r(x,u), \tag{63}$$

where

$$\Delta_u^\rho(x)=f^\rho(x+u)-f^\rho(x)+c\rho u, \tag{64}$$

and

$$\begin{aligned}
r(x,u) &:= \mathbf{E}\left[\int_{\tau(x)}^{\tau(x+u)}\Big(f^\rho(x+u-y(t))+c\rho(x+u+s_\rho)\Big)e^{-\rho t}dt\right] \\
&\quad +\mathbf{E}\left[\Big((K+cy(\tau(x+u))e^{-\rho\tau(x+u)}-(K+cy(\tau(x)))e^{-\rho\tau(x)}\Big)\right] \\
&= \mathbf{E}\left[\int_{\tau(x)}^{\tau(x+u)}\Big(f^\rho(x+u-y(t))+c\rho(x+u+s_\rho-y(\tau(x)))\Big)e^{-\rho t}dt\right] \\
&\quad +\mathbf{E}\left[\Big(cy(\tau(x))\left(e^{-\rho\tau(x)}-e^{-\rho\tau(x+u)}\right)-(K+cy(\tau(x)))e^{-\rho\tau(x)}\Big)\right] \\
&\quad +\mathbf{E}\left[\Big(K+c(y(\tau(x+u))-y(\tau(x)))e^{-\rho\tau(x+u)}+cy(\tau(x))e^{-\rho\tau(x+u)}\Big)\right] \\
&= \mathbf{E}\Bigg[e^{-\rho\tau(x))}\int_0^{\tau(x+u)-\tau(x)}\Big(f^\rho\Big(x+u-y(\tau(x))+(y(t+\tau(x)))-y(\tau(x))\Big) \\
&\qquad\qquad +c\rho(x+u+s_\rho-y(\tau(x))\Big)e^{-\rho(t)}dt\Bigg] \\
&\quad +\mathbf{E}\left[e^{-\rho\tau(x)}\Big(K+c\Big(y(\tau(x+u))-y(\tau(x))\Big)e^{-\rho(\tau(x+u)-\tau(x))}-K\Big)\right].
\end{aligned} \tag{65}$$

If $y(\tau(x))\geq x+u$, then the right-hand side of (65) is equal to zero. If $y(\tau(x))<x+u$, then for fixed $\tau(x)$ and $y(\tau(x))$, the difference $y(\tau(x+u))-y(\tau(x))$ has the same distribution as that of $y(\tau(x+u-y(\tau(x)))$, the difference $\tau(x+u)-\tau(x)$ has the same distribution as that of $\tau(x+u-y(\tau(x))$, and the process $y(t+\tau(x)))-y(\tau(x))$ has the same distribution as that of $y(t)$. It follows from these and (65) that

$$r(x,u)=\mathbf{E}\left[e^{-\rho\tau(x)}\Big(m^\rho(x+u-y(\tau(x)))-K\Big)I_{[y(\tau(x))<x+u]}\right]. \tag{66}$$

It follows from the definition (56) of $\hat{P}_\rho(x)$, (62), (63), and (66) that

$$\begin{aligned}
&\hat{P}_\rho(x+u)-\hat{P}_\rho(x)=m^\rho(x+u)-m^\rho(x)-d_\rho(\varphi_\rho(x+u)-\varphi_\rho(x)) \\
&=\mathbf{E}\left[\int_0^{\tau(x)}\Delta_u^\rho(x-y(t))e^{-\rho t}dt\right]+\mathbf{E}\Bigg[e^{-\rho\tau(x)}\Big(\Big(m_\rho(x+u-y(\tau(x))) \\
&-d_\rho\varphi_\rho(x+u-y(\tau(x)))\Big)-K\Big)I_{[y(\tau(x))<x+u]}\Bigg]=\mathbf{E}\Bigg[\int_0^{\tau(x)}\Delta_u^\rho(x-y(t))e^{-\rho t}dt\Bigg] \\
&+\mathbf{E}\left[e^{-\rho\tau(x)}\Big(\hat{P}_\rho(x+u-y(\tau(x)))-\hat{P}_\rho(x-y(\tau(x)))\Big)\right].
\end{aligned} \tag{67}$$

The last equality holds since $\hat{P}_\rho(x - y(\tau(x))) = K$, and if $y(\tau(x)) > x + u$, then $\hat{P}_\rho(x + u - y(\tau(x))) = K$. This proves Lemma 5 for $v = x$.

For $0 < v < x$, we can rewrite the first expectation in the right-hand side of (67) in the form

$$\mathbf{E}\left[\int_0^{\tau(x)} \Delta_u^\rho(x - y(t))e^{-\rho t}dt\right] = \mathbf{E}\left[\int_0^{\tau(v)} \Delta_u^\rho(x - y(t))e^{-\rho t}dt + e^{-\rho\tau(v)}J\right], \tag{68}$$

where

$$J = \mathbf{E}\left[\int_0^{\tau(x)-\tau(v)} \Delta_u^\rho\Big(x - y(\tau(v) - (y(t+\tau(v)) - y(\tau(v)))\Big)e^{-\rho t}dt\middle|\mathcal{F}_{\tau(v)}\right]. \tag{69}$$

Using (67) with $x - y(\tau(v))$ instead of x, and the same arguments as those for obtaining (66), we get from (69),

$$\begin{aligned} J &= \hat{P}_\rho(x + u - y(\tau(v))) - \hat{P}_\rho(x - y(\tau(v))) \\ &-\mathbf{E}\left[e^{-\rho(\tau(x)-\tau(v))}\Big(\hat{P}_\rho(x + u - y(\tau(x))) - \hat{P}_\rho(x - y(\tau(x)))\Big)\Big|\mathcal{F}_{\tau(v)}\right]. \end{aligned} \tag{70}$$

Substitution of (68) and (70) into (67) completes the proof of Lemma 5.

For $x > \sigma_\rho - s_\rho$, consider the equality (61) from Lemma 5 with $v = x - \sigma_\rho + s_\rho$. Then the integrand in the first term of the right-hand side of (61) is positive since $f(x) + c\rho x$ increases for $x > \sigma_\rho$, $\hat{P}_\rho(x + u - y(\tau(x - \sigma_\rho + s_\rho))) \geq 0$ due to (58), and $\hat{P}_\rho(x - y(\tau(x - \sigma_\rho + s_\rho))) \leq K$ due to (60) and the inequality $x - y(\tau(x - \sigma_\rho + s_\rho)) \leq \sigma_\rho - s_\rho$. This completes the proof of (57), and consequently of Lemma 4.

Remark 3.6 In the deterministic case when $N(t) \equiv 0$, the problem reduces to the EOQ problem considered by Beyer and Sethi (1998) under the same assumptions on the surplus cost function $f(x)$ as here. If $N(t) \equiv 0$, $\rho = 0$, $f(x) = hx$ for $x > 0$, and $f(x) = p|x|$ for $x < 0$, then

$$m(S,Q) = \frac{hS^2}{2D} + \frac{p(S-Q)^2}{2D} + K, \quad \varphi(Q) = \frac{Q}{D}, \quad a_0(S,Q) = \frac{hS^2 + p(S-Q)^2 + 2DK}{2Q}. \tag{71}$$

Minimization of $a_0(S,Q)$ gives the well-known results

$$S_0 = \sqrt{\frac{2pDK}{h(h+p)}}, \quad s_0 = -\sqrt{\frac{2hDK}{p(h+p)}}, \quad a_0 = \sqrt{\frac{2phDK}{(h+p)}}. \tag{72}$$

Furthermore, when no backlogging is allowed, i.e., when $p \to \infty$, we get the classic EOQ square-root formula

$$S_0 = \sqrt{2DK/h} \tag{73}$$

along with $s_0 = 0$ and $a_0 = \sqrt{2hDK}$.

Remark 3.7 In the case when no backlogging is allowed, Sivazlian (1974) shows that the best (s,S)-policy in $\bar{\mathcal{U}}$ for a continuous-review model with the Poisson demand with intensity λ gives $s_0 = 0$ and S_0 satisfies

$$S_0(S_0 - 1) \le 2K/(\lambda h) \le S_0(S_0 + 1), \tag{74}$$

where we note that $1/\lambda$ is the expected demand per unit time associated with the Poisson process. Note that (74) is the discrete-time version of the EOQ formula when the demand equals $1/\lambda$ in each period.

Remark 3.8 In the case when $D = 0$, the function $\varphi(Q)$ still has a derivative which is equal to the limit of (38). All other formulas remain the same.

STEP 4. For each sample path of the process $x^U(t)$ obtained from (5) using an admissible policy U, Dynkin's formula leads to

$$\begin{aligned} e^{-\rho T} P_\rho(x^U(T)) - P_\rho(x^U(0)) &= \int_0^T \left(-\rho P_\rho(x^U(t)) - D\frac{d}{dx} P_\rho(x^U(t)) \right) e^{-\rho t} dt \\ &+ \int_0^T (P_\rho(x^U(t)) - P_\rho(x^U(t-0)) e^{-\rho t} dN_1(t) \\ &+ \sum_{\{i:\theta_i < T\}} (P_\rho(x^U(\theta_i) + u_i) - P_\rho(x^U(\theta_i))) e^{-\rho\theta_i}. \end{aligned} \tag{75}$$

From the definition of the Poisson process and the independence of ξ_i from this process, it follows that

$$\mathbf{E}\left[\int_0^T (P_\rho(x^U(t)) \quad -P_\rho(x^U(t-0))e^{-\rho t}dN_1(t) \right. \\ \left. - \int_0^T (P_\rho(x^U(t-0)-\xi) - P_\rho(x^U(t-0)))e^{-\rho t}\lambda dt\right] = 0. \tag{76}$$

Taking the expectation in (75), using (76), and substituting (52) there, we can rewrite it as

$$\begin{aligned} P_\rho(x^U(0)) \quad &= \mathbf{E}\left[e^{-\rho T}P_\rho(x^U(T)) + \int_0^T A_\rho(x^U(t))e^{-\rho t}dt + \sum_{\{i:\theta_i<T\}}(K+cu_i)e^{-\rho\theta_i}\right] \\ &-\mathbf{E}\left[\sum_{\{i:\theta_i<T\}}(P_\rho(x^U(\theta_i)+u_i) - P_\rho(x^U(\theta_i)+(K+cu_i))e^{-\rho\theta_i}\right]. \end{aligned} \tag{77}$$

It follows from (77), (49), and (53) that

$$P_\rho(x) \le \mathbf{E}\left[\int_0^T (f(x^U(t)) - a_\rho)e^{-\rho t}dt + \sum_{\{i:\theta_i<T\}} c(u_i)e^{-\rho\theta_i} + P_\rho(x^U(t))e^{-\rho T}\right]. \tag{78}$$

Since for $U_\rho^{s_\rho,S}$, the inequality (78) becomes an equality, the proof that $P_\rho(x)$ satisfies (25) is completed.

4 Proof of Theorem 1

For proving Theorem 1, we need the following lemma which states that we can restrict the class of policies to only those which do not result in very large surplus levels at any time during the infinite horizon.

Lemma 6. *There exists $\tilde{x}$ such that for any initial surplus level x and any strategy $U \in \mathcal{U}$, there exists a strategy $\tilde{U} \in \mathcal{U}$ such that $x^{\tilde{U}}(t) < \tilde{x}$ for $t > \theta_1$ and $F_\rho(x,\tilde{U}) \le F_\rho(x,U)$.*

Proof. See Appendix.

Let us now show that Theorem 1, in the case when s_ρ and S_ρ exist, follows from (25), Lemma 5, and the fact that the constructed $W_\rho(x)$ is linear in x for $x < s_\rho$. Indeed, these properties allows us to take the limit as $T \to \infty$ in (25). More specifically for $\rho > 0$, the last term in (25) vanishes as $T \to \infty$, and using (26) we obtain the result.

Likewise for $\rho = 0$, the limit of (25) divided by T as $T \to \infty$ gives

$$F_0(x) = a_0. \tag{79}$$

It remains to prove the existence of s_ρ and S_ρ. Let S_n and Q_n be such that

$$\lim_{n\to\infty} d_\rho(S_n, Q_n) = \inf_{S,Q} d_\rho(S, Q).$$

Since $f(x) + c\rho x$ increases for $x > \sigma_\rho$ (decreases for $x < \sigma_\rho$), it follows from (32) and (33) that if $Q < S - \sigma_\rho$ ($S < \sigma_\rho$ correspondingly), then $d_\rho(S, Q) > d_\rho(Q - \sigma_\rho, Q)$ ($d_\rho(S, Q) > d_\rho(\sigma_\rho, Q)$ correspondingly). That is, we can assume without loss of generality that $Q_n \geq S_n - \sigma_\rho$ and $S_n \geq \sigma_\rho$.

Since $f(x) + c\rho x$ increases to ∞ as $x \to \infty$, it follows from (32), (33) and (10) that if $Q > S - \sigma_\rho$ then $d_\rho(S, Q) \to \infty$ as $S \to \infty$. That is, we can assume without loss of generality that S_n is bounded.

Since $f(x) + c\rho x$ decreases for $x < \sigma_\rho$ from ∞ it follows from (40) and (41) that $\frac{\partial}{\partial Q} d_\rho(S, Q) > 0$ if $Q - S$ is large enough. That is, using boundedness of S_n, we can assume without loss of generality that Q_n is bounded. The existence of s_ρ and S_ρ follows now from the continuity of $d_\rho(S, Q)$.

5 Conclusions

We show the optimality of an (s, S) policy for a continuous-review stochastic inventory model with demand consisting of a compound Poisson process and a constant demand rate. While the incorporation of a constant demand rate into the existing models involving only a compound Poisson demand appear to be innocuous, their optimality proofs do not extend easily to deal with the generalized problem. We, therefore, develop a new approach of showing the optimality of an (s, S) policy for the generalized problem. Importantly, our approach is unified in the sense that it addresses both the discounted cost and average cost criteria. This is done by introduction of new average discounted-cost formulas having appealing intuitive interpretations. Moreover, the approach does not require the assumptions of convexity or quasi-convexity of the surplus cost used in the literature. In addition, we show that the optimal ordering level s is unique. On the other hand, the optimal order-up-to level S is not necessarily unique.

An important byproduct of our generalization is that its optimal (s, S) policy reduces to the well-known EOQ formula as a special case, when the compound Poisson demand is turned off by setting its intensity to zero. Our proof therefore represents an alternative rigorous proof of the EOQ formula, not easily found in the literature.

Appendix

Proof of Lemma 2. Since the process $N(t)$ is right-continuous, we have $y(\tau(Q)) = y(\tau(Q)+0)$. If process $y(t)$ reaches the level Q continuously, then the probability that it has a jump at the same instant is zero, i.e., $\mathbf{P}(y(\tau(Q)-0) = Q,\ y(\tau(Q)+0) > Q) = 0$.

Note that

$$\text{if } y(\tau(Q)) \geq Q+\delta \text{ for some } \delta > 0, \text{ then } \tau(Q+\delta) = \tau(Q) \text{ and } y(\tau(Q+\delta)) = y(\tau(Q)). \tag{80}$$

According to (10) and (80) for $\delta > 0$, we have

$$\begin{aligned} \frac{\varphi_\rho(Q+\delta) - \varphi_\rho(Q)}{\delta} &= \mathbf{E}\left[\frac{e^{-\rho\tau(Q)} - e^{-\rho\tau(Q+\delta)}}{\delta\rho} I_{[Q<y(\tau(Q))<Q+\delta]}\right] \\ &+\mathbf{E}\left[\frac{e^{-\rho\tau(Q)} - e^{-\rho\tau(Q+\delta)}}{\delta\rho} I_{[y(\tau(Q))=Q]} I_{[N(\tau(Q+\delta))\neq N(\tau(Q))]}\right] \\ &+\mathbf{E}\left[\frac{e^{-\rho\tau(Q)} - e^{-\rho\tau(Q+\delta)}}{\delta\rho} I_{[y(\tau(Q))=Q]} I_{[N(\tau(Q+\delta))=N(\tau(Q))]}\right]. \end{aligned} \tag{81}$$

Note that $\tau(Q+\delta)-\tau(Q)\leq \delta/D$. Therefore, $0\leq \dfrac{e^{-\rho\tau(Q)}-e^{-\rho\tau(Q+\delta)}}{\delta}\leq 1/D$. Since $\mathbf{P}\left[N(\tau(Q+\delta))\neq N(\tau(Q)\right]\to 0$ and $\mathbf{P}\left[Q<y(\tau(Q))<Q+\delta)\right]\to 0$ as $\delta\to 0$, the first and the second terms on the right-hand side of (81) tend to zero as $\delta\to 0$. From the fact that $\mathbf{P}\left[N(\tau(Q+\delta))=N(\tau(Q)\right]\to 1$ as $\delta\to 0$ and $\tau(Q+\delta))=\tau(Q)+\delta/D$ if $N(\tau(Q+\delta))=N(\tau(Q))$, the statement of Lemma 2 follows for the right derivative. The proof for the left derivative is analogous.

Proof of Lemma 3. Denote $\tilde{f}(v)=f(S-v)+c\rho S$. According to (11) and (80),

$$\frac{m_\rho(S,Q+\delta)-m_\rho(S,Q)}{\delta}$$
$$=\frac{1}{\delta}\mathbf{E}\left[\left(\int_{\tau(Q)}^{\tau(Q+\delta)}\tilde{f}(y(t))e^{-\rho t}dt+c(y(\tau(Q+\delta)))e^{-\rho\tau(Q+\delta)}-c(y(\tau(Q)))e^{-\rho\tau(Q)}\right)I_{[A_1]}\right]$$
$$+\mathbf{E}\left[\left(\frac{1}{\delta}\int_{\tau(Q)}^{\tau(Q+\delta)}\tilde{f}(y(t))e^{-\rho t}dt+\frac{c(y(\tau(Q+\delta)))e^{-\rho\tau(Q+\delta)}-c(y(\tau(Q)))e^{-\rho\tau(Q)}}{\delta}\right)I_{[A_2]}\right]$$
$$+\mathbf{E}\left[\left(\int_{\tau(Q)}^{\tau(Q+\delta)}\tilde{f}(y(t))e^{-\rho t}dt+c(y(\tau(Q+\delta)))e^{-\rho\tau(Q+\delta)}-c(y(\tau(Q)))e^{-\rho\tau(Q)}\right)I_{[A_3]}\right], \tag{82}$$

where $A_1=\{\omega: Q<y(\tau(Q))<Q+\delta\}$, $A_2=\{\omega: y(\tau(Q))=Q, N(\tau(Q+\delta))=N(\tau(Q)\}$, and $A_3=\{\omega: y(\tau(Q))=Q, N(\tau(Q+\delta))\neq N(\tau(Q))\}$.

The first term on the right-hand side of (82) tends to zero as $\delta\to 0$. If $y(\tau(Q))=Q$ and $N(\tau(Q+\delta))=N(\tau(Q))$, then $\tau(Q+\delta)=\tau(Q)+\delta/D$ and $y(\tau(Q+\delta))=y(\tau(Q)+\delta)=Q+\delta$. Therefore, the second term tends to $\varphi'_\rho(Q)\left(\tilde{f}(Q)-\rho(K+cQ)+Dc\right)$. On account of $\mathbf{P}\left[N(\tau(Q+\delta))=N(\tau(Q)+1\right]/\delta\to\lambda$ and $y(\tau(Q+\delta))=y(\tau(Q))+\xi=Q+\xi$, if $N(\tau(Q+\delta))=N(\tau(Q))+1$ and $y(\tau(Q))=Q$, the third term tends to $\varphi'_\rho(Q)\lambda c\bar{\xi}$. This proves the lemma.

Proof of Lemma 6. Let us show at first that if $U=(\theta_1,u_1,\theta_2,u_2,\ldots)$ is fixed and $x^U(\theta_n)>0$ with positive probability for some $n>0$, then there exists a strategy $\tilde{U}\in\mathcal{U}$ for which $F_\rho(x,\tilde{U})\leq F_\rho(x,U)$. Indeed, consider the following strategy $\tilde{U}$. For $1\leq i<n$, it coincides with U. If $x^U(\theta_n)\leq 0$, it coincides with U also for $i\geq n$.

If $x^U(\theta_n)>0$, we do not order at θ_n, wait for one of the two following events, and behave accordingly as follows. If the new process $x^{\tilde{U}}(t)$ crosses the level 0 before θ_{n+1}, we order $\tilde{u}_n = u_n$ at the time $\tilde{\theta}_n$ of crossing the level 0, and after that we continue to use U. If the new process $x^{\tilde{U}}(t)$ does not cross the level 0 before θ_{n+1}, we order $\tilde{u}_n = u_n + u_{n+1}$ at $\tilde{\theta}_n = \theta_{n+1}$, and after that we continue to use U.

It is evident that $x^{\tilde{U}}(t) = x^U(t)$ for $t \notin [\theta_n, \tilde{\theta}_n]$, and $x^{\tilde{U}}(t) = x^U(t) - u_n$ for $t \in [\theta_n, \tilde{\theta}_n]$. Since $f(x)$ increases for $x > 0$ and $x^{\tilde{U}}(t) > 0$ for $t \in [\theta_n, \tilde{\theta}_n]$, we have $F_\rho(x, \tilde{U}) \leq F_\rho(x, U)$. So we can consider only such strategies that $x^U(\theta_n) \leq 0$ with probability one.

Since $f(x)$ increases to ∞ as x increases to ∞, for any given A there exists $\tilde{x}$ such that

$$\mathbf{E}\left[\int_0^{\tau(A)} f(x - y(t))e^{-\rho t}dt\right] > K + \mathbf{E}\left[\int_0^{\tau(A)} f(A - y(t))e^{-\rho t}dt\right] \text{ for } x > \tilde{x}. \tag{83}$$

Let $U = (\theta_1, u_1, \theta_2, u_2, \ldots)$ be fixed, $x^U(\theta_i) \leq 0$ for all $i > 0$, and $x^U(\theta_n) + u_n > \tilde{x}$ with positive probability for some $n > 0$. Consider the following strategy $\tilde{U}$. For $1 \leq i < n$, it coincides with U. If $x^U(\theta_n) + u_n \leq \tilde{x}$, it coincides with U also for $i \geq n$. In the opposite case, instead of ordering u_n at θ_n, we order only up to the level A, and then wait for one of the two following events and behave accordingly as specified below. If the new process $x^{\tilde{U}}(t)$ crosses level 0 before the process $x^U(t)$ crosses the level $\tilde{x}$, we order at the time of crossing again up to the level A, and wait for a new moment of crossing. If the new process $x^{\tilde{U}}(t)$ does not cross the level 0 before the process $x^U(t)$ crosses at some moment τ the level $\tilde{x}$, we order at this moment τ up to the level $x^U(\tau + 0)$ and continue to use U thereafter. It is evident that $x^{\tilde{U}}(t) = x^U(t)$ for $t \notin [\theta_n, \tau]$, and on account of (83) we have $F_\rho(x, \tilde{U}) \leq F_\rho(x, U)$. This completes the proof of Lemma 6.

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