Are Base-stock Policies Optimal in Inventory Problems with Multiple Delivery Modes?

Qi Feng

School of Management, the University of Texas at Dallas
Richardson, TX 75083-0688, USA

Guillermo Gallego

Department of Industrial Engineering and Operations Research
The Columbia University in the City of New York, USA

Suresh P. Sethi

School of Management, the University of Texas at Dallas
Richardson, TX 75083-0688, USA

Houmin Yan

Department of Systems Engineering and Engineering Management,
The Chinese University of Hong Kong, Shatin, N. T., Hong Kong

Hanqin Zhang

Academy of Mathematics and Systems Sciences, Academia Sinica
Beijing, 100080, China

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We present a periodic review inventory model with multiple delivery modes. While base-stock policies are optimal for one or two consecutive delivery modes, they are not so otherwise. For multiple consecutive delivery modes, we show that only the fastest two modes have optimal base stocks, and provide a simple counterexample to show that the remaining ones do not. We investigate why the base-stock policy is or is not optimal in different situations. This paper is an abridged version of Feng et al. (2004).

Subject Classifications: Inventory/production: uncertainty, multiple delivery modes, base-stock policies.

1. Introduction

In this note, we construct a counterexample to establish that a base-stock policy is not optimal in general for inventory models with multiple delivery modes. In most of the studies on inventory models with replenishment leadtime options, it is assumed that there are two consecutive procurement modes available (e.g., Neuts 1964, Lawson et al. 2000, Sethi et al. 2001, Muharremoglu et. al. 2003). That is, the leadtimes of the two modes vary by exactly one period. This assumption leads to the optimality of a base-stock policy. Whittemore et al. (1997) formulate a problem for general two delivery modes and derive explicit formulas for the optimal ordering quantities when the modes are consecutive. They also comment that the optimal policy for two nonconsecutive modes may not have a simple structure. In the case of three consecutive delivery modes, Fukuda (1964) first considers a special case when orders are placed only every other period, in which case a base-stock policy is shown to be optimal. Zhang (1996) extends Fukuda's model by allowing three consecutive modes ordered every period, and claims the optimality of a base-stock policy under certain conditions. Her claim is untrue as evident from the counterexample presented in this note. Feng et al. (2005) show that there exist optimal base-stock levels for the two faster modes in an inventory model with three consecutive delivery modes and demand forecast updates.

We formulate a general problem in Section 2. We analyze the policy structure through an example in Section 3, and discuss the separability of the cost functions in Section 4. We conclude our study in Section 5.

2. Problem Formulation

We consider a finite horizon periodic review inventory system with N consecutive delivery modes. In the case of non-consecutive delivery modes, one can insert fictitious delivery modes as suggested in Sethi et al. (2001) to transform the problem into one with consecutive modes by setting the cost for any fictitious mode to equal that for the next faster real mode. Setting costs this way implies that we only consider policies that do not issue orders using fictitious modes.

An order Q_ℓ^i in period ℓ via the ith delivery mode, $i = 1, 2, ..., N$, is an order associated with a leadtime of i periods, and is termed the type i order. We denote q_ℓ^i as the realized

value of Q_ℓ^i. A *reference (pre-order) inventory position* for an order Q_ℓ^i is defined as the sum of on-hand inventory in period ℓ plus all previous and current orders placed before Q_ℓ^i and to be delivered by the time Q_ℓ^i is delivered, i.e., at the end of period $\ell + i - 1$. A *post-order inventory position* for an order Q_ℓ^i is defined as the sum of its reference inventory position and Q_ℓ^i. The notation is given below and the inventory positions are presented in Table 1.

Q_k^i = the amount of type i order that is placed at the beginning of period k and will arrive at the end of period $k + i - 1$, $1 \leqslant k \leqslant T$, $1 \leqslant i \leqslant N$;

c_k^i = the unit procurement cost of type i order in period k, $1 \leqslant k \leqslant T$, $1 \leqslant i \leqslant N$;

D_k = the demand in period k (materialized after delivery of orders), $1 \leqslant k \leqslant T$;

x_k = the inventory/backlog level at the beginning of period k, $1 \leqslant k \leqslant T$;

$p_k^i = \sum_{j=i+1}^{N} q_{k+i-j}^j$, the amount of in-transit orders at the beginning of period k that will arrive at the end of period $k + i - 1$, $1 \leqslant k \leqslant T$, $1 \leqslant i \leqslant N$;

y_k = $x_k + p_k^1$, the reference inventory position for Q_k^1 at the beginning of period k, $1 \leqslant k \leqslant T$;

z_k^i = the post-order inventory position for Q_k^i viewed at the beginning of period k, $1 \leqslant k \leqslant T$, $1 \leqslant i \leqslant N$;

$H_k(\cdot)$ = the inventory holding/backlog cost in period k assessed on the beginning inventory x_k, $1 \leqslant k \leqslant T$;

$H_{T+1}(\cdot)$ = the costs for the ending inventory/backlog.

We assume that that $E[D_k] < \infty$ for each k. Furthermore, the inventory cost functions $H_k(x)$ is nonnegative convex with $H_k(0) = 0$ for each k. The objective is to choose a sequence of orders $\{Q_k^1, ..., Q_k^N\}_{1 \leqslant k \leqslant T}$ so as to minimize the total expected cost given by
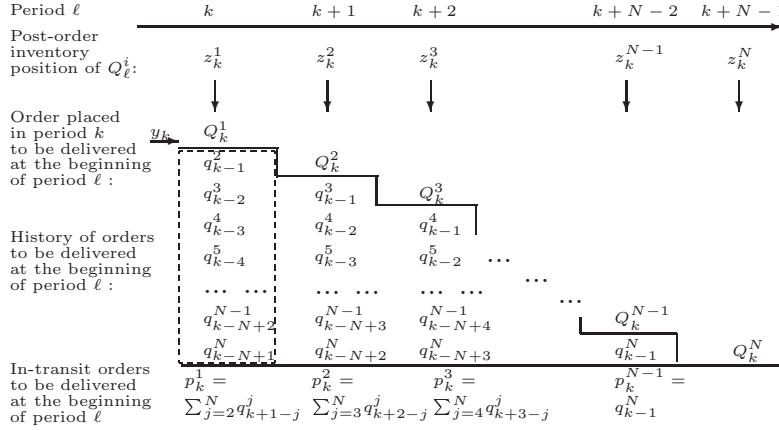
$$J_1\left(x_1, p_1^1, ..., p_1^{N-1}; \{Q_k^1, ..., Q_k^N\}_{1 \leqslant k \leqslant T}\right) = H_1(x_1) + E\left\{\sum_{k=1}^{T}\left[\sum_{j=1}^{N} c_k^j Q_k^j + H_{k+1}(X_{k+1})\right]\right\}, \quad (1)$$

subject to the inventory dynamics

$$X_{k+1} = x_k + p_k^1 + Q_k^1 - D_k, \quad 1 \leqslant k \leqslant T. \quad (2)$$

Let $W_k(y_k, p_k^2, .., p_k^{N-1})$ denote the cost-to-go function in period k. Then the function satisfies

Table 1: Inventory positions viewed at the beginning of period k.

Period ℓ	k	$k+1$	$k+2$	$k+N-2$	$k+N-1$
Post-order inventory position of Q_ℓ^i:	z_k^1	z_k^2	z_k^3	z_k^{N-1}	z_k^N
Order placed in period k to be delivered at the beginning of period ℓ :	y_k Q_k^1				
	q_{k-1}^2	Q_k^2			
	q_{k-2}^3	q_{k-1}^3	Q_k^3		
History of orders to be delivered at the beginning of period ℓ :	q_{k-3}^4	q_{k-2}^4	q_{k-1}^4		
	q_{k-4}^5	q_{k-3}^5	q_{k-2}^5 $\cdots$		
	$\cdots$ $\cdots$	$\cdots$ $\cdots$	$\cdots$ $\cdots$	$\cdots$	
	q_{k-N+2}^{N-1}	q_{k-N+3}^{N-1}	q_{k-N+4}^{N-1}	Q_k^{N-1}	
	q_{k-N+1}^N	q_{k-N+2}^N	q_{k-N+3}^N	q_{k-1}^N	Q_k^N
In-transit orders to be delivered at the beginning of period ℓ	$p_k^1 = \sum_{j=2}^N q_{k+1-j}^j$	$p_k^2 = \sum_{j=3}^N q_{k+2-j}^j$	$p_k^3 = \sum_{j=4}^N q_{k+3-j}^j$	$p_k^{N-1} = q_{k-1}^N$	

the dynamic programming equation

$$W_k(y_k, p_k^2, .., p_k^{N-1}) = \min_{\substack{z_k^1 \geqslant y_k, \\ z_k^j \geqslant z_k^{j-1} + p_k^j, \\ j=2,...,N}} \left\{ c_k^1(z_k^1 - y_k) + \sum_{j=2}^N c_k^j(z_k^j - z_k^{j-1} - p_k^j) + EH_{k+1}(z_k^1 - D_k) \right.$$

$$\left. + EW_{k+1}\left(z_k^2 - D_k, z_k^3 - z_k^2, ..., z_k^N - z_k^{N-1}\right) \right\}, \quad 1 \leqslant k \leqslant T-1. \quad (3)$$

One can show, along the lines of the proof of Theorem 4.3 in Sethi et al. (2001), that there exist functions $z_k^{i*}(y_k, p_k^2, ..., p_k^N)$, $1 \leqslant i \leqslant N$, which minimize the right-hand side in (3).

3.   An Example

In this section, we examine a three period example with three consecutive modes. There are three types of orders, namely, fast Q^1, medium Q^2, and slow Q^3, with stationary ordering costs $c^1 = 3$, $c^2 = 2$, and $c^3 = 1$, respectively. The demand D_1 for the first period follows uniform distribution over $[0, 20]$, and the demands for the last two periods are deterministic $D_2 = D_3 = 20$. Holding and backlog costs at the end of each period are given by

$$H_4(x) = \begin{cases} 0 & \text{if } x \geqslant 0, \\ -10x & \text{if } x < 0; \end{cases} \quad H_3(x) = \begin{cases} 3x & \text{if } x \geqslant 0, \\ -4x & \text{if } x < 0; \end{cases} \quad H_2(x) = \begin{cases} 2x & \text{if } x \geqslant 0, \\ -4x & \text{if } x < 0. \end{cases}$$

3.1   The Optimal Ordering Policies

The optimal cost function for the third period is given by

$$W_3(y_3) = \begin{cases} 0 & \text{if } y_3 \geqslant 20, \\ 60 - 3y_3 & \text{if } y_3 < 20. \end{cases}$$

The optimal base-stock level is $\bar{z}_3^1 = 20$.

The optimal cost function for period 2 is given by

$$W_2(y_2, q_1^3) = \begin{cases} 100 - 2q_1^3 - 3y_2 & \text{if } q_1^3 \leqslant 20, y_2 \leqslant 20, \\ 20 - 2q_1^3 + y_2 & \text{if } 20 < y_2 \leqslant 40 - q_1^3, \\ 60 - 3y_2 & \text{if } q_1^3 > 20, y_2 \leqslant 20, \\ -60 + 3y_2 & \text{if } y_2 > \max\{20, 40 - q_1^3\}. \end{cases}$$

Here, $p_2^2 = q_1^3$. The optimal base-stock levels are $(\bar{z}_2^1, \bar{z}_2^2) = (20, 40)$.

The optimal cost function for period 1 is given by

$W_1(y_1, q_0^3) = \hfill (4)$

$$\begin{cases} \frac{340}{3} - 2q_0^3 - 3y_1 & \text{if } y_1 \leqslant 10, q_0^3 \leqslant \frac{40}{3}, & (a) \\ \frac{385}{3} - 2q_0^3 - 6y_1 + \frac{3}{20}(y_1)^2 & \text{if } 10 < y_1 \leqslant \min\{20, \frac{70}{3} - q_0^3\}, & (b) \\ \frac{205}{3} - 2q_0^3 & \text{if } 20 < y_1 \leqslant \frac{70}{3} - q_0^3, & (c) \\ 150 - 9q_0^3 + \frac{3}{20}(q_0^3)^2 - 7y_1 + \frac{3}{10}q_0^3 y_1 + \frac{3}{20}(y_1)^2 & \text{if } \max\{20, \frac{70}{3} - q_0^3\} < y_1 \leqslant 30 - q_0^3, & (d) \\ 105 - 6q_0^3 + \frac{1}{10}(q_0^3)^2 - 4y_1 + \frac{1}{5}q_0^3 y_1 + \frac{1}{10}(y_1)^2 & \text{if } \max\{20, 30 - q_0^3\} < y_1 \leqslant 40 - q_0^3, & (e) \\ -55 + 2q_0^3 + 4y_1 & \text{if } \max\{20, 40 - q_0^3\} < y_1 \leqslant 50 - q_0^3, & (f) \\ 70 - 3q_0^3 + \frac{1}{20}(q_0^3)^2 - y_1 + \frac{1}{10}q_0^3 y_1 + \frac{1}{20}(y_1)^2 & \text{if } \max\{20, 50 - q_0^3\} < y_1 \leqslant 60 - q_0^3, & (g) \\ -110 + 3q_0^3 + 5y_1 & \text{if } y_1 > \max\{20, 60 - q_0^3\}, & (h) \\ \frac{380}{3} - 4q_0^3 + \frac{3}{40}(q_0^3)^2 - 3y_1 & \text{if } y_1 \leqslant \frac{50}{3} - \frac{1}{2}q_0^3, \frac{40}{3} < q_0^3 \leqslant \frac{80}{3}, & (i) \\ 210 - 9q_0^3 + \frac{3}{20}(q_0^3)^2 - 13y_1 + \frac{3}{10}q_0^3 y_1 + \frac{3}{10}(y_1)^2 & \text{if } \max\{\frac{50}{3} - \frac{1}{2}q_0^3, \frac{70}{3} - q_0^3\} < y_1 \\ & \qquad \leqslant \min\{20, 30 - q_0^3\}, & (j) \\ 165 - 6q_0^3 + \frac{1}{10}(q_0^3)^2 - 10y_1 + \frac{1}{5}q_0^3 y_1 + \frac{1}{4}(y_1)^2 & \text{if } \max\{14 - \frac{2}{5}q_0^3, 30 - q_0^3\} < y_1 \\ & \qquad \leqslant \min\{20, 40 - q_0^3\}, & (k) \\ 5 + 2q_0^3 - 2y_1 + \frac{3}{20}(y_1)^2 & \text{if } \max\{0, 40 - q_0^3\} < y_1 \leqslant \min\{20, 50 - q_0^3\}, & (l) \\ 130 - 3q_0^3 + \frac{1}{20}(q_0^3)^2 - 7y_1 + \frac{1}{10}q_0^3 y_1 + \frac{1}{5}(y_1)^2 & \text{if } \max\{0, 50 - q_0^3\} < y_1 \leqslant \min\{20, 60 - q_0^3\}, & (m) \\ -50 + 3q_0^3 - y_1 + \frac{3}{20}(y_1)^2 & \text{if } y_1 > \max\{0, 60 - q_0^3\}, & (n) \\ \frac{85}{2} + q_0^3 - 3y_1 & \text{if } y_1 \leqslant 35 - q_0^3, q_0^3 > 35, & (o) \\ 116 - \frac{18}{5}q_0^3 + \frac{3}{50}(q_0^3)^2 - 3y_1 & \text{if } y_1 \leqslant 14 - \frac{5}{2}q_0^3, \frac{80}{3} < q_0^3 \leqslant 35, & (p) \\ 165 - 6q_0^3 + \frac{1}{10}(q_0^3)^2 - 10y_1 + \frac{1}{5}q_0^3 y_1 + \frac{1}{10}(y_1)^2 & \text{if } 35 - q_0^3 < y_1 \leqslant \min\{0, 30 - q_0^3\}, & (q) \\ 5 + 2q_0^3 - 2y_1 & \text{if } 30 - q_0^3 < y_1 \leqslant \min\{0, 40 - q_0^3\}, & (r) \\ 130 - 3q_0^3 + \frac{1}{20}(q_0^3)^2 - 7y_1 + \frac{1}{10}q_0^3 y_1 + \frac{1}{20}(y_1)^2 & \text{if } 50 - q_0^3 < y_1 \leqslant \min\{0, 60 - q_0^3\}, & (s) \\ -50 + 3q_0^3 - y_1 & \text{if } 60 - q_0^3 < y_1 \leqslant 0. & (t) \end{cases}$$

In Fig. 1., we depict the various regions involved in defining $W_1(y_1, q_0^3)$ in (4). Here, $p_1^2 = q_0^3$.

There are optimal base-stock levels for the first two modes, which are given by

$$(\bar{z}_1^1, \bar{z}_1^2) = \begin{cases} (35 - q_0^3, 35) & \text{if } q_0^3 \geqslant 35, \\ (14 - \frac{2}{5}q_0^3, 14 + \frac{3}{10}q_0^3) & \text{if } \frac{80}{3} \leqslant q_0^3 < 35, \\ (\frac{50}{3} - \frac{1}{2}q_0^3, \frac{50}{3} + q_0^3) & \text{if } \frac{40}{3} \leqslant q_0^3 < \frac{80}{3}, \\ (10, \frac{70}{3}) & \text{if } 0 \leqslant q_0^3 < \frac{40}{3}. \end{cases}$$

3.2   Discussion

We focus on analyzing the optimal decision for Q_1^{3*}. Fig. 2 shows how Q_1^{3*} changes for given the reference inventory position z_1^{2*}, which does not follow a base-stock structure (This also
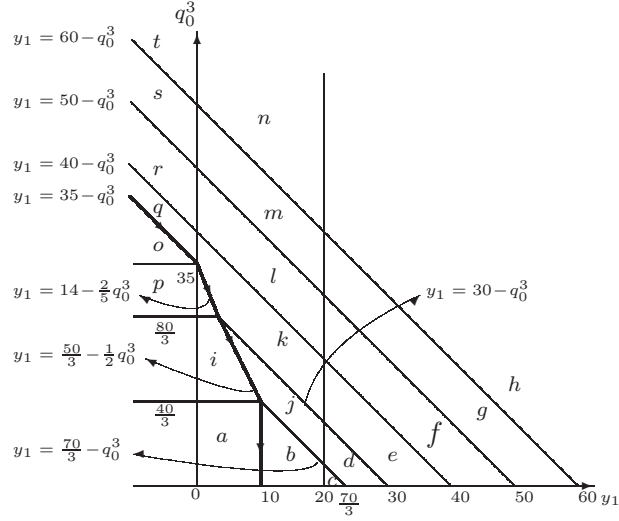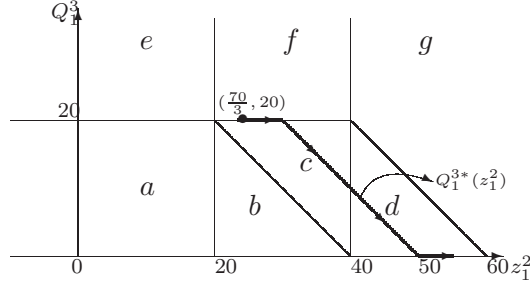
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Figure 1: The cost function for period 1.

$$y_1 = 60 - q_0^3 \quad t$$
$$y_1 = 50 - q_0^3 \quad s$$
$$y_1 = 40 - q_0^3 \quad r$$
$$y_1 = 35 - q_0^3 \quad q$$
$$o$$
$$y_1 = 14 - \tfrac{2}{5}q_0^3 \quad p \quad 35$$
$$\tfrac{80}{3}$$
$$y_1 = \tfrac{50}{3} - \tfrac{1}{2}q_0^3 \quad i$$
$$\tfrac{40}{3}$$
$$y_1 = \tfrac{70}{3} - q_0^3 \quad a$$

q_0^3 (vertical axis), $y_1 = 30 - q_0^3$

regions: n, m, l, k, j, h, g, f, b, d, e

axis: $0 \quad 10 \quad 20 \; \tfrac{70}{3} \quad 30 \quad 40 \quad 50 \quad 60 \; y_1$

Figure 2: The function $Q_1^{3*}(z_1^2)$.

Q_1^3 (vertical axis)

regions: e, f, g, a, b, c, d

20, $\left(\tfrac{70}{3}, 20\right)$, $Q_1^{3*}(z_1^2)$

axis: $0 \quad 20 \quad 40 \quad 50 \quad 60 \; z_1^2$

gives a counterexample to Lemma 1 in Zhang 1996). To further explore the reason, we consider the optimal post-order inventory positions in period 1:

$$(z_1^{1*}, z_1^{2*}, z_1^{3*}) = \tag{5}$$

$$
\begin{cases}
\begin{cases}
(\ 10, & \tfrac{70}{3}, & \tfrac{130}{3}\ ) & \text{if } y_1 \leqslant 10, \\
(\ y_1, & \tfrac{70}{3}, & \tfrac{130}{3}\ ) & \text{if } 10 < y_1 \leqslant \tfrac{70}{3} - q_0^3, \\
(\ y_1, & y_1 + q_0^3, & y_1 + q_0^3 + 20) & \text{if } \tfrac{70}{3} - q_0^3 < y_1 \leqslant 30 - q_0^3, \\
(\ y_1, & y_1 + q_0^3, & 50\ ) & \text{if } 30 - q_0^3 < y_1 \leqslant 50 - q_0^3, \\
(\ y_1, & y_1 + q_0^3, & y_1 + q_0^3\ ) & \text{if } y_1 \geqslant 50 - q_0^3,
\end{cases} & \text{if } 0 \leqslant q_0^3 < \tfrac{40}{3}; \\[3em]
\begin{cases}
(\tfrac{50}{3} - \tfrac{1}{2}q_0^3, & \tfrac{50}{3} + q_0^3, & \tfrac{110}{3} + q_0^3\ ) & \text{if } y_1 \leqslant \tfrac{50}{3} - \tfrac{1}{2}q_0^3, \\
(\ y_1, & y_1 + q_0^3, & y_1 + q_0^3 + 20) & \text{if } \tfrac{50}{3} - \tfrac{1}{2}q_0^3 < y_1 \leqslant 30 - q_0^3, \\
(\ y_1, & y_1 + q_0^3, & 50\ ) & \text{if } 30 - q_0^3 < y_1 \leqslant 50 - q_0^3, \\
(\ y_1, & y_1 + q_0^3, & y_1 + q_0^3\ ) & \text{if } y_1 \geqslant 50 - q_0^3,
\end{cases} & \text{if } \tfrac{40}{3} \leqslant q_0^3 < \tfrac{80}{3}; \\[3em]
\begin{cases}
(14 - \tfrac{2}{5}q_0^3, & \tfrac{50}{3} + q_0^3, & 50\ ) & \text{if } y_1 \leqslant 14 - \tfrac{2}{5}q_0^3, \\
(\ y_1, & y_1 + q_0^3, & 50\ ) & \text{if } 14 - \tfrac{2}{5}q_0^3 < y_1 \leqslant 50 - q_0^3, \\
(\ y_1, & y_1 + q_0^3, & y_1 + q_0^3) & \text{if } y_1 \geqslant 50 - q_0^3,
\end{cases} & \text{if } \tfrac{80}{3} \leqslant q_0^3 < 35; \\[3em]
\begin{cases}
(35 - q_0^3, & 35, & 50\ ) & \text{if } y_1 \leqslant 35 - q_0^3, \\
(\ y_1, & y_1 + q_0^3, & 50\ ) & \text{if } 35 - q_0^3 < y_1 \leqslant 50 - q_0^3, \\
(\ y_1, & y_1 + q_0^3, & y_1 + q_0^3) & \text{if } y_1 \geqslant 50 - q_0^3,
\end{cases} & \text{if } q_0^3 \geqslant 35;
\end{cases}
$$

It seems that the ideal order-up-to level for Q_1^3 would be 50, if the ordering policy for the slow mode were to follow a base-stock policy. However, this cannot always be achieved when q_0^3 is low ($q_0^3 \leqslant 80/3$).

Table 2 illustrates the optimal ordering decisions when $0 \leqslant q_0^3 < 40/3$. If $y_1 < 70/3 - q_0^3$, we have to order a positive Q_2^{1*} in period 2 and 20 units of Q_1^{3*}. However, the inventory position z_1^{3*} does not account for the value of Q_2^{1*}. Thus, the optimal decision is not to bring z_1^{3*} up to 50. When $y_1 \geqslant 70/3 - q_0^3$, there is no fast order placed in period 2. As a result, the optimal inventory position z_1^{3*} should be as close to the target level 50 as possible. One may think that if we modify the reference inventory position for Q_1^3 to include the anticipated order Q_2^1, then we could restore the base-stock policy. However, this would not work in the case of general stochastic demand.

Table 2: When $0 \leqslant q_0^3 < \frac{40}{3}$

Period	y_1	$(-\infty, 10]$	$(10, \frac{70}{3} - q_0^3)$	$[\frac{70}{3} - q_0^3, 30 - q_0^3]$	$(30 - q_0^3, 40 - q_0^3]$	$(40 - q_0^3, 50 - q_0^3]$	$(50 - q_0^3, \infty)$
1	$\bar{z}_1^1$	10	10	10	10	10	10
	Q_1^{1*}	$10 - y_1$	0	0	0	0	0
	z_1^{1*}	10	y_1	y_1	y_1	y_1	y_1
	$z_1^{1*} + q_0^3$	$10 + q_0^3$	$y_1 + q_0^3$	$y_1 + q_0^3$	$y_1 + q_0^3$	$y_1 + q_0^3$	$y_1 + q_0^3$
	$\bar{z}_1^2$	$\frac{70}{3}$	$\frac{70}{3}$	$\frac{70}{3}$	$\frac{70}{3}$	$\frac{70}{3}$	$\frac{70}{3}$
	Q_1^{2*}	$\frac{40}{3} - q_0^3$	$\frac{70}{3} - y_1 - q_0^3$	0	0	0	0
	z_1^{2*}	$\frac{70}{3}$	$\frac{70}{3}$	$y_1 + q_0^3$	$y_1 + q_0^3$	$y_1 + q_0^3$	$y_1 + q_0^3$
	Q_1^{3*}	20	20	20	$50 - y_1 - q_0^3$	$50 - y_1 - q_0^3$	0
	z_1^{3*}	$\frac{130}{3}$	$\frac{130}{3}$	$y_1 + q_0^3 + 20$	50	50	$y_1 + q_0^3$
Period 2	y_2	$\frac{70}{3} - d_1$	$\frac{70}{3} - d_1$	$y_1 + q_0^3 - d_1$	$y_1 + q_0^3 - d_1$	$y_1 + q_0^3 - d_1$	$y_1 + q_0^3 - d_1$
	$\bar{z}_2^1$	20	20	20	20	20	20
	EQ_2^{1*}	$\frac{125}{18}$	$\frac{125}{18}$	$\frac{(40 - y_1 - q_0^3)^2}{40}$	$\frac{(40 - y_1 - q_0^3, 0)^2}{40}$	0	0
	$\bar{z}_2^2$	40	40	40	40	40	40
	EQ_2^{2*}	0	0	0	0	2.5	$\frac{[\max\{60 - y_1 - q_0^3, 0\}]^2}{40}$

When q_0^3 is lower than $80/3$, there is a positive probability that $Q_2^{1*} > 0$. Since the reference inventory position for the type 3 order in period 1 does not take this order into account, the optimal ordering policy in period 1 is not a base-stock policy. When q_0^3 is greater than $80/3$, we always have $Q_2^{1*} = 0$ and the optimal policy in period 1 follows a base-stock policy.

We also observe from our example that when the type 3 orders do not follow base-stock policy, the order quantity Q_1^{3*} is constant 20. However, this observation cannot be generalized. If we modify the holding/backlog cost in period 3 to be $H_3(x) = 0.5x^2$, then Q_1^{3*} does not have this property any longer. The corresponding $Q_1^{3*}(z_1^2)$ is given in Fig. 3.

Figure 3: $Q_1^{3*}(z_1^2)$ for nonlinear holding costs.
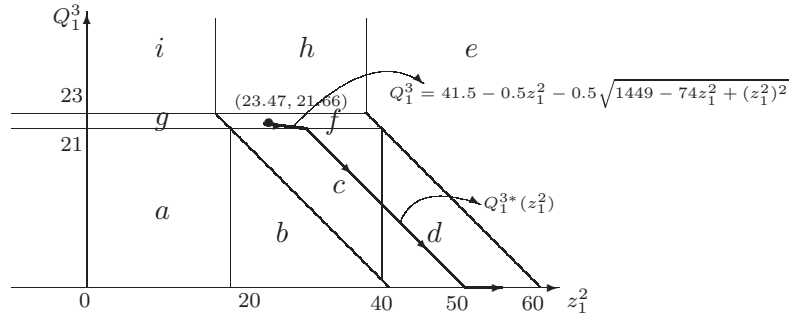
4. Separability and the Base-Stock Policies

The example in the last section shows that the optimal ordering policies for inventory models with more than two modes are no longer base-stock policies in general. Next, we examine the relationship between an optimal base-stock policy and the separability of the cost functions. The next result establishes the existence of optimal base-stock levels for the first two modes in general inventory systems with multiple delivery modes.

PROPOSITION 1 *The minimand in the right-hand side of* (3) *can be expressed as* $G_1(z_k^1) + G_2(z_k^2, ..., z_k^N)$, *where* $G_1(z_k^1)$ *is convex in* z_k^1 *and* $G_2(z_k^2, ..., z_k^N)$ *is jointly convex in* $(z_k^2, ..., z_k^N)$. *Let* $(z_k^{1*}, ..., z_k^{N*})$ *be the minimizing vector in* (3). *Then there exist two base stocks* $\bar{z}_k^1$ *and* $\bar{z}_k^2$ *such that*

$$z_k^{1*} = \bar{z}_k^1 \vee y_k, \quad z_k^{2*} = \bar{z}_k^2 \vee (z_k^{1*} + p_k^2).$$

In general, the cost-to-go function defined in (3) is not separable in reference inventory positions $(z_k^2, ..., z_k^N)$. If it were, it can be easily shown that there would exist optimal base-stock levels in period k for type 3 and higher orders. This may happen in some very special cases. One such example is Fukuda's (1964) model of three consecutive modes where orders are placed in every other period. There the resulting cost-to-go function is separable in the reference inventory positions, and the optimal ordering policy is a base-stock policy. Furthermore, the follwoing proposition represents a generalization of Fukuda's (1964) result.

PROPOSITION 2 *If the first mode in period* $k + 1$ *is not used, then the first three modes in period* k *have optimal base-stock levels.*

REMARK 1 *Note that the proof cannot be generalized for more than three modes. That is, if the first two modes in period $k + 1$ are not used, then the fourth mode in period k may not have an optimal base stock.*

5. Concluding Remarks

We have shown that the optimality of a base-stock policy is closely related to the structure of the cost-to-go function. Since the cost-to-go function for each period is separable in the post-order inventory positions for the first two modes, there exist optimal base-stock levels for the first two modes. In general, the cost-to-go function for a given period is not separable in post-order inventory positions of higher modes. Thus, the optimal ordering decision for type 3 or higher order may not have a base-stock structure. Our discussion shows that the base-stock policy fails to be optimal even under very restrictive conditions, e.g., uniform demand, linear holding costs. The intuitive reason is that the optimal post-order inventory position for a mode of type 3 or higher cannot, in general, anticipate the future order quantities. Moreover, the dependence of the optimal order quantities on the reference inventory position and in-transit orders may be quite complex.

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Appendix

The proof of Proposition 1 uses the following lemma.

LEMMA 1 *Suppose $G(x^1, ..., x^n)$ is jointly convex in $(x^1, ..., x^n)$. Define $G^o(x^1, ..., x^k) = \min$ $\{G(x^1, ..., x^n)|x^j \geqslant x^{j-1} + a_j$ for $j = k+1, ..., n\}$ for $k = 1, ..., N$. Then $G^o(x^1, ..., x^k)$ is jointly convex in $(x^1, ..., x^k)$.*

PROOF. We show that $G^o(x^1, ..., x^k)$ is jointly convex by induction on k. Clearly, when $k = N$, $G^o(x^1, ..., x^N) = G(x^1, ..., x^N)$ is jointly convex. Suppose that $G^o(x^1, ..., x^{k+1})$ is jointly convex in $(x^1, ..., x^{k+1})$. Then,

$$
\begin{aligned}
G^o(x^1, ..., x^k) &= \min\{G(x^1, ..., x^N)|x^j \geqslant x^{j-1} + a_j \text{ for } j = k+1, ..., N\} \\
&= \min\{G^o(x^1, ..., x^{k+1})|x^{k+1} \geqslant x^k + a_{k+1}\}.
\end{aligned}
$$

Now define $\tilde{x}^{k+1}(x^1, ..., x^k) = \arg\min_{x^{k+1}} G^o(x^1, ..., x^{k+1})$, then

$$
G^o(x^1, ..., x^k) = G^o\Big(x^1, ..., \tilde{x}^{k+1}(x^1, ..., x^k) \vee (x^k + a_{k+1})\Big).
$$

Clearly, $G^o\Big(x^1, ..., \tilde{x}^{k+1}(x^1, ..., x^k)\Big)$ is convex since the lower envelope of a convex function is convex. Thus, $G^o(x^1, ..., x^k)$ is convex.

$\square$

PROOF OF PROPOSITION 1.   The minimand in the right-hand side of (3) is easily seen to be separable in z_k^1 and $(z_k^2, ... z_k^N)$. By Lemma 1, we have

$$
W_k(y_k, p_k^2, ..., p_k^{N-1})
$$

$$= \min\{G_1(z_k^1) + G_2(z_k^2, ..., z_k^N) | z_k^1 \geqslant y_k, z_k^j \geqslant z_k^{j-1} + p_k^j, j = 2, ..., N\}$$

$$= \min\{G_1(z_k^1) + G_2^o(z_k^2) | z_k^1 \geqslant y_k, z_k^2 \geqslant z_k^1 + p_k^2\}.$$

Denote $\tilde{z}_k^1$, $\tilde{z}_k^2$ and $\hat{z}_k^1$ as the unconstrained minimizer to $G_1(z_k^1)$, $G_2^o(z_k^2)$, and $G_1(z_k^1) + G_2^o(z_k^1 + p_k^2)$, respectively. Let $\bar{z}_k^1 = \tilde{z}_k^1 \wedge \hat{z}_k^1$ and $\bar{z}_k^2 = \tilde{z}_k^2$. Also note that

$$\min_{\substack{z_k^1 \geqslant y_k \\ z_k^2 \geqslant z_k^1 + p_k^1}} \left\{ G_1(z_k^1) + G_2^o(z_k^2) \right\} = \min_{z_k^1 \geqslant y_k} \left\{ G_1(z_k^1) + G_2^o\left(\tilde{z}_k^2 \vee (z_k^1 + p_k^2) \right) \right\}.$$

The function inside "min" on the right-hand side is convex in z_k^1. We consider two cases.

Case 1: $\tilde{z}_k^1 + p_k^2 \leqslant \tilde{z}_k^2$. First note that in this case $\tilde{z}_k^1 \leqslant \hat{z}_k^1$, $\bar{z}^1 = \tilde{z}_k^1$. We have $\bar{z}_k^1 + p_k^1 \leqslant \bar{z}_k^2$. Thus, for any $z_k^1 \geqslant y_k$,

$$G_1(\tilde{z}_k^1 \vee y_k) + G_2^o\left(\tilde{z}_k^2 \vee (y_k \vee \tilde{z}_k^1 + p_k^2) \right) \leqslant G_1(z_k^1 \vee y_k) + G_2^o\left(\tilde{z}_k^2 \vee (y_k \vee z_k^1 + p_k^2) \right).$$

To see the last inequality, we note that the right-hand side is convex in z_k^1. If $y_k \leqslant \bar{z}_k^1$, the right-hand side is minimized at $z_k^1 = \tilde{z}_k^1 = \bar{z}_k^1$. If $y_k > \bar{z}_k^1$, the right-hand side is minimized at $z_k^1 = y_k$. Hence, the minimizer is $z_k^{1*} = \tilde{z}_k^1 \vee y_k = \bar{z}_k^1 \vee y_k$ and $z_k^{2*} = \tilde{z}_k^2 \vee (z_k^{1*} + p_k^2)$.

Case 2: $\tilde{z}_k^1 + p_k^2 > \tilde{z}_k^2$. In this case the minimizer must be such that $z_k^{1*} + p_k^2 = z_k^{2*}$. Note that $\hat{z}_k^1 < \tilde{z}_k^1$ and $\tilde{z}_k^2 < \hat{z}_k^1 + p_k^2$. So $\bar{z}_k^1 = \hat{z}_k^1$ and $\bar{z}_k^2 = \tilde{z}_k^2 < \hat{z}_k^1 + p_k^2$. Thus, for any $z_k^1 \geqslant y_k$,

$$G_1(\hat{z}_k^1 \vee y_k) + G_2^o(\hat{z}_k^1 \vee y_k + p_k^2) = G_1(\hat{z}_k^1 \vee y_k) + G_2^o\left(\tilde{z}_k^2 \vee (\hat{z}_k^1 \vee y_k + p_k^2) \right)$$

$$\leqslant G_1(z_k^1 \vee y_k) + G_2^o\left(\tilde{z}_k^2 \vee (z_k^1 + p_k^2) \right).$$

Similar to Case 1, if $y_k \leqslant \hat{z}_k^1$, then the right-hand side is minimized at $\hat{z}_k^1$. Otherwise, it is minimized at y_k. Hence, the minimizer is $z_k^{1*} = \hat{z}_k^1 \vee y_k = \bar{z}_k^1 \vee y_k$ and $z_k^{2*} = z_k^{1*} + p_k^2 = \bar{z}_k^2 \vee (z_k^{1*} + p_k^2)$.

$\square$

PROOF OF PROPOSITION 2. The cost function for period k can be written as

$$J_k(z_k^1, ... z_k^N) = c_k^1(z_k^1 - y_k) + \sum_{j=2}^{N} c_k^j(z_k^j - z_k^{j-1} - p_k^j) + EH_{k+1}(z_k^1 - D_k)$$

$$+ EW_{k+1}(z_k^2 - D_k, z_k^3 - z_k^2, ..., z_k^N - z_k^{N-1})$$

$$= c_k^1(z_k^1 - y_k) + \sum_{j=2}^{N} c_k^j(z_k^j - z_k^{j-1} - p_k^j) + EH_{k+1}(z_k^1 - D_k) +$$

$$+E \min_{\substack{z^3_{k+1} \geqslant z^4_k - D_k, \\ z^j_{k+1} \geqslant z^{j-1}_{k+1} + z^{j+1}_k - z^j_k,}} \left\{ c^3_{k+1}(z^3_{k+1} - z^4_k + D_k) + \sum_{j=4}^{N} c^j_{k+1}(z^j_{k+1} - z^{j-1}_{k+1} - z^{j+1}_k + z^j_k) \right.$$

$$+EH_{k+2}(z^2_k - D_k - D_{k+1})$$

$$\left. +EW_{k+2}(z^3_k - D_k - D_{k+1}, z^3_{k+1} - z^3_k + D_k + D_{k+1}, ..., z^N_{k+1} - z^{N-1}_{k+1}) \right\}.$$

Clearly, the last expression is separable in (z^1_k, z^2_k, z^3_k). Hence, the result follows from Proposition 5 of Feng et al. (2004).

$\square$