Generic and Brand Advertising Strategies in a Dynamic Duopoly

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March 2004

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Abstract

To increase the sales of their products through advertising, firms can use brand advertising to capture market share from their competitors or increase primary demand for the category through generic advertising. This paper examines the issues of whether, when, and how much brand advertising versus generic advertising should be done. The context is a dynamic duopoly where firms decide on brand and generic advertising expenditures. Using differential game theory, optimal advertising decisions are obtained for the cases of symmetric and asymmetric competitors. Thereafter, comparative statics are presented and the managerial implications of the results are discussed.

Keywords: *Advertising; Generic advertising*; *Differential games; Dynamic duopoly; Optimal control*

1. Introduction

From the relationship, "Product Sales = Category Sales × Market Share," it follows that marketing decisions can increase sales only by increasing category sales volume, i.e., primary demand[1], or by increasing market share. When the relevant marketing instrument is advertising, we define advertising whose effect is to increase category sales as "generic advertising" and advertising whose effect is to gain market share as "brand advertising." Operationally, generic advertising generates new sales by targeting beliefs about the product category while downplaying or oftentimes not mentioning the sponsoring brand. In contrast, brand advertising provides consumers with information about the brand's value proposition that differentiates it from its competitors, thereby encouraging consumers to choose the advertised brand over competing brands (Krishnamurthy 2000; 2001).

Allocating funds to generic advertising has an interesting consequence for the firm. Since generic advertising promotes the general qualities of the product category, it benefits all the firms in the category regardless of whether or not they paid for the advertising. Competing firms can benefit from the firm's contribution by free-riding or "cheap-riding," i.e., by not spending significant amounts of their own money on generic advertising (Krishnamurthy 2000). In this paper, we determine how much different firms should contribute towards generic advertising, and how firm and competitive factors affect their contributions.

Characterizing the optimal advertising policies in the presence of competitive effects is of great interest to researchers. However, the extant research on this topic is limited to either static models that incorporate generic and brand advertising (e.g., Krishnamurthy 2000) or dynamic models that do not explicitly consider generic advertising (e.g., Chintagunta and Vilcassim 1992;

[1] The terms "primary demand" and "category demand" are used interchangeably.

Erickson 1992; Prasad and Sethi 2004a; Sorger 1989). Thus, an important contribution of this paper is to examine dynamic competition in brand and generic advertising. We will examine the relevant literature in detail in the next section.

The current research answers the following questions: (1) What should be a firm's generic and brand advertising budget, and how should it be dynamically allocated? (2) How do the nature of competition (symmetric or asymmetric) and other firm and market characteristics affect this allocation? Answering these questions contributes to the substantive literature on optimal advertising strategies in the presence of competition. We model market expansion by specifying a modified form of the Lanchester model of combat (Little 1979). We model the two types of sales growth in a dynamic duopoly – via market expansion (size of the pie) through generic advertising and market share allocation (slice of the pie) through brand advertising. Since both advertising and sales are time-varying, dynamic optimization techniques are utilized to analyze the situation (Sethi and Thompson 2000). Specifically, the "value function approach" is used to obtain explicit closed-loop Nash equilibrium solutions for the optimal advertising decisions (Villas-Boas 1999). We examine both symmetric and asymmetric competition.

The rest of the paper is organized as follows: In Sections 2 and 3, we review the existing literature. Section 4 presents the model. Section 5 deals with the analysis. Section 6 presents the results for symmetric and asymmetric firms. Section 7 examines extensions of the basic model. Finally, Section 8 concludes with a summary and directions for future research.

2. Literature Review

The effect of advertising on sales is an oft-researched topic (e.g., Bass and Parsons 1969). The bulk of this literature is devoted to brand advertising, justified by casual empirical evidence which suggests it to be the more common form of advertising. Thus, we first elaborate on the less known merits of generic advertising. We will then be in a better position to consider the relevant managerial decisions of budgeting and allocation of brand versus generic advertising.

Generic advertising increases primary demand by attracting new consumers, increasing per capita consumption of the product, and lengthening the product life cycle (Friedman and Friedman 1976). Let us consider these cases in more detail:

a) New product categories: Generic advertising is particularly effective in the introductory stage of the product life cycle. Consumers are unaware of the product's uses and benefits and need to be informed and educated. For example, when P&G introduced Pampers diapers, it tried to enhance product acceptance by highlighting the advantages of using disposable diapers. Consider also advertisements by Sirius and XM, competitors in the nascent market for satellite radio. Douglas Wilsterman, Marketing VP of Sirius, says, "You've got to do a little bit of both [*viz. brand and generic ads*]. You can't just talk about yourself without people knowing what you represent in terms of a revolutionarily dynamic change" (Beardi 2001). Along similar lines, Steve Cook, XM's VP of Sales and Marketing, adds, "XM's ads will be about continuing to "grow the whole category pie," rather than competing with Sirius" (Boston and Halliday 2003). Other examples of generic advertising in new product categories, such as high-definition TV and the recordable DVD format, show advertisements promoting the advantages of these new standards without touting brand-specific features. Although the brand name may be mentioned, ads ask customers to compare the new technology with their existing technology without differentiating each firm's brand from competitors offering the same technology.

b) Increased penetration of mature products: Firms use generic advertising to market mature products by promoting new uses. Examples include Arm and Hammer's informing the public of new ways to use baking soda and Skippy showing consumers nontraditional ways to enjoy peanut butter. Other attempts to increase the sales of established products can be seen in De Beers' advertisement urging customers to buy diamonds for all occasions (Bates 2001), Dannon promoting the benefits of yogurt consumption, Norelco's "Gotcha!" campaign about the advantages of electric shavers over razors, and the Trojan condoms advertising campaign in the 1970s stressing the importance of family planning (Friedman and Friedman 1976).

c) Commodities: The critical role of generic advertising in the promotion of commodities can be traced to the 1950s when producers of tea and butter used generic advertising to compete with makers of coffee and margarine, respectively. Over the years, dairy producers have invested hundreds of millions of dollars in promoting the consumption of milk and other dairy products. These include the well-known "milk mustache" campaign by the California Milk Advisory Board, advertisements for eggs by the American Egg Board, and the "Pork: The Other White Meat" campaign. The advertisement for cotton, "The Fabric of Our Lives," that for plastics by the American Plastics Council, and the California Raisin Advisory Board's promotion of raisins as a wholesome natural snack for children also come to mind in this regard. Other associations have used generic advertising to promote lamb, grapes, oranges, savings bank, life insurance, and the importance of having regular eye checkups (Friedman and Friedman 1976). When firms form an association to manage the generic advertising campaign, the basic rule is one of *co-opetition*, i.e., cooperate first and then compete, which suggests that firms can gain advantage by means of a judicious mixture of cooperation and competition (Brandenburger and Nalebuff 1997).

These examples show the range of uses of generic advertising. However, from these examples, no intuition emerges about when to emphasize generic advertising spending (initial or later stages of the product life cycle), under which circumstances to use it (market characteristics, e.g., high-tech, commodity, or other markets), or what effect competitive structure (symmetric or asymmetric competition) has on the decision. Consequently, a modeling approach is required. In this paper, we will broadly cover the issues of whether, when, and how much generic versus brand advertising should be undertaken.

In what follows, we will first examine models of brand advertising and then consider models that incorporate generic advertising in addition to brand advertising.

3. Modeling Background

3.1 Models of Brand Advertising

Researchers have studied competitive advertising strategies of firms using the theory of differential games (e.g., Chintagunta and Vilcassim 1992; Deal 1979; Deal, Sethi, and Thompson 1979; Erickson 1985; Jorgensen 1982; Prasad and Sethi 2004a). An example of such a model is

$$\dot{x}_i(t) = \rho_i u_i(t)(1 - x_i(t)) - \rho_j u_j(t) x_i(t), \quad x_i(0) = x_{i0}, \ \ \forall \ i, j \in \{1, 2\}, \ i \neq j, \tag{1}$$

where $x_i(t)$ is the market share of firm i at time t, $u_i(t)$ is the brand advertising of firm i, and ρ_i is the effectiveness of firm i's advertising. The model is a competitive extension of the Vidale-Wolfe (1957) model of advertising in which firm i uses brand advertising to capture firm j's market share, and vice versa.

A variant of (1) was used by Sethi (1983) to derive optimal advertising policies in a monopoly, and by Sorger (1989) and Prasad and Sethi (2004a; 2004b) to model brand advertising competition. This model is specified as

$$\dot{x}_i(t) = \rho_i u_i(t)\sqrt{1 - x_i(t)} - \rho_j u_j(t)\sqrt{x_i(t)}, \quad x_i(0) = x_{i0}. \tag{2}$$

Sorger (1989) compares the model in (2) to other models of brand advertising, derives solutions for the optimal advertising expenditures so as to maximize each firm's profit, and also discusses various desirable properties of this model, notably the diminishing marginal returns to advertising and the fact that the structure can be made to resemble word-of-mouth and excess advertising models. (The word-of-mouth effects refer to the communication between adopters and non-adopters of a brand. Excess advertising reflects the fact that a firm will gain market share if its advertising expenditure exceeds that of its rival.) Chintagunta and Jain (1995) find that this specification fits the data from four product-markets (pharmaceutical, soft drink, beer, and detergent) well.

We next examine models that include both generic and brand advertising.

3.2 Models of Generic Advertising

Krishnamurthy (2000) studies the relationship between generic and brand advertising under several scenarios. Firm i's profit function is

$$\Pi_i = M(G)\frac{A_i(B_i)}{\sum_k A_k(B_k)} m_i - G_i - B_i, \tag{3}$$

where $M(G)$ is the size of the market, G_i is firm i's contribution to the generic advertising campaign, $G = \sum_i G_i$ is the total generic advertising spending, m_i is the margin, B_i is firm i's brand advertising expenditure, and A_i is an attraction function dependent on B_i. The analysis suggests that if there is a dominant firm in the industry, the Nash equilibrium is for that firm to contribute everything and for the remaining firms to contribute nothing. Krishnamurthy (2001) provides an extension to the above model to alleviate the free-riding problem. In this contribution scheme, called the "provision point voluntary contribution mechanism," a target budget called the "provision point" is announced with the condition that the generic advertising campaign will be mounted if and only if the contributions equal or exceed this point.

It should be noted that these models are static and do not consider the sales and advertising dynamics.

Fruchter (1999) proposed a model of advertising competition with market expansion due to competitive advertising. The model is

$$\dot{S}_i(t) = \rho_i u_i(t)\big(\overline{S}(t) - S_i(t)\big) - S_i(t)\sum_{\substack{j=1 \\ j\neq i}}^{n} \rho_j u_j(t), \quad S_i(0) = S_{i0}, \tag{4}$$

where S_i is the sales of firm i, $u_i(t)$ is firm i's advertising, and ρ_i is the advertising effectiveness. Firm i's profit function is given by

$$\Pi_i = \int_0^{\infty} e^{-rt}(m_i S_i(t) - u_i^2(t))dt, \tag{5}$$

where m_i is the margin and r the discount rate for firm i.

Analysis of (5) reveals that the firm should decrease its advertising expenditure as its sales increases, and that the ratio of the advertising expenditures of any two firms in competition is time-invariant, and depends only on the ratio of their advertising effectiveness, gross margins, and potential sales (Fruchter 1999). However, in this model, both generic and brand advertising are modeled using a single advertising variable, so their separate effects on sales are not distinguished. Since sales responds differently to generic and brand advertising, their effects should ideally be modeled separately.

From this review, we see that generic and brand advertising decisions have not been modeled together in a dynamic context. The present study specifically addresses this gap in the literature. In the next section, we build on these extant models of advertising competition, and develop a model of generic and brand advertising competition in a dynamic duopoly.

4. Model

We consider a dynamic duopoly with firms 1 and 2. Whenever we use i and j to represent the two firms, then $i=1$ implies $j=2$ and vice versa. We begin by listing the main notation.

$S_i(t)$	Sales of firm i at time t.
$u_i(t)$	Brand advertising of firm i at time t.
$a_i(t)$	Generic advertising of firm i at time t.
m_i	Per unit profit margin for firm i.
c_i	Advertising cost parameter for firm i.
ρ_i	Effectiveness of brand advertising of firm i.
k_i	Effectiveness of generic advertising of firm i.
θ_i	Proportion of marginal sales increase due to generic advertising allocated to firm i. We use $\theta_1=\theta$ and $\theta_2=1-\theta$.
r_i	Discount rate for firm i.
$V_i(S_i,\ S_j\ ,t)$	Value (or profit) function of firm i at time t.

The market is one where advertising is the dominant marketing mix variable and other marketing mix decisions are less important or non-strategic. An example of a market with such features is the cola industry, dominated by Coke, Pepsi, and their “Cola Wars” (Chintagunta and Vilcassim 1992; Erickson 1992). We start by modeling the effect of generic advertising on category demand. The change in primary demand $\dot{Q}(t)=\dot{S}_i(t)+\dot{S}_j(t)$ is given by

$$\frac{dQ(t)}{dt}=\dot{Q}(t)=\dot{S}_i(t)+\dot{S}_j(t)=k_i a_i(t)+k_j a_j(t), \quad i, j\in\{1,2\},\ i\neq j, \tag{6}$$

where $\dot{S}_i(t)$ is the rate of change of firm i's sales, $a_i(t)$ is the generic advertising of firm i, and k_i is the effectiveness of firm i's generic advertising.

The increase in the category demand as a result of generic advertising is shared unequally by the two firms. Let θ_i , a parameter representing brand strength, denote the proportion of the sales increase that is transferred to firm i . The effect of generic advertising on firm $i's$ sales, denoted $\dot{S}_{i,\,g}(t)$, is then

$$\dot{S}_{i,\,g}(t) = \theta_i\big(k_i a_i(t) + k_j a_j(t)\big). \tag{7}$$

To model the effect of brand advertising on sales, we modify the model given by (2) into a model of sales. The effect of brand advertising on firm i ’s sales, denoted $\dot{S}_{i,\,b}(t)$, is

$$\dot{S}_{i,\,b}(t) = \rho_i u_i(t)\sqrt{Q(t) - S_i(t)} - \rho_j u_j(t)\sqrt{S_i(t)}, \tag{8}$$

where $u_i(t)$ is the brand advertising decision of firm i and ρ_i is the effectiveness of that advertising. Hence, the brand advertising model is based on Sethi (1983) and other papers as discussed in the literature review. The total change in firm i ’s sales is $\dot{S}_i(t) = \dot{S}_{i,\,g}(t) + \dot{S}_{i,\,b}(t)$. Adding equations (7) and (8), the total effect of generic and brand advertising on firm i ’s sales rate is

$$\dot{S}_i(t) = \rho_i u_i(t)\sqrt{Q(t) - S_i(t)} - \rho_j u_j(t)\sqrt{S_i(t)} + \theta_i\big(k_i a_i(t) + k_j a_j(t)\big), \quad S_i(0) = S_{i0}, \tag{9}$$

where S_{i0} is the initial sales of firm i . Rewriting the equations of motion for the two firms using $S_j(t) = Q(t) - S_i(t)$ yields

$$\begin{aligned} \dot{S}_1(t) &= \rho_1 u_1(t)\sqrt{S_2(t)} - \rho_2 u_2(t)\sqrt{S_1(t)} + \theta\big(k_1 a_1(t) + k_2 a_2(t)\big), \quad S_1(0) = S_{10}, \\ \dot{S}_2(t) &= \rho_2 u_2(t)\sqrt{S_1(t)} - \rho_1 u_1(t)\sqrt{S_2(t)} + (1-\theta)\big(k_1 a_1(t) + k_2 a_2(t)\big), \quad S_2(0) = S_{20}. \end{aligned} \tag{10}$$

The two equations in (10) are intuitive in that the change in sales of one firm is given by the gain in sales due to its brand advertising ($\rho_i u_i(t)\sqrt{S_j(t)}$) minus the loss in sales due to the rival's brand advertising ($\rho_j u_j(t)\sqrt{S_i(t)}$) plus the gain in sales due to market expansion ($\theta_i(k_i a_i(t)+k_j a_j(t))$).

The control variables available to each firm are its generic and brand advertising decisions. Firm i's discounted profit maximization problem is

$$\max_{u_i(t),\ a_i(t)} V_i = \int_0^\infty e^{-r_i t}\left(m_i S_i(t) - C(u_i(t),\ a_i(t))\right)dt\,, \tag{11}$$

where r_i is the discount rate of firm i, m_i is the gross margin of firm i, and $C(u_i(t),\ a_i(t))$ is the total advertising spending of firm i. Firm i's total advertising expense is specified as

$$C(u_i(t),\ a_i(t)) = \frac{c_i}{2}\left(a_i(t)^2 + u_i(t)^2\right). \tag{12}$$

As in most of the literature, the cost of advertising is assumed to be quadratic (e.g., Roberts and Samuelson 1988; Sorger 1989). (Alternatively, one can use linear advertising costs and have advertising appear as a square root in the state equations.)

The discounted profit maximization problems of the two firms can now be rewritten as the differential game

$$\begin{aligned} &\max_{u_1(t),\ a_1(t)} V_1 = \int_0^\infty e^{-r_1 t}\left(m_1 S_1(t) - \frac{c_1}{2}\left(a_1(t)^2 + u_1(t)^2\right)\right)dt, \\ &\max_{u_2(t),\ a_2(t)} V_2 = \int_0^\infty e^{-r_2 t}\left(m_2 S_2(t) - \frac{c_2}{2}\left(a_2(t)^2 + u_2(t)^2\right)\right)dt, \end{aligned} \tag{13}$$

$$\text{s.t.}\quad \begin{aligned} &\dot{S}_1(t) = \rho_1 u_1(t)\sqrt{S_2(t)} - \rho_2 u_2(t)\sqrt{S_1(t)} + \theta\left(k_1 a_1(t) + k_2 a_2(t)\right), \quad S_1(0) = S_{10}, \\ &\dot{S}_2(t) = \rho_2 u_2(t)\sqrt{S_1(t)} - \rho_1 u_1(t)\sqrt{S_2(t)} + (1-\theta)\left(k_1 a_1(t) + k_2 a_2(t)\right), \quad S_2(0) = S_{20}, \end{aligned} \tag{14}$$

where V_i is firm i's profit function, also known as the value function.

5. Analysis

The advertising differential game in (13-14) can be analyzed to yield either open-loop or closed-loop equilibria. In an open-loop Nash equilibrium, the equilibrium advertising paths are functions of time only and not of sales or market shares. In contrast, a closed-loop Nash equilibrium is one in which the optimal advertising paths depend on time as well as on realized sales levels or market shares. A closed-loop equilibrium is more robust and is sub-game perfect. It is clear that managers would find closed-loop strategies more useful since these strategies allow them to monitor the market and modify their advertising trajectories to respond to sudden changes in the marketplace (Erickson 1992). Chintagunta and Vilcassim (1992) and Erickson (1992) provide evidence that a closed-loop solution fits empirical data better than its open-loop counterpart. Thus, we adopt the closed-loop solution concept.

The optimal advertising policies are given in Proposition 1 (all proofs in Appendix):

Proposition 1*: The differential game* (13-14) *has a unique closed-loop Nash equilibrium solution for the two firms. For firm* i*, the optimal decisions are* ($\forall\ i,\ j \in \{1,\ 2\},\ i \neq j$)*:*

a) Brand advertising: $$u_i^* = \frac{\rho_i}{c_i}(\beta_i - \gamma_i)\sqrt{S_j}\,, \tag{15}$$

b) Generic advertising: $$a_i^* = \frac{k_i}{c_i}(\theta_i\beta_i + \theta_j\gamma_i), \tag{16}$$

c) Value function: $$V_i = \alpha_i + \beta_i S_i + \gamma_i S_j\,, \tag{17}$$

where α_i, α_j, β_i, β_j, γ_i*, and* γ_j *solve the simultaneous equations*

$$r_i\alpha_i - \frac{k_i^2}{2c_i}(\theta_i\beta_i + \theta_j\gamma_i)^2 - \frac{k_j^2}{c_j}(\theta_i\beta_i + \theta_j\gamma_i)(\theta_i\gamma_j + \theta_j\beta_j) = 0, \tag{18a}$$

$$r_i\beta_i - m_i + \frac{\rho_j^2}{c_j}(\beta_i - \gamma_i)(\beta_j - \gamma_j) = 0, \tag{18b}$$

$$r_i\gamma_i - \frac{\rho_i^2}{2c_i}(\beta_i - \gamma_i)^2 = 0\,. \tag{18c}$$

From Proposition 1(a), each firm's brand advertising is increasing in its competitor's sales. Given that the purpose of brand advertising is to capture market share from the rival, this result is intuitive because the smaller firm has a larger target of potential customers. The expression for the generic advertising decision in Proposition 1(b) suggests that the optimal generic advertising spending is constant. This finding of stationary solutions is consistent with the analysis of an infinite horizon setting (Horsky and Mate 1988; Villas-Boas 1999).

In an expanding market, an increase in brand advertising by one firm prompts an increase in brand advertising by the rival. Consistent with this finding, Erickson (1985, 1992) reports both market expansion and increasing brand advertising expenditures by Anheuser-Busch and Miller in the beer industry. This pattern of increasing brand advertising spending is also seen in the "Cola Wars" between Coke and Pepsi (Chintagunta and Vilcassim 1992).

It is important to recognize that increasing brand advertising by both competitors is likely only if the total market is expanding, and that this effect would be absent in the case of a mature market. By ignoring market expansion, extant models of market share competition, which suggest decreasing advertising by one firm in response to the rival's increasing advertising, might lead to incorrect prescriptions, as noted by Roberts and Samuelson (1988). In a mature market, where advertising is primarily competitive, one firm's profit would increase in its sales, but decrease in the rival's sales.

In the next section, we solve the differential game in (13-14) to obtain the generic and brand advertising decisions for both symmetric and asymmetric firms.

6. Symmetric and Asymmetric Firms

Explicit solutions for the advertising decisions can be obtained by solving the set of simultaneous equations in (18a-c). Note that α_i, β_i, and γ_i are functions of the model parameters only and not of time. We first solve the problem for asymmetric competitors in Section 6.1, examine free-riding in Section 6.2, and then consider symmetric firms in Section 6.3.

6.1. Asymmetric Firms

The comparative statics for the parameters on the variables of interest are presented in Table 1.

<Insert Table 1 about here>

One can see that there are a few ambiguous effects because of the large number of parameters in the asymmetric case. However, for most cases, the results are clear. These include the increase of firm i's optimal advertising expenditures and value function with a decrease in its discount rate and an increase in its gross margin. In addition, the value function and the optimal generic advertising spending of the firm increase with an increase in the effectiveness of its generic advertising and the proportion of increase in its sales as a result of generic advertising.

For the competitive effects, note that an increase in the rival's advertising cost parameter, a decrease in the rival's gross margin, an increase in the effectiveness of the rival's generic advertising, an increase in the rival's discount rate, and a decrease in the proportion of the rival's sales as a result of generic advertising all increase the firm's value function. The increase of one firm's value function, a measure of profit, with the effectiveness of the rival's generic advertising can be seen as evidence of the free-riding of generic advertising expenditures – the greater the effort exerted by the rival in developing better generic advertising copy, the better off the first firm is likely to be. The parameter for effectiveness of generic advertising and the proportion of sales increase due to generic advertising do not have an effect on the optimal brand advertising policies.

From the optimal advertising expenditures, one can compute the sales trajectories of the two firms and their long-run equilibrium market shares, which are listed in the following proposition:

Proposition 2*: For the differential game given by equations* (13-14)*, the long-run market shares of the two firms are given by*

$$\left(\bar{x}_1 = \frac{\frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1)}{\frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1) + \frac{\rho_2^2}{c_2}(\beta_2 - \gamma_2)}, \quad \bar{x}_2 = \frac{\frac{\rho_2^2}{c_2}(\beta_2 - \gamma_2)}{\frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1) + \frac{\rho_2^2}{c_2}(\beta_2 - \gamma_2)} \right), \tag{19}$$

where β_i *and* γ_i *are obtained from Proposition* 1.

The form of the long-run equilibrium market shares in equation (19) resembles the market attraction models ($us/[us + them]$) encountered in the marketing literature. The market share attraction model given by (19) takes into account the effectiveness of brand advertising, the advertising cost parameter, and the value function parameters as measures of a firm's attractiveness. The equilibrium market share of the firm increases with that firm's brand advertising effectiveness and the rival's advertising cost parameter, and decreases with its advertising cost parameter and the effectiveness of the rival's brand advertising. Therefore, improving the effectiveness of advertising by developing better advertising copy has not only a short-term effect on the firm's sales, but also a long-run impact on the market share of the firm.

Note that the long run market shares are unaffected by the parameter θ_i, which captures the proportion of the marginal increase in market size obtained by firm i. This is because the market share calculation is driven primarily by the share of the existing market size and not by the share of the marginal increase (except during the introductory phase when the established market is quite small). The latter is determined by the brand characteristics and brand advertising.

6.2. Free-riding

Next, we determine the impact of free-riding or cheap-riding on the profitability of the two firms. Let us define the "stronger" firm as the one with more favorable model parameters, i.e., higher effectiveness of generic and brand advertising, higher gross margin, lower discount rate, lower advertising cost parameter, and higher proportion of sales increase from generic advertising, than the "weaker" firm. We numerically compute the equilibrium profits of the two firms to see whether the weaker firm can profit by taking advantage of the stronger firm's investment in generic advertising. It is seen that, for a wide range of parameter values, the weaker firm initially has higher profit than the stronger firm (Figure 1).

<Insert Figure 1 about here>

This is because the weaker firm benefits from the stronger firm's generic advertising spending, while it takes some time for the stronger firm to recoup its investment. However, in the long run, the stronger firm still wins out, implying that free-riding does not offer a long-term market share advantage to the weaker firm. This is referred to as the "big pig, little pig" problem (the game of "Boxed Pigs") wherein the "little pig" (market follower) profits from the efforts of the "big pig" (market leader). Campbell Soup's advertisements about the benefits of soup consumption and that soup can be used in casseroles and other dinner dishes, and commercials by IBM and De Beers illustrate this phenomenon.

Observation: The stronger firm should tolerate free-riding of generic advertising by the weaker firm. This free-riding has a great impact only on the stronger firm's short-run profitability but not on its long-run profitability. The stronger firm obtains less profit in the initial periods than the weaker firm because it is spending much more on generic advertising. However, this is not true in a de facto monopoly, i.e., the larger firm is always more profitable in this extreme case.

In a de facto monopoly, i.e., if one of the firms has near-total control of the market both in terms of its sales and its attractiveness as measured by the model parameters, we find that most of the generic advertising is done by the stronger firm. In this case, the weaker firm benefits from the stronger firm's generic advertising, but the gains from free-riding are tempered by the lower attractiveness of the weaker firm.

In the next section, we turn our attention to the case of symmetric competitors.

6.3. Symmetric Firms

For symmetric firms, the optimal advertising decisions and profit are given in Proposition 3.

Proposition 3*: For symmetric firms, for firm* i*, where* $i, j \in \{1, 2\}$*,* $i \neq j$*, the closed loop NE is characterized as follows:*

a) *Brand advertising:* $u_i^* = \frac{\rho}{c}(\beta - \gamma)\sqrt{S_j}$, (20)

b) *Generic advertising :* $a_i^* = \frac{k}{2c}(\beta + \gamma)$, (21)

c) *Profit:* $V_i = \alpha + \beta S_i + \gamma S_j$, (22)

where the parameters α , β , *and* γ *have the following explicit solutions:*

$$\alpha = \frac{k^2\left(c^2r^4 - 3cmr^2\rho^2 + 18m^2\rho^4 + (6m\rho^2 - cr^2)\sqrt{cr^2(cr^2 + 6m\rho^2)}\right)}{108cr^3\rho^4}, \quad (23a)$$

$$\beta = \frac{3m\rho^2 - 2cr^2 + 2\sqrt{cr^2(cr^2 + 6m\rho^2)}}{9r\rho^2}, \quad (23b)$$

$$\gamma = \frac{3m\rho^2 + cr^2 - \sqrt{cr^2(cr^2 + 6m\rho^2)}}{9r\rho^2}. \quad (23c)$$

The comparative statics are given in Table 2. The results in Table 2 indicate that an increase in the effectiveness of brand advertising, ρ , increases the optimal brand advertising spending of the firm. This is because an increase in ρ offers a greater incentive for the firm to spend more money on brand advertising. An increase in the brand advertising spending translates into a corresponding decrease in the generic advertising spending. In addition, an increase in ρ decreases the firm's value function. This is because both S_1 and S_2 appear in the firm's value function, and ρ increases the sales of the first firm by taking sales away from the second firm.

<Insert Table 2 about here>

The decrease of both generic and brand advertising spending with an increase in the advertising cost parameter, c , make sense economically. However, the firm's value function increases as c increases. This can again be attributed to the competitive effects of brand advertising in taking sales away from the rival. An increase in the gross margin, m, increases the optimal advertising expenditures because it results in increased revenue if advertising is increased. Although some of this advertising might be wasted, it can also increase the size of the pie, and, hence, the value function.

As the discount rate, r, increases, the optimal advertising expenditures decrease. A high value of r implies less weight on future profits, resulting in decreasing optimal advertising expenditures and the value function with an increase in the discount rate. In the limiting case, when the discount rate is extremely high, the firm acts myopically by trying to maximize only current period profits. As the effectiveness of generic advertising, k, increases, the optimal generic advertising spending also increases. However, brand advertising is unaffected by k because k increases the sales of both firms without greatly affecting market shares.

We now extend the above analysis to the extreme case of a pure monopoly to see how the generic advertising strategies for a monopolist would differ from those obtained in Section 6.1. The results are summarized in the following proposition:

Proposition 4*: When both brands are owned by the same firm, total generic advertising is higher than in the competitive case.*

Note that this case, which illustrates a firm's coordinated decision making with regard to the two types of advertising, is an example of category management for different brands in the firm's product line (Zenor 1994). The increase in generic advertising relative to the competitive case follows from the fact that there is no free-riding. Therefore, all the gains from generic advertising accrue to the monopolist.

7. Extensions

We consider a few extensions of the basic model to determine the robustness of the results. First, we incorporate the idea of a market potential to provide an upper bound for the market demand. In another extension, we relax the assumption that brand advertising is done solely for offensive reasons, to capture the rival's market share, and include the possibility of advertising to retain the firm's current clientele. We also examine an endogenous specification of θ, the proportion of the benefit from generic advertising accruing to one firm.

7.1. Market Potential

The primary demand model is modified by introducing a market potential $\bar{Q}$ as follows:

$$\dot{Q}(t) = \left(k_1 a_1(t) + k_2 a_2(t)\right)\sqrt{\bar{Q} - Q(t)}, \quad Q(0) = Q_0\,, \tag{24}$$

where the square root form resembles the Sethi (1979) model. As before, generic advertising by both the firms combines to increase the primary demand. The Sorger (1989) model is used for the market shares. If $x_i(t)$ is the market share of firm i, the state equation is

$$\dot{x}_i(t) = \rho_i u_i(t)\sqrt{1 - x_i(t)} - \rho_j u_j(t)\sqrt{x_i(t)}, \;\; x_i(0) = x_{i0}, \;\forall\; i,\; j = \{1, 2\},\; i \neq j. \tag{25}$$

In equation (25), the first term is the gain in market share due to firm i's brand advertising and the second term is the loss in market share due to firm j's brand advertising.

Firm i's objective is given by

$$\max_{u_i(t),\, a_i(t)} V_i = \int_0^\infty e^{-r_i t}\left(m_i x_i(t)\dot{Q}(t) - \frac{c_i}{2}\left(u_i(t)^2 + a_i(t)^2\right)\right)dt\,, \tag{26}$$

where the remaining notation is defined as before.

We write the differential game as

$$\max_{u_i(t),\, a_i(t)} V_i = \int_0^{\infty} e^{-r_i t}\left(m_i x_i(t)\left(k_i a_i(t) + k_j a_j(t)\right)\sqrt{\bar{Q} - Q(t)} - \frac{c_i}{2}\left(u_i(t)^2 + a_i(t)^2\right)\right)dt, \ \forall\ i,\ j \in \{1, 2\}, \quad (27)$$

$$\text{s.t.} \quad \dot{x}_i(t) = \rho_i u_i(t)\sqrt{1 - x_i(t)} - \rho_j u_j(t)\sqrt{x_i(t)}, \quad x_i(0) = x_{i0}, \quad \forall\ i,\ j \in \{1, 2\}, \quad (28)$$

$$\dot{Q}(t) = \left(k_1 a_1(t) + k_2 a_2(t)\right)\sqrt{\bar{Q} - Q(t)}, \quad Q(0) = Q_0. \quad (29)$$

The analysis of the closed-loop strategies using the Hamiltonian maximization is presented in the Appendix. The optimal advertising decisions are given by

$$u_1(t) = \frac{\rho_1}{c_1}\lambda_1(t)\sqrt{1 - x_1(t)}, \quad a_1(t) = \frac{k_1}{c_1}(m_1 x_1(t) + \lambda_2(t))\sqrt{\bar{Q} - Q(t)}, \quad (30)$$

$$u_2(t) = -\frac{\rho_2}{c_2}\mu_1(t)\sqrt{1 - x_2(t)}, \quad a_2(t) = \frac{k_2}{c_2}(m_2 x_2(t) + \mu_2(t))\sqrt{\bar{Q} - Q(t)}, \quad (31)$$

where the unknowns $\lambda_1(t)$, $\lambda_2(t)$, $\mu_1(t)$, and $\mu_2(t)$ and the trajectories $x_i(t)$ and $Q(t)$ are obtained by solving simultaneous differential equations given in the Appendix. Since these equations do not permit closed-form solutions, we use numerical methods. The solutions can then be used in (30-31) to obtain the optimal levels of brand and generic advertising investments.

To study the effect of market potential on the optimal advertising policies, extensive simulations were run for the two extreme cases – one in which almost none of the market potential has been realized and another in which there is near-total market penetration. In these simulations, we used a reasonably large finite horizon to approximate the infinite horizon.

The generic advertising is significantly greater when only a small fraction of the market potential has been realized than when most of the potential has been realized. This is because if most of the market is untapped, the two firms have a huge incentive to invest more money in generic advertising, and these investments decrease over time as the market becomes saturated. Since the larger firm has more to gain from expanding the market, it is the more aggressive generic advertiser.

As more of the market potential is tapped, the two firms decrease their brand advertising since the marginal returns to advertising are not as high as before on account of the relatively slow change in primary demand and market shares. Similarly, for the optimal generic advertising, we find that the firms decrease their generic advertising over time, and as the planning horizon increases, the generic advertising goes to zero.

The results for the two types of advertising strategies are very similar to those obtained earlier in the paper, with the smaller firm being the more aggressive advertiser with regard to brand advertising and the larger firm being more aggressive with regard to generic advertising. The decrease of advertising with decreasing potential sales or increasing market penetration is consistent with the findings of Fruchter (1999). In summary, the optimal generic and brand advertising strategies of the two firms depend on how much of the market potential has been realized, but the results from the earlier sections are robust with respect to the market potential.

7.2. Other Extensions

To model the fact that brand advertising can be used to retain market share (defensive advertising), assume that a proportion ω_i of firm i's brand advertising is spent purely for offensive reasons and $1-\omega_i$ is the proportion of brand advertising spent for defensive reasons. Moreover, let θ_i, the proportion of the benefit from primary demand expansion accruing to firm i, be endogenized in terms of the sales levels of the two firms as $\theta_i = \frac{S_i^\delta}{S_i^\delta + S_j^\delta}$, with $\delta < 1$ capturing diminishing returns. In other words, the firm with greater sales captures a greater fraction of the gain from market expansion.

The discounted profit maximization problem can now be rewritten as

$$\max_{u_1(t),\, a_1(t)} V_1 = \int_0^\infty e^{-r_1 t}\left(m_1 S_1(t) - \frac{c_1}{2}\left(a_1(t)^2 + u_1(t)^2\right)\right)dt,$$
$$\max_{u_2(t),\, a_2(t)} V_2 = \int_0^\infty e^{-r_2 t}\left(m_2 S_2(t) - \frac{c_2}{2}\left(a_2(t)^2 + u_2(t)^2\right)\right)dt, \tag{32}$$

$$\text{s.t.} \quad \dot{S}_1(t) = \rho_1 u_1(t)\left(\omega_1\sqrt{S_2(t)} + (1-\omega_1)\sqrt{S_1(t)}\right) - \rho_2 u_2(t)\left(\omega_2\sqrt{S_1(t)} + (1-\omega_2)\sqrt{S_2(t)}\right)$$
$$+\frac{S_1(t)^\delta}{S_1(t)^\delta + S_2(t)^\delta}\left(k_1 a_1(t) + k_2 a_2(t)\right), \quad S_1(0) = S_{10},$$
$$\dot{S}_2(t) = \rho_2 u_2(t)\left(\omega_2\sqrt{S_1(t)} + (1-\omega_2)\sqrt{S_2(t)}\right) - \rho_1 u_1(t)\left(\omega_1\sqrt{S_2(t)} + (1-\omega_1)\sqrt{S_1(t)}\right)$$
$$+\frac{S_2(t)^\delta}{S_1(t)^\delta + S_2(t)^\delta}\left(k_1 a_1(t) + k_2 a_2(t)\right), \quad S_2(0) = S_{20}. \tag{33}$$

The analysis is presented in the Appendix, and only the results from the simulation are discussed here. First, consider the effect of endogenizing the proportion of the benefit from generic advertising, θ_i, in terms of the sales of the two firms. From the numerical results, we find that as δ increases from 0 to 1 in the symmetric case, i.e., when θ_i changes from 0.5 to the market shares of the two firms, the larger firm allocates a greater chunk of its advertising budget toward generic advertising. This is because the larger firm gains proportionately more from generic advertising since the value of θ for the larger firm is always greater than 0.5. Consequently, the brand advertising spending of the larger firm decreases as δ increases. The smaller firm's strategies, on the other hand, are not greatly affected by this change in δ because the value of θ for the smaller firm is relatively low, so a greater fraction of the smaller firm's sales come as a result of its high brand advertising outlay.

Moving to the effect of defensive advertising on the optimal advertising decisions, the simulation reveals that as ω decreases from 1 to 0, i.e., when more of the firm's brand advertising budget is expended for defensive reasons (retaining its customers), both firms increase their brand advertising significantly, with the larger firm spending more money doing so because it has a greater share to defend. Corresponding to this increase in the brand advertising, the two firms decrease their generic advertising outlays. In the absence of defensive advertising, as in Section 4, the smaller firm would invest more in brand advertising since it has more to gain from the larger firm.

8. Conclusions

To increase sales of its brand, a firm can use generic advertising to expand the entire market or brand advertising to win market share. The benefits of generic advertising are conferred to all firms regardless of who contributed. As a result, the generic advertising strategy of a firm must be integrated with its brand advertising strategy, necessitating a thorough understanding of the relationship between the two. However, the market expansion role of advertising has been under-studied relative to its share expansion role.

This paper explicitly considers market expansion and market share effects. In it, we derive the closed-loop Nash equilibrium strategies for a dynamic duopoly where firms make decisions on generic and brand advertising. Explicit solutions are obtained for symmetric and asymmetric competitor scenarios. The effects of the model parameters on the optimal advertising policies and profits were computed. A general conclusion is that generic and brand advertising must be properly coordinated, and neglecting one of the two will lead to sub-optimal allocation of the advertising budget.

Analysis shows that generic advertising plays an important role over the entire product life cycle (Proposition 1). The optimal brand advertising can increase or decrease with time depending on the values of the model parameters. Normative results are that the firm’s profit and advertising decisions should increase with its gross margin and decrease with its discount rate

(Table 1). Moreover, they should increase with the rival's advertising cost parameter and its discount rate, and decrease with an increase in the competitor's gross margin and the effectiveness of its brand advertising.

We also examined free-riding in generic advertising and its effect on the long-run profitability of the two firms and found that although there is free-riding, the stronger firm would be better off tolerating this free-riding since this does not affect its long-term profitability greatly. We also found that in a monopoly, generic advertising is higher than in the competitive case.

Three extensions to the basic model were examined. The first dealt with the inclusion of a market potential, the second with generalizing the allocation of gains from generic advertising, and the third with brand advertising being used for defensive reasons. Numerical analyses of these extensions provide evidence of the robustness of the basic model.

There are several avenues for future research. In addition to advertising, pricing is an important decision variable in many markets and should be incorporated in an extended model (Teng and Thompson 1984). One may also extend the model to study advertising competition in an oligopoly along the lines of Erickson (1995) and Villas-Boas (1993). In addition, the comparative statics results can be tested empirically.

Appendix

Proof of Proposition 1

The Hamilton-Jacobi-Bellman (HJB) equation for firm i, $i, j \in \{1,2\}$, $i \neq j$, is given by

$$r_i V_i = \max_{u_i,\, a_i} \begin{cases} m_i S_i - \dfrac{c_i}{2}(a_i^2 + u_i^2) + \dfrac{\partial V_i}{\partial S_i}\left(\rho_i u_i \sqrt{S_j} - \rho_j u_j \sqrt{S_i} + \theta_i (k_i a_i + k_j a_j)\right) \\ + \dfrac{\partial V_i}{\partial S_j}\left(\rho_j u_j \sqrt{S_i} - \rho_i u_i \sqrt{S_j} + \theta_j (k_i a_i + k_j a_j)\right). \end{cases} \tag{A1}$$

From this, the first-order conditions for u_i and a_i yield, respectively,

$$u_i^* = \frac{\rho_i}{c_i}\left(\frac{\partial V_i}{\partial S_i} - \frac{\partial V_i}{\partial S_j}\right)\sqrt{S_j}, \quad a_i^* = \frac{k_i}{c_i}\left(\theta_i \frac{\partial V_i}{\partial S_i} + \theta_j \frac{\partial V_i}{\partial S_j}\right). \tag{A2}$$

Substituting (A2) into (A1) yields

$$\begin{aligned} r_i V_i = m_i S_i &+ \frac{k_i^2}{2c_i}\left(\theta_i \frac{\partial V_i}{\partial S_i} + \theta_j \frac{\partial V_i}{\partial S_j}\right)^2 + \frac{\rho_i^2}{2c_i}\left(\frac{\partial V_i}{\partial S_i} - \frac{\partial V_i}{\partial S_j}\right)^2 S_j \\ &- \frac{\rho_j^2}{c_j}\left(\frac{\partial V_i}{\partial S_i} - \frac{\partial V_i}{\partial S_j}\right)\left(\frac{\partial V_j}{\partial S_j} - \frac{\partial V_j}{\partial S_i}\right) S_i + \frac{k_j^2}{c_j}\left(\theta_i \frac{\partial V_i}{\partial S_i} + \theta_j \frac{\partial V_i}{\partial S_j}\right)\left(\theta_i \frac{\partial V_j}{\partial S_i} + \theta_j \frac{\partial V_j}{\partial S_j}\right). \end{aligned} \tag{A3}$$

The linear value function $V_i = \alpha_i + \beta_i S_i + \gamma_i S_j$ satisfies (A3). The optimal brand and generic advertising decisions in (A2) may now be rewritten as

$$u_i^* = \frac{\rho_i}{c_i}(\beta_i - \gamma_i)\sqrt{S_j},\ a_i^* = \frac{k_i}{c_i}(\theta_i \beta_i + \theta_j \gamma_i). \tag{A4}$$

Substituting $V_i = \alpha_i + \beta_i S_i + \gamma_i S_j$ and (A4) into (A3) and simplifying, we have

$$\begin{aligned} r_i \alpha_i + r_i \beta_i S_i + r_i \gamma_i S_j = m_i S_i &+ \frac{k_i^2}{2c_i}(\theta_i \beta_i + \theta_j \gamma_i)^2 + \frac{\rho_i^2}{2c_i}(\beta_i - \gamma_i)^2 S_j \\ &- \frac{\rho_j^2}{c_j}(\beta_i - \gamma_i)(\beta_j - \gamma_j) S_i + \frac{k_j^2}{c_j}(\theta_i \beta_i + \theta_j \gamma_i)(\theta_i \gamma_j + \theta_j \beta_j). \end{aligned} \tag{A5}$$

Equating the coefficients of S_i, S_j, and the constant in equation (A5) results in the following simultaneous equations to solve for α_i, β_i, and γ_i, $i, j \in \{1,2\}$, $i \neq j$:

$$r_i \alpha_i = \frac{k_i^2}{2c_i}(\theta_i \beta_i + \theta_j \gamma_i)^2 + \frac{k_j^2}{c_j}(\theta_i \beta_i + \theta_j \gamma_i)(\theta_i \gamma_j + \theta_j \beta_j), \tag{A6}$$

$$r_i\beta_i = m_i - \frac{\rho_j^2}{c_j}(\beta_i - \gamma_i)(\beta_j - \gamma_j), \tag{A7}$$

$$r_i\gamma_i = \frac{\rho_i^2}{2c_i}(\beta_i - \gamma_i)^2. \tag{A8}$$

Writing out these equations explicitly for $i = 1,\ j = 2$ and $i = 2,\ j = 1$, we get

$$r_1\alpha_1 = \frac{k_1^2}{2c_1}(\theta\beta_1 + (1-\theta)\gamma_1)^2 + \frac{k_2^2}{c_2}(\theta\beta_1 + (1-\theta)\gamma_1)(\theta\gamma_2 + (1-\theta)\beta_2), \tag{A9}$$

$$r_1\beta_1 = m_1 - \frac{\rho_2^2}{c_2}(\beta_1 - \gamma_1)(\beta_2 - \gamma_2), \tag{A10}$$

$$r_1\gamma_1 = \frac{\rho_1^2}{2c_1}(\beta_1 - \gamma_1)^2, \tag{A11}$$

$$r_2\alpha_2 = \frac{k_2^2}{2c_2}(\theta\beta_2 + (1-\theta)\gamma_2)^2 + \frac{k_1^2}{c_1}(\theta\beta_1 + (1-\theta)\gamma_1)(\theta\gamma_2 + (1-\theta)\beta_2), \tag{A12}$$

$$r_2\beta_2 = m_2 - \frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1)(\beta_2 - \gamma_2), \tag{A13}$$

$$r_2\gamma_2 = \frac{\rho_2^2}{2c_2}(\beta_2 - \gamma_2)^2. \tag{A14}$$

To solve, subtract (A11) from (A10) and (A14) from (A13). Let $y = \beta_1 - \gamma_1$ and $z = \beta_2 - \gamma_2$. Upon simplifying, we have the following simultaneous quadratic equations in y and z:

$$\frac{\rho_1^2}{2c_1}y^2 + \frac{\rho_2^2}{c_2}yz + r_1 y - m_1 = 0, \tag{A15}$$

$$\frac{\rho_2^2}{2c_2}z^2 + \frac{\rho_1^2}{c_1}yz + r_2 z - m_2 = 0. \tag{A16}$$

Substituting for z in terms of y in equation (A16) after solving for z in (A15) yields the following quartic equation in y:

$$\eta_1 y^4 + \eta_2 y^3 + \eta_3 y^2 + \eta_4 y - \eta_5 = 0, \tag{A17}$$

where $\eta_1 = 3c_2\rho_1^4$, $\eta_2 = 4c_1c_2\rho_1^2(r_1 + r_2)$, $\eta_3 = 4c_1\left(c_1c_2(2r_1r_2 - r_1^2) + 2c_1m_2\rho_2^2 - c_2m_1\rho_1^2\right)$, $\eta_4 = 8c_1^2c_2m_1(r_1 - r_2)$, and $\eta_5 = 4c_1^2c_2m_1^2$.

The roots of equation (A17) can be computed using *Mathematica v4.0*. Of the four solutions for y, two are imaginary, one is negative, and only one is always real and positive. This is the unique Nash equilibrium of the differential game. That solution is given by

$$y = -\frac{\eta_2}{4\eta_1} + \frac{1}{2}\left(\sqrt{\omega_1 + \omega_5 + \omega_6} + \sqrt{2\omega_1 - \omega_5 - \omega_6 + \frac{\omega_7}{4\sqrt{\omega_1 + \omega_5 + \omega_6}}}\right), \quad \text{(A18)}$$

where $\omega_1 = \frac{\eta_2^2}{4\eta_1^2} - \frac{2\eta_3}{3\eta_1}$, $\omega_2 = \eta_3^2 - 3\eta_2\eta_4 - 12\eta_1\eta_5$, $\omega_3 = 2\eta_3^3 - 9\eta_2\eta_3\eta_4 + 27\eta_1\eta_4^2 - 27\eta_5\eta_2^2 + 72\eta_1\eta_3\eta_5$,

$\omega_4 = (\omega_3 + \sqrt{\omega_3^2 - 4\omega_2^3})^{1/3}$, $\omega_5 = \frac{2^{1/3}\omega_2}{3\eta_1\omega_4}$, $\omega_6 = \frac{\omega_4}{32^{1/3}\eta_1}$, and $\omega_7 = \frac{4\eta_2\eta_3}{\eta_1^2} - \frac{\eta_2^3}{\eta_1^3} - \frac{8\eta_4}{\eta_1}$.

Knowing y, we can compute the following:

$$\gamma_1 = \frac{\rho_1^2}{2c_1r_1}y^2, \ \beta_1 = y + \frac{\rho_1^2}{2c_1r_1}y^2, \quad \text{(A19)}$$

$$z = \frac{c_2}{\rho_2^2 y}\left(m_1 - \frac{\rho_1^2}{2c_1}y^2 - r_1 y\right), \ \gamma_2 = \frac{\rho_2^2}{2c_2r_2}z^2, \ \beta_2 = \frac{\rho_2^2}{2c_2r_2}z^2 + z. \quad \text{(A20)}$$

One can see from (A19-20) that $\beta_i > 0$, $\gamma_i > 0$, and $\beta_i > \gamma_i$, resulting in positive values for the controls.

<u>Comparative Statics</u>

Due to space constraints, the proofs for the comparative statics (Tables 1 and 2) are available upon request from the authors.

<u>Proof of Proposition 2</u>

To derive the optimal sales paths, we substitute the results from Proposition 1 into the two state equations to obtain the following system of differential equations:

$$\begin{aligned}
\dot{S}_1 &= \frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1)S_2 - \frac{\rho_2^2}{c_2}(\beta_2 - \gamma_2)S_1 + \theta\left(\frac{k_1^2}{c_1}(\theta\beta_1 + (1-\theta)\gamma_1) + \frac{k_2^2}{c_2}(\theta\gamma_2 + (1-\theta)\beta_2)\right), \ S_1(0) = S_{10}, \\
\dot{S}_2 &= \frac{\rho_2^2}{c_2}(\beta_2 - \gamma_2)S_1 - \frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1)S_2 + (1-\theta)\left(\frac{k_1^2}{c_1}(\theta\beta_1 + (1-\theta)\gamma_1) + \frac{k_2^2}{c_2}(\theta\gamma_2 + (1-\theta)\beta_2)\right), \ S_2(0) = S_{20}.
\end{aligned} \quad \text{(B1)}$$

For expositional ease, denote

$$\psi_1 = \frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1),\ \psi_2 = \frac{\rho_2^2}{c_2}(\beta_2 - \gamma_2),\ \psi_3 = \frac{k_1^2}{c_1}(\theta\beta_1 + (1-\theta)\gamma_1) + \frac{k_2^2}{c_2}(\theta\gamma_2 + (1-\theta)\beta_2). \quad \text{(B2)}$$

The differential equations can now be rewritten as

$$\begin{aligned} \dot{S}_1 &= \psi_1 S_2 - \psi_2 S_1 + \theta\psi_3, \quad S_1(0) = S_{10}, \\ \dot{S}_2 &= \psi_2 S_1 - \psi_1 S_2 + (1-\theta)\psi_3, \quad S_2(0) = S_{20}. \end{aligned} \quad \text{(B3)}$$

Note from (B3) that there is no long-run equilibrium in sales, i.e., $\dot{S}_1$ and $\dot{S}_2$ need not go to zero. The long-run equilibrium market shares resulting from the equations in (B3) are given by

$$\left(\bar{x}_1 = \lim_{t\to\infty} \frac{S_1(t)}{S_1(t) + S_2(t)}, \quad \bar{x}_2 = \lim_{t\to\infty} \frac{S_2(t)}{S_1(t) + S_2(t)} \right). \quad \text{(B4)}$$

Simplifying, we have

$$\left(\bar{x}_1 = \frac{\frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1)}{\frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1) + \frac{\rho_2^2}{c_2}(\beta_2 - \gamma_2)}, \quad \bar{x}_2 = \frac{\frac{\rho_2^2}{c_2}(\beta_2 - \gamma_2)}{\frac{\rho_1^2}{c_1}(\beta_1 - \gamma_1) + \frac{\rho_2^2}{c_2}(\beta_2 - \gamma_2)} \right). \quad \text{(B5)}$$

<u>Proof of Proposition 3</u>

To solve the simultaneous equations:

$$r\alpha = \frac{3k^2}{8c}(\beta + \gamma)^2, \quad \text{(B6)}$$

$$r\beta = m - \frac{\rho^2}{c}(\beta - \gamma)^2, \quad \text{(B7)}$$

$$r\gamma = \frac{\rho^2}{2c}(\beta - \gamma)^2, \quad \text{(B8)}$$

multiply (B8) by 2 and add to (B7). This yields $\beta = \frac{m}{r} - 2\gamma.$ Substituting this into (B8) yields the following quadratic equation in γ:

$$\frac{\rho^2}{2c}(\frac{m}{r} - 3\gamma)^2 - r\gamma = 0 \quad \Rightarrow 9r^2\rho^2\gamma^2 - 2r(3m\rho^2 + cr^2)\gamma + m^2\rho^2 = 0. \quad \text{(B9)}$$

The solutions of this quadratic equation are

$$\gamma = \frac{3m\rho^2 + cr^2 \pm \sqrt{cr^2(cr^2 + 6m\rho^2)}}{9r\rho^2}. \quad \text{(B10)}$$

To find out which of the two roots to choose, we use the test that $\gamma = 0$ when $m = 0$. This is because the value function should be identically equal to zero when the gross margin is zero since the firm makes zero profit in this case. Checking with (B10), it is easy to see that

$$\gamma = \frac{3m\rho^2 + cr^2 - \sqrt{cr^2(cr^2 + 6m\rho^2)}}{9r\rho^2} \tag{B11}$$

is the only root that satisfies this condition.

Knowing γ, we can compute β and α using $\beta = \frac{m}{r} - 2\gamma$ and (B6), respectively. We have

$$\beta = \frac{3m\rho^2 - 2cr^2 + 2\sqrt{cr^2(cr^2 + 6m\rho^2)}}{9r\rho^2}, \tag{B12}$$

$$\alpha = \frac{k^2\left(c^2r^4 - 3cmr^2\rho^2 + 18m^2\rho^4 + (6m\rho^2 - cr^2)\sqrt{cr^2(cr^2 + 6m\rho^2)}\right)}{108cr^3\rho^4}. \tag{B13}$$

An examination of equations (B11-13) reveals that $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\beta > \gamma$, so the controls and the value functions are positive.

<u>Proof of Proposition 4</u>

If the firm owns both brands, its decision problem is

$$\max_{u_1(t),\, a_1(t),\, u_2(t),\, a_2(t)} V = \int_0^\infty e^{-rt}\left(m_1S_1(t) + m_2S_2(t) - \frac{c_1}{2}\left(a_1(t)^2 + u_1(t)^2\right) - \frac{c_2}{2}\left(a_2(t)^2 + u_2(t)^2\right)\right)dt, \tag{B14}$$

$$\text{s.t.} \quad \begin{aligned} \dot{S}_1(t) &= \rho_1u_1(t)\sqrt{S_2(t)} - \rho_2u_2(t)\sqrt{S_1(t)} + \theta\left(k_1a_1(t) + k_2a_2(t)\right), \quad S_1(0) = S_{10}, \\ \dot{S}_2(t) &= \rho_2u_2(t)\sqrt{S_1(t)} - \rho_1u_1(t)\sqrt{S_2(t)} + (1-\theta)\left(k_1a_1(t) + k_2a_2(t)\right), \quad S_2(0) = S_{20}, \end{aligned} \tag{B15}$$

where V is the value function of the firm, a_i and u_i are the generic and brand advertising decisions, respectively, of brand i, S_i is the sales of brand i, c_i is the advertising cost parameter and m_i the margin of brand i, and r is the firm's discount rate. The rest of the notation is as described earlier.

The HJB equation is

$$rV=\max_{u_1,\,a_1,\,u_2,\,a_2}\begin{cases} m_1S_1+m_2S_2-\dfrac{c_1}{2}(u_1^2+a_1^2)-\dfrac{c_2}{2}(u_2^2+a_2^2) \\ +\dfrac{\partial V}{\partial S_1}\left(\rho_1u_1\sqrt{S_2}-\rho_2u_2\sqrt{S_1}+\theta(k_1a_1+k_2a_2)\right) \\ +\dfrac{\partial V}{\partial S_2}\left(\rho_2u_2\sqrt{S_1}-\rho_1u_1\sqrt{S_2}+(1-\theta)(k_1a_1+k_2a_2)\right). \end{cases} \quad \text{(B16)}$$

The first-order conditions yield

$$u_1^*=\frac{\rho_1}{c_1}\left(\frac{\partial V}{\partial S_1}-\frac{\partial V}{\partial S_2}\right)\sqrt{S_2},\quad a_1^*=\frac{k_1}{c_1}\left(\theta\frac{\partial V}{\partial S_1}+(1-\theta)\frac{\partial V}{\partial S_2}\right), \quad \text{(B17)}$$

$$u_2^*=\frac{\rho_2}{c_2}\left(\frac{\partial V}{\partial S_2}-\frac{\partial V}{\partial S_1}\right)\sqrt{S_1},\quad a_2^*=\frac{k_2}{c_2}\left(\theta\frac{\partial V}{\partial S_1}+(1-\theta)\frac{\partial V}{\partial S_2}\right). \quad \text{(B18)}$$

As before, substituting the solutions from (B17-18) into (B16) suggests that a linear value function $V=\alpha_m+\beta_mS_1+\gamma_mS_2$ will solve the resulting partial differential equation. The optimal advertising decisions can now be rewritten as

$$u_1^*=\max\{0,\ \frac{\rho_1}{c_1}(\beta_m-\gamma_m)\sqrt{S_2}\},\quad a_1^*=\frac{k_1}{c_1}(\theta\beta_m+(1-\theta)\gamma_m), \quad \text{(B19)}$$

$$u_2^*=\max\{0,\ \frac{\rho_2}{c_2}(\gamma_m-\beta_m)\sqrt{S_1}\},\quad a_2^*=\frac{k_2}{c_2}(\theta\beta_m+(1-\theta)\gamma_m). \quad \text{(B20)}$$

The monopolist can choose the optimal advertising decisions to ensure the value function in the monopoly case can never be less than that in the competitive one. In other words, $V\geq V_1+V_2$, where V_1 and V_2 are the profits in the competitive case. We, therefore, have

$$\alpha_m+\beta_mS_1+\gamma_mS_2\geq\alpha_1+\beta_1S_1+\gamma_1S_2+\alpha_2+\gamma_2S_1+\beta_2S_2. \quad \text{(B21)}$$

Equation (B21) can be rewritten to yield

$$\alpha_m-(\alpha_1+\alpha_2)+\left(\beta_m-(\beta_1+\gamma_2)\right)S_1+\left(\gamma_m-(\beta_2+\gamma_1)\right)S_2\geq 0. \quad \text{(B22)}$$

Since equation (B22) holds $\forall\ S_1\geq 0,\ S_2\geq 0$, it must be the case that

$$\alpha_m\geq(\alpha_1+\alpha_2),\ \beta_m\geq\beta_1+\gamma_2,\ \gamma_m\geq\beta_2+\gamma_1, \quad \text{(B23)}$$

where each of the above coefficients is non-negative.

From equation (A4), the total generic advertising in the competitive case is

$$\frac{k_1}{c_1}(\theta\beta_1+(1-\theta)\gamma_1)+\frac{k_2}{c_2}(\theta\gamma_2+(1-\theta)\beta_2), \quad \text{(B24)}$$

while that in the monopoly case is, from equations (B19-20),

$$\frac{k_1}{c_1}(\theta\beta_m+(1-\theta)\gamma_m)+\frac{k_2}{c_2}(\theta\beta_m+(1-\theta)\gamma_m). \tag{B25}$$

Subtracting equation (B24) from (B25), the difference between the total generic advertising in the monopoly case and that in the competitive case is

$$\frac{k_1}{c_1}(\theta(\beta_m-\beta_1)+(1-\theta)(\gamma_m-\gamma_1))+\frac{k_2}{c_2}(\theta(\beta_m-\gamma_2)+(1-\theta)(\gamma_m-\beta_2)), \tag{B26}$$

which, from equation (B23), is greater than zero. Therefore, the monopolist's total generic advertising is greater than that under competition.

Extension: Market Potential

The current-value Hamiltonian for firm i, $i, j \in \{1,2\}$, $i \neq j$, is

$$\begin{aligned} H_i &= m_i(k_i a_i + k_j a_j)x_i\sqrt{\bar{Q}-Q}-\frac{c_i}{2}(u_i^2+a_i^2) \\ &\quad +\lambda_i\left(\rho_i u_i\sqrt{1-x_i}-\rho_j u_j\sqrt{x_i}\right)+\mu_i\left((k_i a_i+k_j a_j)\sqrt{\bar{Q}-Q}\right). \end{aligned} \tag{C1}$$

The first-order conditions for firm i are

$$\frac{\partial H_i}{\partial u_i}=-c_i u_i+\rho_i\lambda_i\sqrt{1-x_i}=0, \tag{C2}$$

$$\frac{\partial H_i}{\partial a_i}=m_i k_i x_i\sqrt{\bar{Q}-Q}-c_i a_i+k_i\mu_i\sqrt{\bar{Q}-Q}=0. \tag{C3}$$

Using equations (C2-3), one can write

$$u_i(t)=\frac{\rho_i}{c_i}\lambda_i(t)\sqrt{1-x_i(t)}, \quad a_i(t)=\frac{k_i}{c_i}\left(m_i x_i(t)+\mu_i(t)\right)\sqrt{\bar{Q}-Q(t)}. \tag{C4}$$

The closed-loop adjoint equations for the shadow prices $\lambda_i(t)$ and $\mu_i(t)$ for firm i are

$$\begin{aligned} \dot{\lambda}_i &= r_i\lambda_i-\frac{\partial H_i}{\partial x_i}-\frac{\partial H_i}{\partial u_j}\frac{\partial u_j}{\partial x_i}-\frac{\partial H_i}{\partial a_j}\frac{\partial a_j}{\partial x_i} \\ &= r_i\lambda_i-m_i(k_i a_i+k_j a_j)\sqrt{\bar{Q}-Q}+\mu_i\left(\frac{\rho_i u_i}{2\sqrt{1-x_i}}+\frac{\rho_j u_j}{2\sqrt{x_i}}\right)-\frac{\rho_j^2}{2c_j}\lambda_i\lambda_j+\frac{k_j^2}{c_j}m_j(m_i x_i+\mu_i)(\bar{Q}-Q), \end{aligned} \tag{C5}$$

$$\dot{\mu}_i = r_i\mu_i - \frac{\partial H_i}{\partial Q} - \frac{\partial H_i}{\partial u_j}\frac{\partial u_j}{\partial Q} - \frac{\partial H_i}{\partial a_j}\frac{\partial a_j}{\partial Q}$$
$$= r_i\mu_i + (m_i x_i + \mu_i)(k_i a_i + k_j a_j) + \frac{k_j^2}{2c_j}(m_i x_i + \mu_i)(m_j x_j + \mu_j). \tag{C6}$$

Rewriting the above equations by substituting the optimal advertising decisions, we have

$$\dot{x}_i = \frac{\rho_i^2}{c_i}\lambda_i x_j + \frac{\rho_j^2}{c_j}\lambda_j x_i, \quad x_i(0) = x_{i0}, \tag{C7}$$

$$\dot{Q} = \Big(\frac{k_i^2}{c_i}(m_i x_i + \mu_i) + \frac{k_j^2}{c_j}(m_j x_j + \mu_j)\Big)(\bar{Q} - Q), \quad Q(0) = Q_0, \tag{C8}$$

$$\dot{\lambda}_i = r_i\lambda_i - m_i\Big(\frac{k_i^2}{c_i}(m_i x_i + \mu_i) + \frac{k_j^2}{c_j}(m_j x_j + \mu_j)\Big)(\bar{Q} - Q)$$
$$+ \lambda_i\Big(\frac{\rho_i^2}{2c_i}\lambda_i - \frac{\rho_j^2}{c_j}\lambda_j\Big) + \frac{k_j^2}{c_j}m_j(m_i x_i + \mu_i)(\bar{Q} - Q), \tag{C9}$$

$$\dot{\mu}_i = r_i\mu_i + \frac{1}{2}(m_i x_i + \mu_i)\Big(\frac{k_i^2}{c_i}(m_i x_i + \mu_i) + \frac{k_j^2}{c_j}(m_j x_j + \mu_j)\Big)$$
$$+ \frac{k_j^2}{2c_j}(m_i x_i + \mu_i)(m_j x_j + \mu_j). \tag{C10}$$

We use numerical methods to solve for $x_i(t)$, $Q(t)$, $\lambda_i(t)$, and $\mu_i(t)$, using a reasonably large finite horizon to approximate the infinite horizon.

Other Extensions

The current-value Hamiltonian for firm i, $i, j \in \{1, 2\}$, $i \neq j$, is

$$H_i = m_i S_i - \frac{c_i}{2}(a_i^2 + u_i^2)$$
$$+ \lambda_i\Big(\rho_i u_i\big(\omega_i\sqrt{S_j} + (1-\omega_i)\sqrt{S_i}\big) - \rho_j u_j\big(\omega_j\sqrt{S_i} + (1-\omega_i)\sqrt{S_j}\big) + \frac{S_i^\delta}{S_i^\delta + S_i^\delta}(k_i a_i + k_j a_j)\Big)$$
$$+ \mu_i\Big(\rho_j u_j\big(\omega_j\sqrt{S_i} + (1-\omega_i)\sqrt{S_j}\big) - \rho_i u_i\big(\omega_i\sqrt{S_j} + (1-\omega_i)\sqrt{S_i}\big) + \frac{S_j^\delta}{S_i^\delta + S_i^\delta}(k_i a_i + k_j a_j)\Big). \tag{C11}$$

The first-order conditions for firm i are

$$\frac{\partial H_i}{\partial u_i} = -c_i u_i + \lambda_i\left((\omega_i\sqrt{S_j} + (1-\omega_i)\sqrt{S_i})\right) - \mu_i\left((\omega_i\sqrt{S_j} + (1-\omega_i)\sqrt{S_i})\right) = 0, \quad \text{(C12)}$$

$$\frac{\partial H_i}{\partial a_i} = -c_i a_i + \frac{k_i\lambda_i S_i^\delta}{S_i^\delta + S_j^\delta} + \frac{k_i\mu_i S_j^\delta}{S_i^\delta + S_j^\delta} = 0. \quad \text{(C13)}$$

Rewriting equations (C12-13) to obtain the optimal advertising decisions yields

$$u_i = \frac{\rho_i}{c_i}(\omega_i\sqrt{S_j} + (1-\omega_i)\sqrt{S_i})(\lambda_i - \mu_i), \quad a_i = \frac{k_i}{c_i}(\frac{\lambda_i S_i^\delta}{S_i^\delta + S_j^\delta} + \frac{\mu_i S_j^\delta}{S_i^\delta + S_j^\delta}). \quad \text{(C14)}$$

The adjoint equations for the shadow prices $\lambda_i(t)$ and $\mu_i(t)$ are

$$\begin{aligned}\dot{\lambda}_i &= r_i\lambda_i - \frac{\partial H_i}{\partial S_i} - \frac{\partial H_i}{\partial u_j}\frac{\partial u_j}{\partial S_i} - \frac{\partial H_i}{\partial a_j}\frac{\partial a_j}{\partial S_i} \\ &= r_i\lambda_i - m_i - (\frac{\rho_i(1-\omega_i)u_i - \rho_j\omega_j u_j}{2\sqrt{S_i}} + \frac{(k_i a_i + k_j a_j)\delta S_j^\delta}{S_i^{1-\delta}(S_i^\delta + S_j^\delta)^2})(\lambda_i - \mu_i) \\ &\quad + \frac{1}{c_j}\left(\rho_j(\omega_j\sqrt{S_i} + (1-\omega_j)\sqrt{S_j})(\lambda_i - \mu_i)\right)(\frac{\rho_j\omega_j}{2\sqrt{S_i}})(\mu_j - \lambda_j) \\ &\quad + \frac{k_j^2}{c_j}(\frac{\lambda_i S_i^\delta}{S_i^\delta + S_j^\delta} + \frac{\mu_i S_j^\delta}{S_i^\delta + S_j^\delta})(\frac{\delta S_j^\delta}{S_i^{1-\delta}(S_i^\delta + S_j^\delta)^2})(\mu_j - \lambda_j),\end{aligned} \quad \text{(C15)}$$

$$\begin{aligned}\dot{\mu}_i &= r_i\mu_i - \frac{\partial H_i}{\partial S_j} - \frac{\partial H_i}{\partial u_j}\frac{\partial u_j}{\partial S_j} - \frac{\partial H_i}{\partial a_j}\frac{\partial a_j}{\partial S_j} \\ &= r_i\mu_i - (\frac{\rho_i\omega_i u_i - \rho_j(1-\omega_j)u_j}{2\sqrt{S_j}} + \frac{(k_i a_i + k_j a_j)\delta S_i^\delta}{S_j^{1-\delta}(S_i^\delta + S_j^\delta)^2})(\lambda_i - \mu_i) \\ &\quad + \frac{\rho_j}{c_j}\left((\omega_j\sqrt{S_i} + (1-\omega_i)\sqrt{S_j})(\lambda_i - \mu_i)\right)(\frac{\rho_j(1-\omega_j)}{2\sqrt{S_j}})(\mu_j - \lambda_j) \\ &\quad - \frac{k_j^2}{c_j}(\frac{\lambda_i S_i^\delta}{S_i^\delta + S_j^\delta} + \frac{\mu_i S_j^\delta}{S_i^\delta + S_j^\delta})(\frac{\delta S_i^\delta}{S_j^{1-\delta}(S_i^\delta + S_j^\delta)^2})(\mu_j - \lambda_j).\end{aligned} \quad \text{(C16)}$$

Since these differential equations do not permit closed-form solutions, we will have to use numerical methods to solve for $x_i(t)$, $Q(t)$, $\lambda_i(t)$, and $\mu_i(t)$. The solutions can then be used in (C14) to obtain the optimal brand and generic advertising decisions for the two firms.

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Table 1: Comparative Statics Results for the Asymmetric Case

Variables	c_i	c_j	m_i	m_j	k_i	k_j	ρ_i	ρ_j	r_i	r_j	θ_i	θ_j
α_i	?	↑	↑	↓	↑	↑	?	↓	↓	↑	↑	↓
β_i	?	↑	↑	↓	=	=	?	↓	↓	↑	=	=
γ_i	?	↑	↑	↓	=	=	?	↓	↓	↑	=	=
u_i^*	↓	↑	↑	↓	=	=	↑	↓	↓	↑	=	=
a_i^*	↓	↑	↑	↓	↑	=	?	↓	↓	↑	↑	↓
V_i	?	↑	↑	↓	↑	↑	?	↓	↓	↑	↑	↓

Legend: ↑ increase; ↓ decrease; = unchanged; ? ambiguous.

Table 2: Comparative Statics Results for the Symmetric Case

Variables	c	m	k	ρ	r
α	↑	↑	↑	↓	↓
β	↑	↑	=	↓	↓
γ	↓	↑	=	↑	↓
u_i^*	↓	↑	=	↑	↓
a_i^*	↓	↑	↑	↓	↓
V_i	↑	↑	↑	↓	↓

Legend: ↑ increase; ↓ decrease; = unchanged; ? ambiguous.

Figure 1: Profits of the Two Firms

$(r_1 = 0.05, r_2 = 0.05, m_1 = 1.5, m_2 = 1.5, \rho_1 = 0.2, \rho_2 = 0.3, c_1 = 0.4, c_2 = 0.4, k_1 = 0.7, k_2 = 0.8,$

$\theta = 0.4,\ S_{10} = 100, S_{20} = 200)$