Chapter 10

Applications to Natural Resources

The rapid increase of world population is causing a corresponding increase in the demand for consumption of natural resources. As a consequence the optimal management and utilization of natural resources is becoming increasingly important. There are two main kinds of natural resource models:

- Renewable resources such as fish, food, timber, etc.

  Section 10.1: a fishery resource model.

  Section 10.2: an optimal forest thinning model.

- Nonrenewable or exhaustible resources such as petroleum, minerals, etc.

  Section 10.3: an exhaustible resource model.

We introduce the following notation and terminology:

$$
\begin{aligned}
\rho &= \text{the discount rate,} \\
x(t) &= \text{the biomass of fish population at time } t, \\
g(x) &= \text{the natural growth function,} \\
u(t) &= \text{the rate of fishing effort at time } t;\ 0 \le u \le U, \\
q &= \text{the catchability coefficient,} \\
p &= \text{the unit price of landed fish,} \\
c &= \text{the unit cost of effort.}
\end{aligned}
$$

Assume that the growth function g is differentiable and concave, and it satisfies

$$g(0) = 0, \quad g(X) = 0, \quad g(x) > 0 \ \text{ for } \ 0 < x < X, \tag{1}$$

where X denotes the *carrying capacity*, i.e., the maximum sustainable fish biomass.

The model equation due to Gordon (1954) and Schaefer (1957) is

$$\dot{x} = g(x) - qux, \quad x(0) = x_0 \tag{2}$$

and the instantaneous profit rate is

$$\pi(x, u) = (pqx - c)u. \tag{3}$$

From (1) and (2), it follows that x will stay in the closed interval $0 \leq x \leq X$ provided x_0 is in the same interval.

An *open access fishery* is one in which exploitation is completely uncontrolled. Gordon (1954) analyzed this model, also known as the Gordon-Schaefer model, and showed that the fishing effort tends to reach an equilibrium, called *bionomic equilibrium*, at the level at which total revenue equals total cost. In other words, the so-called economic rent is completely dissipated. From (3) and (2), this level is simply

$$x_b = \frac{c}{pq} \quad \text{and} \quad u_b = \frac{g(x_b)p}{c}. \tag{4}$$

We assume $u_b \leq U$. The economic basis for this result is as follows: If the fishing effort $u > u_b$ is made, then total costs exceed total revenues so that at least some fishermen will lose money, and eventually some will drop out, thus reducing the level of fishing effort. On the other hand, if fishing effort $u < u_b$ is made, then total revenues exceed total costs, thereby attracting additional fishermen, and increasing the fishing effort.

The Gordon-Schaefer model is a static model which does not (in general) maximize the present value of the total profits which can be obtained from the fish resources.

The bionomic equilibrium solution obtained from the open access fishery model usually implies severe biological overfishing. Suppose a fishing regulatory agency is established to improve the operation of the fishing industry. In determining the objective of the agency, it is convenient to think of it as a sole owner who has complete rights to exploit the fishing resource. It is reasonable to assume that the agency attempts to

$$\text{maximize}\left\{J = \int_0^\infty e^{-\rho t}(pqx - c)u\,dt\right\} \tag{5}$$

subject to (2).

The solution method presented in this section generalizes the one based on Green's theorem used in Section 7.2.2. Solving (2) for u we obtain

$$u = \frac{g(x) - \dot{x}}{qx}, \tag{6}$$

which we substitute into (3) to get

$$J = \int_0^\infty e^{-\rho t}(pqx - c)\frac{g(x) - \dot{x}}{qx}dt. \tag{7}$$

Rewriting, we have

$$J = \int_0^\infty e^{-\rho t}[M(x) + N(x)\dot{x}]dt, \tag{8}$$

where

$$N(x) = -p + \frac{c}{qx} \text{ and } M(x) = (p - \frac{c}{qx})g(x). \tag{9}$$

We note that we can write $\dot{x}dt = dx$ so that (8) becomes the following line integral

$$J_B = \int_B [e^{-\rho t} M(x) dt + e^{-\rho t} N(x) dx], \tag{10}$$

where B is a state trajectory in the (x, t) space, $t \in [0, \infty)$.

In this section we are only interested in the infinite horizon solution. The Green's theorem method achieves such a solution by first solving a finite horizon problem as in Section 7.2.2, and then determining the infinite horizon solution for which you are asked to verify that the maximum principle holds in Exercise 10.1. See also Sethi (1977b).

In order to apply Green's Theorem to (10), let Γ denote a simple closed curve in the (x, t) space surrounding a region R in the space. Then,

$$\begin{aligned} J_\Gamma &= \oint_\Gamma [e^{-\rho t} M(x) dt + e^{-\rho t} N(x) dx] \\ &= \int\!\!\int_R \left\{ \frac{\partial}{\partial t} [e^{-\rho t} N(x)] - \frac{\partial}{\partial x} [e^{-\rho t} M(x)] \right\} dt dx \\ &= \int\!\!\int_R -e^{-\rho t} [\rho N(x) + M'(x)] dt dx. \end{aligned} \tag{11}$$

If we let

$$I(x) = -[\rho N(x) + M'(x)],$$

we can rewrite (11) as

$$J_\Gamma = \iint_R e^{-\rho t} I(x) dt dx.$$

We can now conclude, as we did in Sections 7.2.2 and 7.2.4, that the turnpike level $\bar{x}$ is given by setting the integrand of (11) to 0. That is,

$$-I(x) = [g'(x) - \rho](p - \frac{c}{qx}) + \frac{cg(x)}{qx^2} = 0. \qquad (12)$$

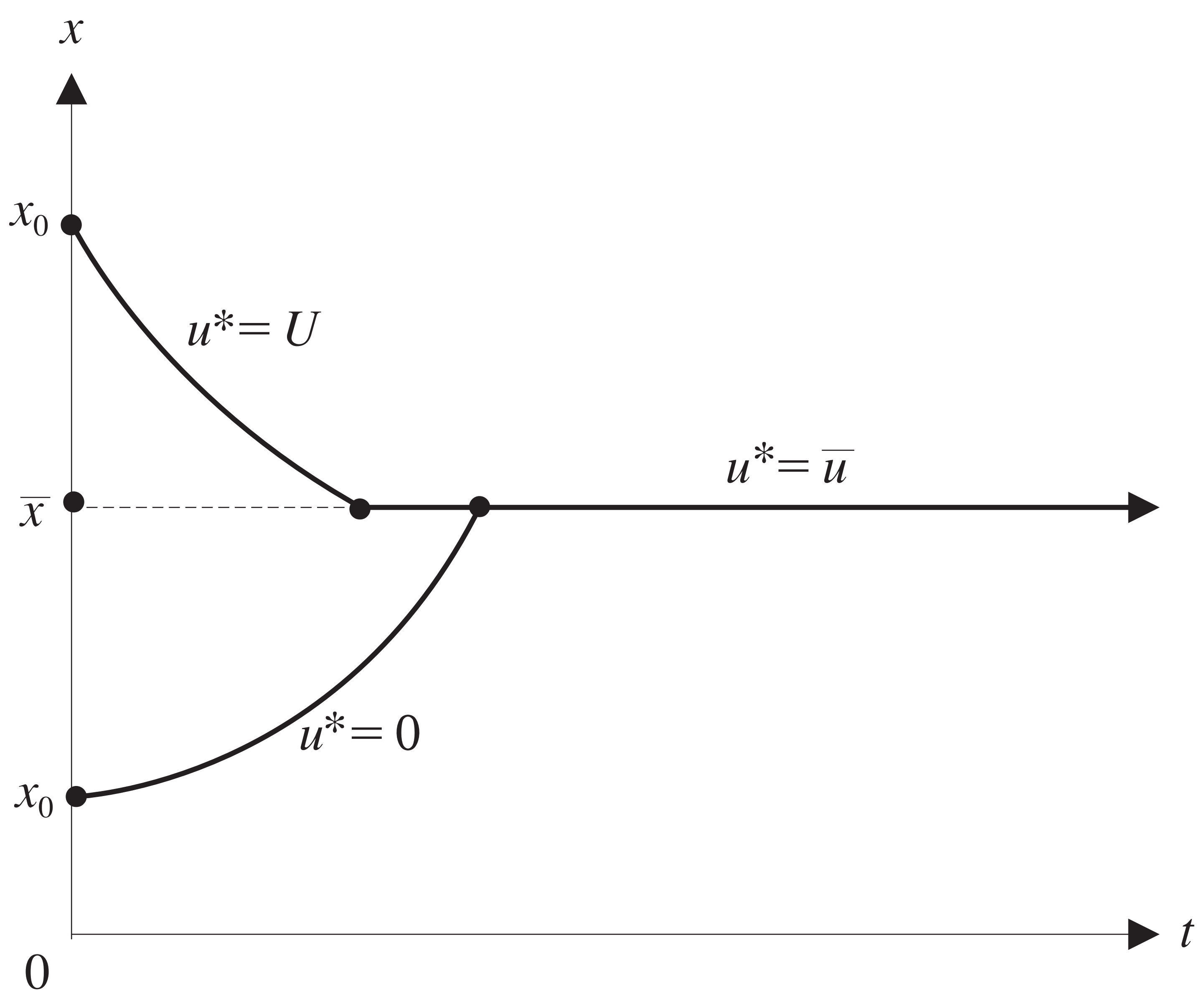
A second-order condition (see comparison Lemma 7.1) must be satisfied for the solution $\bar{x}$ of (12) to be a turnpike solution:

$$I(x) < 0 \text{ for } x < \bar{x} \text{ and } I(x) > 0 \text{ for } x > \bar{x}.$$

Let $\bar{x}$ be the unique solution to (12) satisfying the second-order condition. The procedure can be extended to the case of nonunique solutions as in Sethi (1977b).

The corresponding value $\bar{u}$ of the control which would maintain the fish stock level at $\bar{x}$ is $g(\bar{x})/q\bar{x}$. In Exercise 10.2 you are asked to show that $\bar{x} \in (x_b, X)$ and also that $\bar{u} < U$. In Figure 10.1 optimal trajectories are shown for two different initial values: $x_0 < \bar{x}$ and $x_0 > \bar{x}$.

Figure 9.1: Optimal Policy for the Sole Owner Fishery Model



Let

$$\pi(x) = \frac{g(x)(pqx - c)}{qx}.$$

Then, equation (12) can be expressed as the condition

$$\frac{d\pi(x)}{dx} = \rho\left(\frac{pqx - c}{qx}\right). \tag{13}$$

The interpretation of $\pi(x)$ is that it is the *sustainable economic rent* at fish stock level x. This can be seen by substituting $u = g(x)/qx$ into (3), where $u = g(x)/qx$, obtained using (2), is the fishing effort required to maintain the fish stock at level x. Suppose we have attained the equilibrium level $\bar{x}$ given by (12), and suppose we reduce this level to $\bar{x} - \varepsilon$ by using fishing effort of $\varepsilon/q\bar{x}$.

The immediate marginal revenue, *MR*, from this action is

$$MR = (pq\bar{x} - c)\frac{\varepsilon}{q\bar{x}}.$$

However, this causes a decrease in the sustainable economic rent which equals

$$\pi'(\bar{x})\varepsilon.$$

Over the infinite future, the present value of this stream, i.e., the marginal cost, *MC*, is

$$MC = \int_0^\infty e^{-\rho t}\pi'(\bar{x})\varepsilon dt = \frac{\pi'(\bar{x})\varepsilon}{\rho}.$$

Equating *MR* and *MC*, we obtain (13), which is also (12).

When the discount rate is zero, equation (13) reduces to

$$\pi'(x) = 0,$$

so that it will give the equilibrium fish stock level $\bar{x}$ for $\rho = 0$, which maximizes the instantaneous profit rate $\pi(x)$. This is called in economics the *golden rule level*.

When $\rho = \infty$, we can assume that $\pi'(x)$ is bounded. From (13) we have $pqx - c = 0$, which gives

$$\bar{x} \,|_{\rho=\infty} = x_b = c/pq.$$

The latter is the bionomic equilibrium attained in the open access fishery solution; see (4).

The sole owner solution $\bar{x}$ satisfies $\bar{x} > x_b = c/pq$. If we regard a government regulatory agency as the sole owner responsible for operating the fishery at level $\bar{x}$, then it can impose restrictions, such as gear regulations, catch limitations, etc., which increase the fishing cost c. If c is increased to the level $pq\bar{x}$, then the fishery can be turned into an open access fishery subject to those regulations, and it will attain the bionomic equilibrium at level $\bar{x}$.

We introduce the following notation:

t_0 = the initial age of the forest,

ρ = the discount rate,

$x(t)$ = the volume of usable timber in the forest at time t,

$u(t)$ = the rate of thinning at time t,

p = the constant price per unit volume of timber,

c = the constant cost per unit volume of thinning,

$f(x)$ = the growth function, which is positive, concave, and has a unique maximum at x^m; we assume $f(0) = 0$,

$g(t)$ = the growth coefficient which is positive, decreasing function of time.

The specific function form for the forest growth used in Kilkki and Vaisanen (1969) is as follows:

$$f(x) = xe^{-\alpha x}, \;\; 0 \le x \le \frac{2}{\alpha},$$

where α is a positive constant. Note that f is concave in the relevant range and that $x_m = 1/\alpha$. They use the growth coefficient of the form

$$g(t) = at^{-b},$$

where a and b are positive constants.

The forest growth equation is

$$\dot{x} = g(t)f(x) - u(t), \;\; x(t_0) = x_0. \tag{14}$$

The objective function is to

$$\text{maximize}\left\{J=\int_{t_0}^{\infty}e^{-\rho t}(p-c)u\,dt\right\} \tag{15}$$

subject to (14) and the state and control constraints

$$x(t)\geq 0 \ \text{ and } \ u(t)\geq 0. \tag{16}$$

The control constraint in (16) implies that there is no replanting in the forest. In Section 10.2.3 we extend this model to incorporate the successive replantings of the forest each time it is clear cut.

10.2.2 Determination of Optimal Thinning

We solve the forest thinning model by using the maximum principle. The Hamiltonian is

$$H = (p - c)u + \lambda[gf(x) - u] \tag{17}$$

with the adjoint equation

$$\dot{\lambda} = \lambda[\rho - gf'(x)]. \tag{18}$$

The optimal control is

$$u^* = \text{bang}[0,\ \infty;\ p - c - \lambda]. \tag{19}$$

We do not use the Lagrangian form of the maximum principle to include constraints (16) because, as we shall see, the forestry problem has a natural ending at a time T for which $x(T) = 0$.

To get the singular control solution triple $\{\bar{x}, \bar{\lambda}, \bar{u}\}$, we must observe that due to the time dependence of $g(t)$, $\bar{x}$ and $\bar{u}$ will be functions of time. From (19), we have

$$\bar{\lambda} = p - c, \tag{20}$$

which is a constant so that $\dot{\bar{\lambda}} = 0$. From (18),

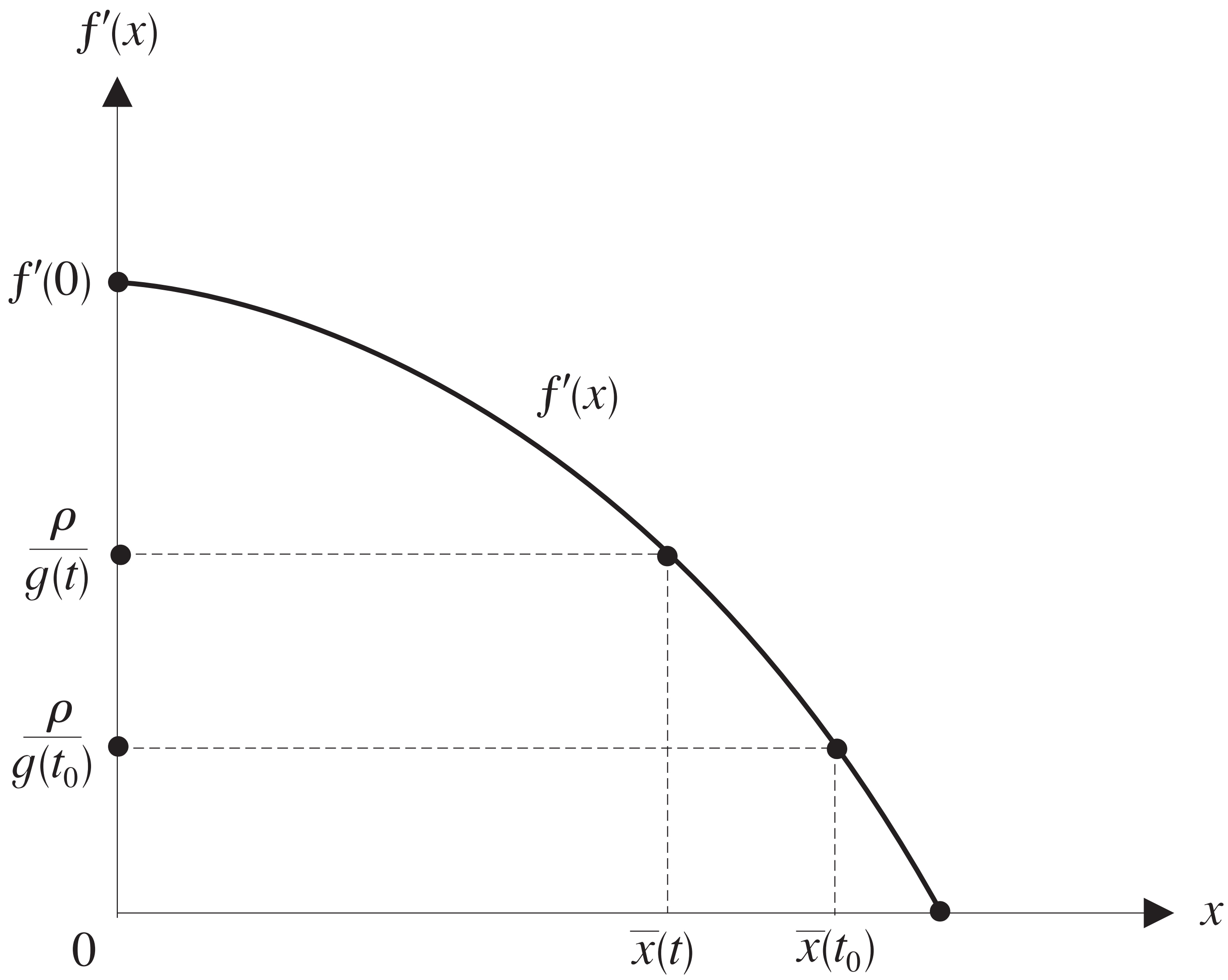
$$f'(\bar{x}(t)) = \frac{\rho}{g(t)} \quad \text{or} \quad \bar{x}(t) = f'^{-1}(\rho/g(t)). \tag{21}$$

Then from (13),

$$\bar{u}(t) = g(t)f(\bar{x}(t)) - \dot{\bar{x}}(t) \tag{22}$$

gives the singular control.

Figure 10.2: Singular Usable Timber Volume $\bar{x}(t)$



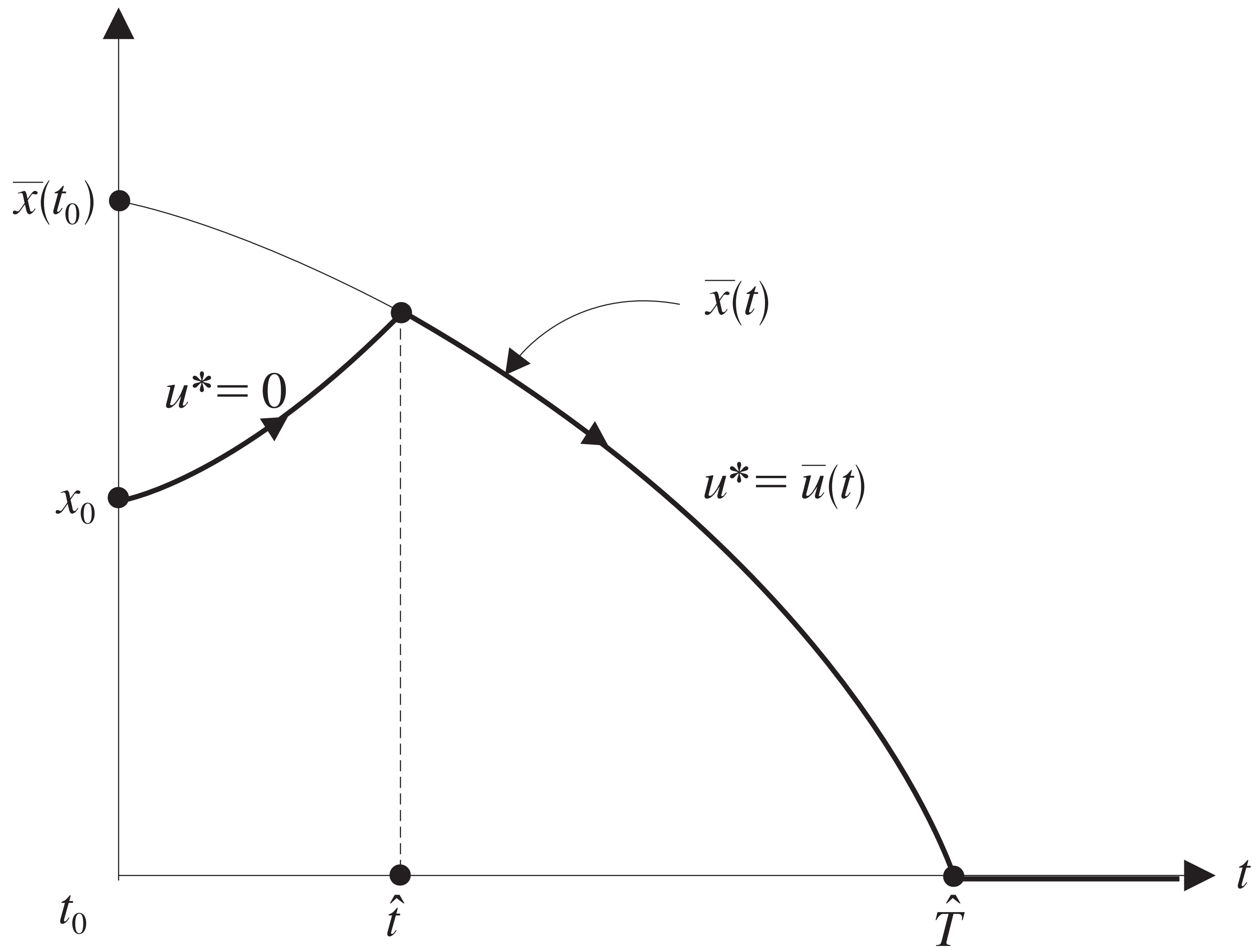
Since $g(t)$ is a decreasing function of time, it is clear from Figure 10.2 that $\bar{x}(t)$ is a decreasing function of time, and then by (22), $\bar{u}(t) \geq 0$. We see from (21) that $\bar{x}(\hat{T}) = 0$ at time $\hat{T}$, where $\hat{T}$ is given by

$$\frac{\rho}{g(\hat{T})} = f'(0).$$

This, in view of $f'(0) = 1$, gives

$$\hat{T} = e^{-(1/b)\ln(\rho/a)}. \tag{23}$$

Figure 10.3: Optimal Policy for the Forest Thinning Model when $x_0 < \bar{x}(t_0)$



In Figure 10.3 we plot $\bar{x}(t)$ as a function of time t. The figure also contains an optimal control trajectory for the case in which $x_0 < \bar{x}(t_0)$. To determine the switching time $\hat{t}$, we first solve (13) with $u = 0$. Let $x(t)$ be the solution. Then, $\hat{t}$ is the time at which the $x(t)$ trajectory intersects the $\bar{x}(t)$ curve; see Figure 10.3.

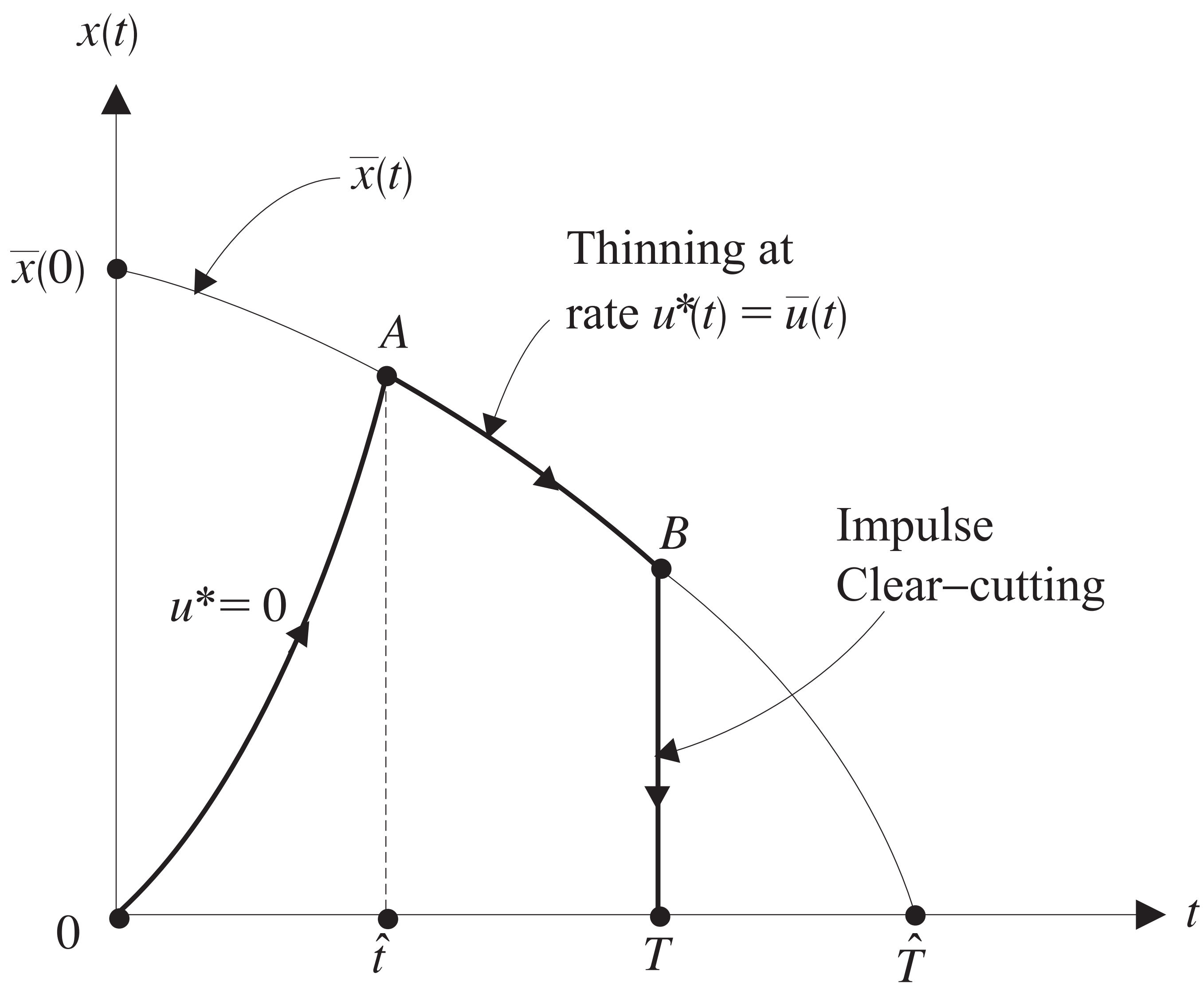
For $x_0 > \bar{x}(t_0)$, the optimal control at t_0 will be the *impulse* cutting to bring the level from x_0 to $\bar{x}(t_0)$ instantaneously. To complete the infinite horizon solution, set $u^*(t) = 0$ for $t \geq \hat{T}$. In Exercise 10.10 you are asked to obtain $\lambda(t)$ for $t \in [0, \infty)$.

This extension is similar to the chain of machines model of Section 9.3. We shall assume that successive plantings (forest rotations) take place at equal intervals.

Let T be the rotation period, i.e., the time from planting to clear-cutting which is to be determined. During the nth rotation, the dynamics of the forest is given by (13) with $t \in [(n-1)T, nT]$ and $x[(n-1)T] = 0$. The objective function to be maximized is given by

$$\begin{aligned} J(T) &= \sum_{k=1}^{\infty} e^{-k\rho T} \int_0^T e^{-\rho t}(p-c)u dt \\ &= \frac{1}{1-e^{-\rho T}} \int_0^T e^{-\rho t}(p-c)u dt. \end{aligned} \tag{24}$$

FIGURE 10.4: OPTIMAL POLICY FOR THE CHAIN OF FORESTS MODEL WHEN $T > \hat{t}$



With the solution in Figure 10.4, we can write the $J^*(T)$ of (24) for a given T as

$$J^*(T) = \frac{1}{1-e^{-\rho T}}\left[\int_{\hat{t}}^{T-} e^{-\rho t}(p-c)\bar{u}dt + \int_{T-}^{T} e^{-\rho t}(p-c)u^*(t)dt\right]$$
$$= \frac{1}{1-e^{-\rho T}}\left[\int_{\hat{t}}^{T} e^{-\rho t}(p-c)\bar{u}(t)dt + e^{-\rho T}(p-c)\bar{x}(T)\right].$$

Note that $u^*(T) = \mathrm{imp}[\bar{x}, 0; T]$ in the second integral is an impulse control bringing the forest from value $\bar{x}(t)$ to 0 by a clearcutting operation; see Exercise 10.11.

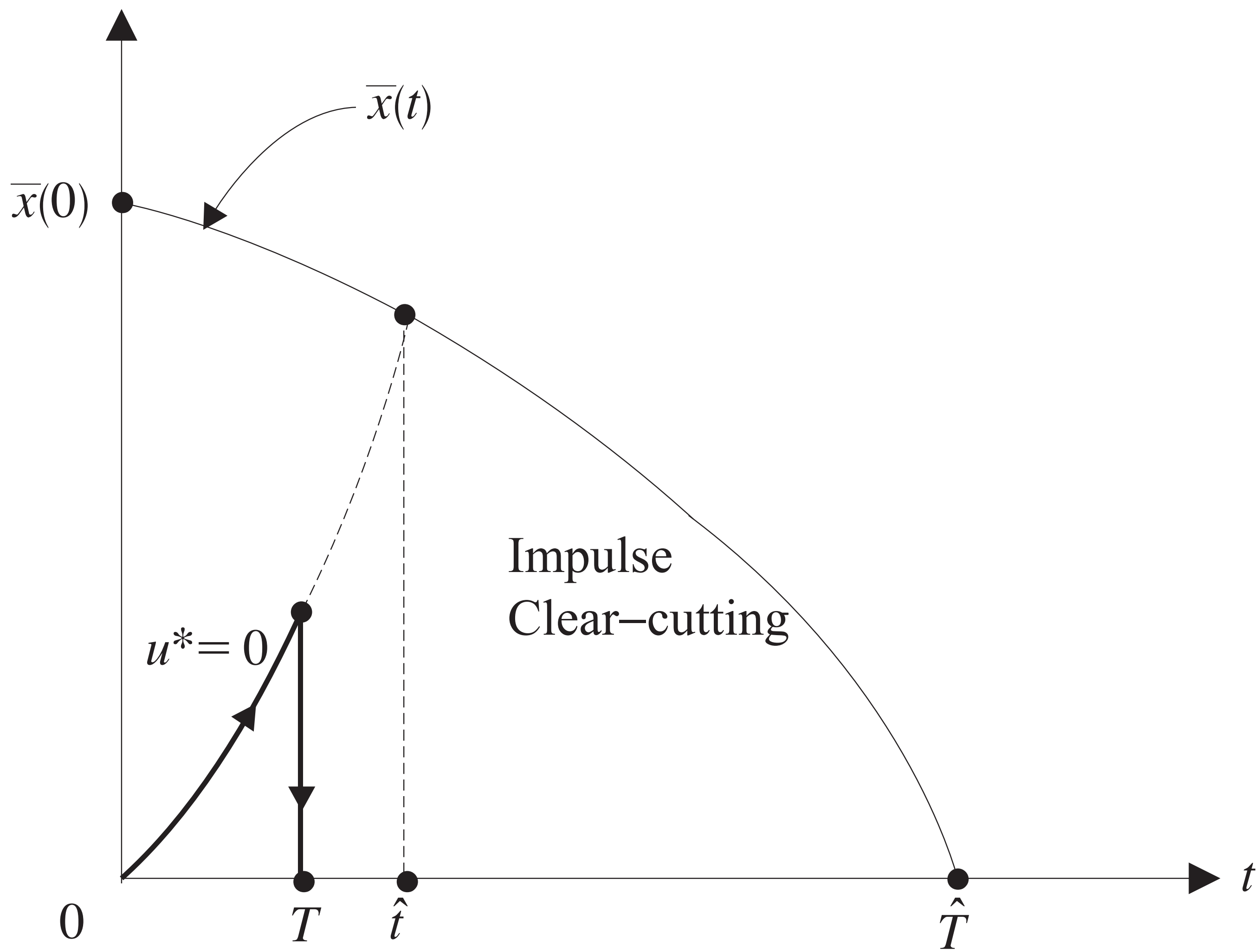
To find the optimal value of T for this case, we differentiate (25) with respect to T, equate the result to zero, and simplify, obtaining (see Exercise 10.12)

$$(1 - e^{-\rho T})g(T)f[\bar{x}(T)] - \rho\bar{x}(T) - \rho\int_{\hat{t}}^{T} e^{-\rho t}\bar{u}(t)dt = 0. \quad (25)$$

If the solution T lies in $(\hat{t}, \hat{T}]$, keep it; otherwise set $T = \hat{T}$.

Note that (25) can also be derived by using the transversality condition (3.14); see Exercise 3.5.

Figure 10.5: Optimal Policy for the Chain of Forests Model when $T \le \hat{t}$



The solution for $x(T)$ is obtained by integrating (14) with $u = 0$ and $x_0 = 0$. Let this solution be denoted as $x^*(t)$. Here (24) becomes

$$J^*(T) = \frac{e^{-\rho T}}{1 - e^{-\rho T}}(p - c)x^*(T). \tag{26}$$

To find the optimal value of T for this case, we differentiate (26) with respect to T and equate $dJ^*(T)/dT$ to zero. We obtain (see Exercise 10.12)

$$(1 - e^{-\rho T})g(T)f[x^*(T)] - \rho x^*(T) = 0. \tag{27}$$

If the solution lies in the interval $[0, \hat{t}]$ keep it; otherwise set $T = \hat{t}$.

The optimal value T^* can be obtained by computing $J^*(T)$ from both cases and selecting whichever is larger.

In this section we discuss a simple model taken from Sethi (1979a). This paper analyzes optimal depletion rates by maximizing a social welfare function which involves consumers' surplus and producers' surplus with various weights. Here we select a model having the equally weighted criterion function.

We assume that at a high enough price, say $\bar{p}$, a substitute, preferably renewable, will become available. We introduce the following notation:

$p(t)$ = the price of the resource at time t,

q = $f(p)$ is the demand function, i.e., the quantity demanded at price p; $f' \leq 0$, $f(p) > 0$ for $p < \bar{p}$, and $f(p) = 0$ for $p \geq \bar{p}$, where $\bar{p}$ is the price at which the substitute completely replaces the resource. A typical graph of the demand function is shown in Figure 10.6,

$$
\begin{aligned}
c &= \text{$G(q)$ is the cost function; $G(0)=0,\ G(q)>0$ for $q>0,\ G'>0$ and $G''\geq 0$ for $q\geq 0$, and $G'(0)<\bar{p}$. The latter assumption makes it possible for the producers to make positive profit at a price p below $\bar{p}$,}
\end{aligned}
$$

c = $G(q)$ is the cost function; $G(0) = 0,\ G(q) > 0$ for $q > 0,\ G' > 0$ and $G'' \geq 0$ for $q \geq 0$, and $G'(0) < \bar{p}$. The latter assumption makes it possible for the producers to make positive profit at a price p below $\bar{p}$,

$Q(t)$ = the available stock or reserve of the resource at time t; $Q(0) = Q_0 > 0$,

ρ = the social discount rate; $\rho > 0$,

T = the horizon time, which is the latest time at which the substitute will become available regardless of the price of the natural resource; $T > 0$.

Let

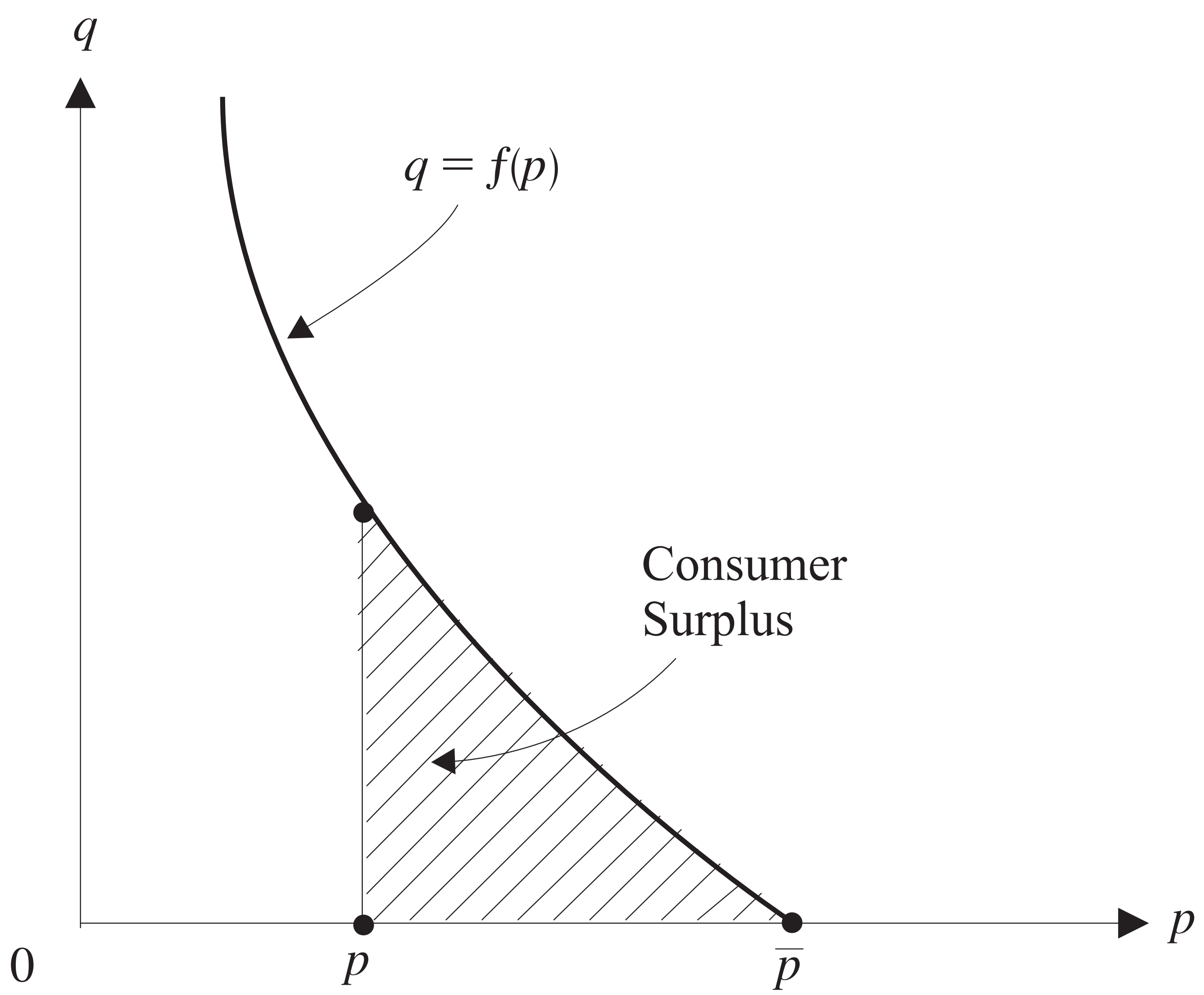
$$c = G[f(p)] = g(p), \tag{28}$$

for which it is obvious that $g(p) > 0$ for $p < \bar{p}$ and $g(p) = 0$ for $p \geq \bar{p}$. Let

$$\pi(p) = pf(p) - g(p) \tag{29}$$

denote the profit function of the producers, i.e., the *producers' surplus*. Let $\underline{p}$ be the smallest price at which $\pi(p)$ is nonnegative. Assume further that $\pi(p)$ is a concave function in the range $[\underline{p}, \bar{p}]$ as shown in Figure 10.7. In the figure the point p^m indicates the price which maximizes $\pi(p)$.

Figure 10.6: The Demand Funtion



We also define

$$\psi(p) = \int_{p}^{\bar{p}} f(y)dy \tag{30}$$

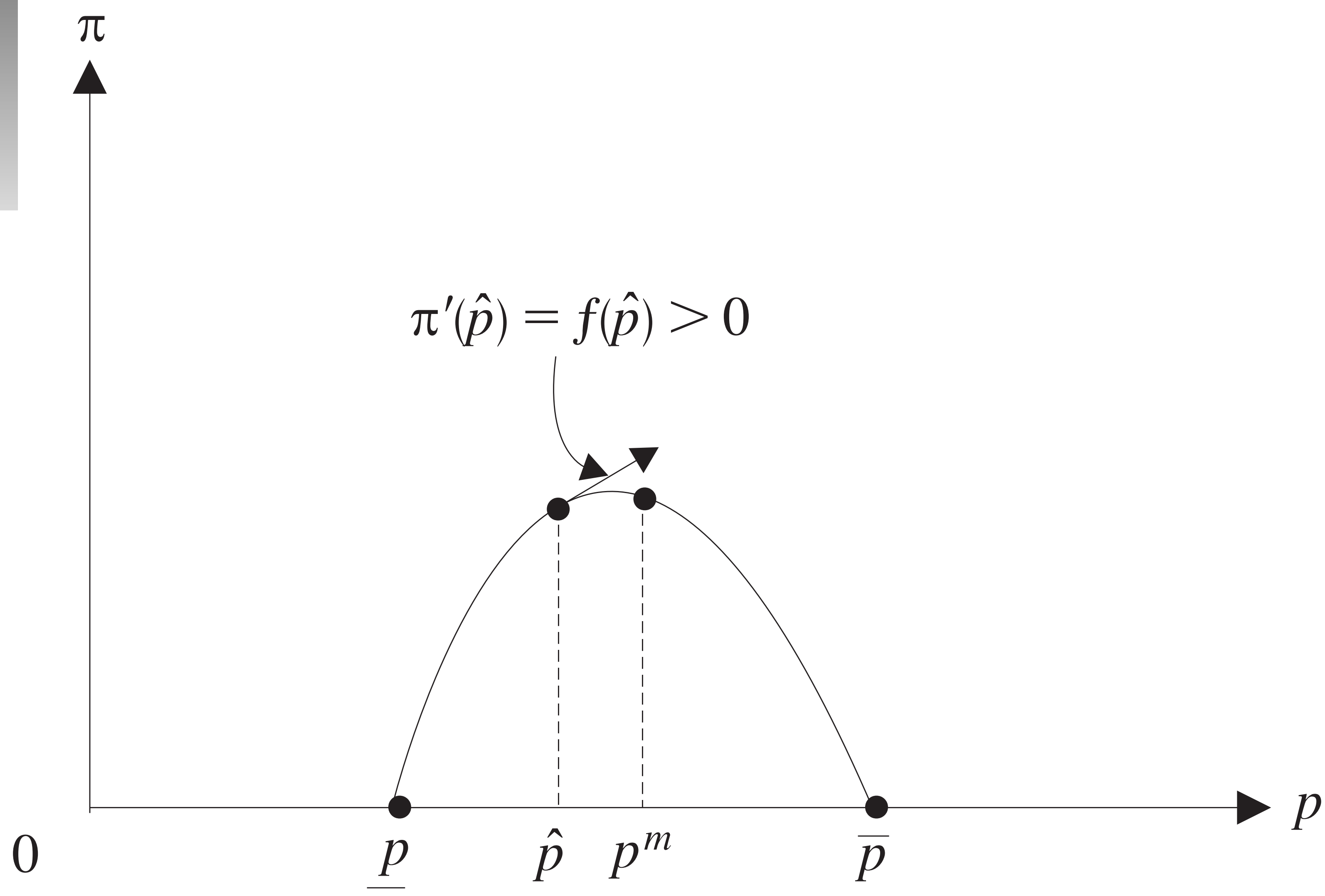
as the *consumers' surplus*. This quantity represents the total excess amount consumers would be willing to pay. In other words, consumers pay $pf(p)$, while they would be willing to pay

$$\int_{\bar{p}}^{p} yf'(y)dy = pf(p) + \psi(p).$$

The instantaneous rate of consumers' surplus and producers' surplus is the sum $\psi(p) + \pi(p)$. Let $\hat{p}$ denote the maximum of this sum, i.e., $\hat{p}$ solves

$$\psi'(\hat{p}) + \pi'(\hat{p}) = \hat{p}f'(\hat{p}) - g'(\hat{p}) = 0. \tag{31}$$

Figure 10.7: The Profit Function



The optimal control problem is:

$$\max\left\{J=\int_0^T[\psi(p)+\pi(p)]e^{-\rho t}dt\right\} \tag{32}$$

subject to

$$\dot{Q}=-f(p),\ \ Q(0)=Q_0, \tag{33}$$

$$Q(T)\geq 0, \tag{34}$$

and $p\in\Omega=[\underline{p},\bar{p}]$. Recall that the sum $\psi(p)+\pi(p)$ is concave in p.

Form the current-value Hamiltonian

$$H(Q, p, \lambda) = \psi(p) + \pi(p) + \lambda[-f(p)], \tag{35}$$

where λ satisfies the relation

$$\dot{\lambda} = \rho\lambda, \quad \lambda(T) \geq 0, \quad \lambda(T)Q(T) = 0, \tag{36}$$

which implies

$$\lambda(t) = \begin{cases} 0 & \text{if } Q(T) \geq 0 \text{ is not binding,} \\ \lambda(T)e^{\rho(t-T)} & \text{if } Q(T) \geq 0 \text{ is binding.} \end{cases} \tag{37}$$

To obtain the optimal control, the Hamiltonian maximizing condition, which is both necessary and sufficient in this case (see Theorem 2.1), is

$$\frac{\partial H}{\partial p} = \psi' + \pi' - \lambda f' = (p - \lambda) f' - g' = 0. \tag{38}$$

To show that the solution $s(\lambda)$ for p of (38) actually maximizes the Hamiltonian, it is enough to show that the second derivative of the Hamiltonian is negative at $s(\lambda)$. Differentiating (38) gives

$$\frac{\partial^2 H}{\partial p^2} = f' - g'' + (p - \lambda) f''.$$

Using (38) we have

$$\frac{\partial^2 H}{\partial p^2} = f' - g'' + \frac{g'}{f'} f''. \tag{39}$$

From the definition of G in (28), we can obtain

$$G'' = \frac{f'g'' - g'f''}{f'^3},$$

which, when substituted into (39), gives

$$\frac{\partial^2 H}{\partial p^2} = f' - G'' f'^2. \tag{40}$$

The right-hand side of (40) is strictly negative because $f' < 0$, and $G'' \geq 0$ by assumption. We remark that $\hat{p} = s(0)$ using (31) and (38), and hence the second-order condition for $\hat{p}$ of (31) to give the maximum of H is verified.

Case 1: $Q(T) \geq 0$ Is not Binding.

From (37), $\lambda(t) \equiv 0$ so that from (38) and (31),

$$p^* = \hat{p}. \tag{41}$$

With this value, the total consumption of the resource is $Tf(\hat{p})$, which must be $\leq Q_0$ so that the constraint $Q(T) \geq 0$ is not binding. Hence,

$$Tf(\hat{p}) \leq Q_0 \tag{42}$$

characterizes Case 1 and its solution is given in (41).

$Tf(\hat{p}) > Q_0$ so that $Q(T) \geq 0$ Is Binding

To obtain the solution requires finding a value of $\lambda(T)$ such that

$$\int_0^{t^*} f(s[\lambda(T)e^{\rho(t-T)}])dt = Q_0, \tag{43}$$

where

$$t^* = \min\left\{T, T + \frac{1}{p}\ln\left[\frac{\bar{p} - G'(0)}{\lambda(T)}\right]\right\}. \tag{44}$$

The time t^* , if it is less than T, is the time at which $s[\lambda(T)e^{\rho(t^*-T)}] = \bar{p}$. From Exercise 10.15,

$$\lambda(T)e^{\rho(t^*-T)} = \bar{p} - G'(0) \tag{45}$$

which, when solved for t^*, gives the second argument of (44).

One method to obtain the optimal solution is to define $\bar{T}$ as the longest time horizon during which the resource can be optimally used. Such a $\bar{T}$ must satisfy

$$\lambda(\bar{T}) = \bar{p} - G'(0),$$

and therefore,

$$\int_0^{\bar{T}} f\left(s\left[\{\bar{p} - G'(0)\}e^{\rho(t-\bar{T})}\right]\right) dt = Q_0, \tag{46}$$

which is a transcendental equation for $\bar{T}$. We now have two subcases.

- **Subcase 2a:** $T \geq \bar{T}$.

The optimal control is

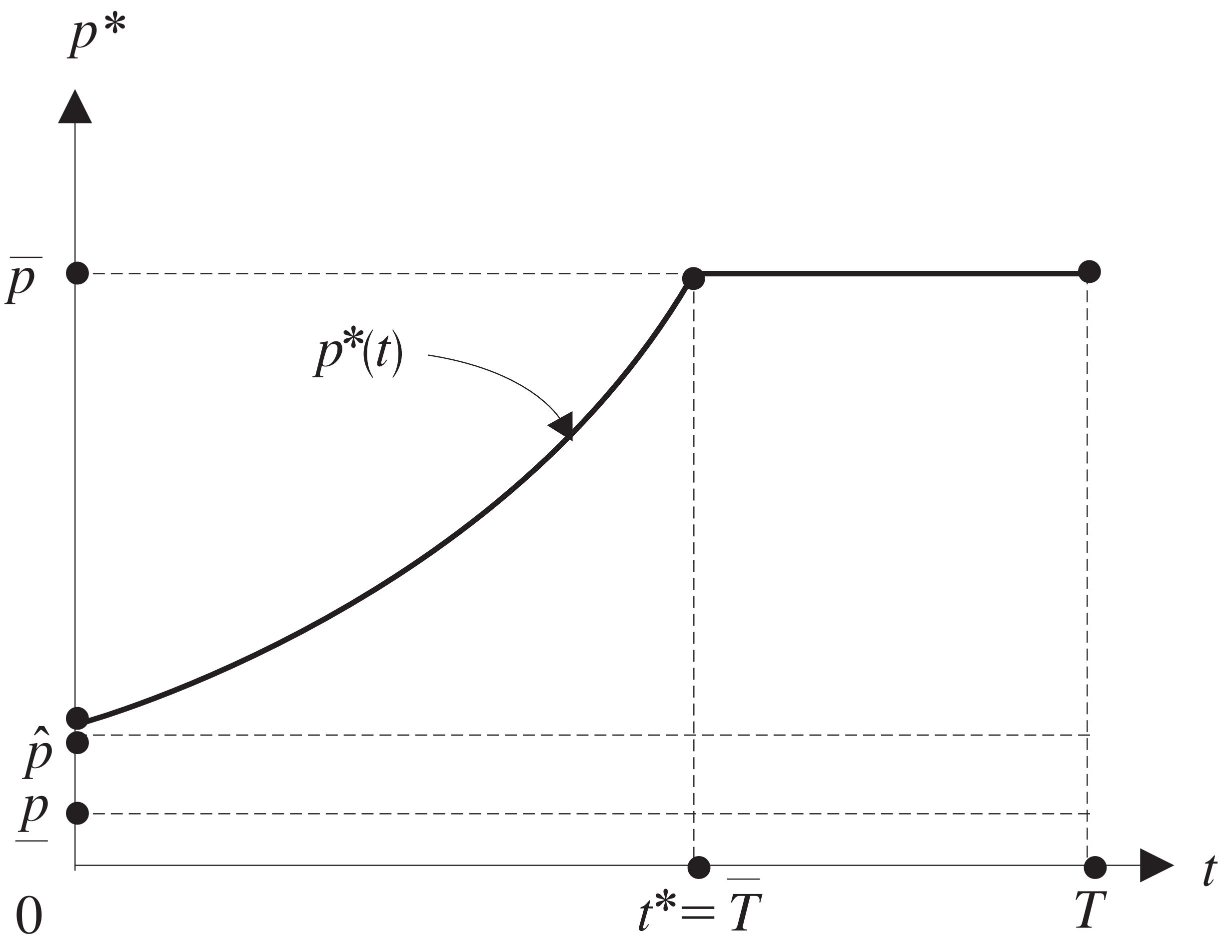
$$p^*(t) = \begin{cases} s\left(\{\bar{p} - G'(0)\}e^{\rho(t-\bar{T})}\right) & \text{for } t \leq \bar{T}, \\ \bar{p} & \text{for } t > \bar{T}. \end{cases} \tag{47}$$

Clearly in this subcase, $t^* = \bar{T}$ and

$$\lambda(T) = [\bar{p} - G'(0)]e^{-\rho(\bar{T}-T)}.$$

A sketch of (47) is shown in Figure 10.8.

FIGURE 10.8: OPTIMAL PRICE TRAJECTORY FOR $T \geq \bar{T}$



- **Subcase 2b:** $T < \bar{T}$.

Here the optimal price trajectory is

$$p^*(t) = s\left[\lambda(T)e^{\rho(t-T)}\right], \tag{48}$$

where $\lambda(T)$ is to be obtained from the transcendental equation

$$\int_0^T f\left(s\left[\lambda(T)e^{\rho(t-T)}\right]\right)dt = Q_0. \tag{49}$$

A sketch of (48) is shown in Figure 10.9.

Figure 10.9: Optimal Price Trajectory for $T < \bar{T}$