Math 6313 Midterm Exam 10/19/15
Dr. Minkoff

Name:

Instructions: You may not use notes or books on this exam. Don't spend too much time on any one problem. Show your work!

NAME:

1 /~~10~~ 8	2 /17	3 /18	4 /15
5 /15	6 /12	T /~~100~~ 85	

[~~10~~ 8 pts](1) Is 1/9 a machine number on the Marc 32? If 1/9 is not a machine number on the Marc 32, determine the binary machine number just to the right of 1/9 on the Marc 32 machine.

$1/9 = .\overline{111}_{10}$

Marc 32 has 24 bit mantissa (in 1+ form)

So no, 1/9 is not a machine number on Marc 32. (no finite representation)

$.1111\ldots \times 2 = .2222$
$\times 2 = .4444$
$\times 2 = .8888$
$\times 2 = 1.77$
$.7777\ldots \times 2 = 1.5555$
$.5555\ldots \times 2 = 1.111$

$a_0 = 0$
$a_1 = 0$
$a_2 = 0$
$a_3 = 1$
$a_4 = 1$
$a_5 = 1$

So $1/9 = .\overline{000111}_2$

$= .111000\overline{111000}_2 \times 2^{-3}$

$= 1.\underbrace{11000}_{6}\ \underbrace{111000}_{6}\ \underbrace{111000}_{6}\ \underbrace{\overline{111000}}_{6}{}_2 \times 2^{-4}$

= normalized 1+ floating pt form. This number does not fit exactly on Marc 32.

Number to right is
$1.11000111000111000111001_2 \times 2^{-4}$

[17 pts](2a) Find an accurate way to calculate $f(x) = \sqrt{1 + 1/x} - 1$ when x is large.

$$f(x) = \sqrt{1 + \frac{1}{x}} - 1$$

$$= \sqrt{\frac{x+1}{x}} - 1$$

$$= \sqrt{\frac{x+1}{x}} - \frac{\sqrt{x}}{\sqrt{x}} = \frac{1}{\sqrt{x}}\left(\sqrt{x+1} - \sqrt{x}\right)$$

$$\Rightarrow \frac{1}{\sqrt{x}}\left(\sqrt{x+1} - \sqrt{x}\right) \cdot \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x}}\left(\frac{x+1-x}{\sqrt{x+1} + \sqrt{x}}\right) = \boxed{\frac{1}{\sqrt{x}\left(\sqrt{x+1} + \sqrt{x}\right)} = g(x)}$$

which does not lead to loss of significance in calculating the function.

(b) If we are only comfortable losing one binary digit of accuracy with our formula from Part (a) for f, for what x values should we use the original formula and for what x values should we use your modified formula?

(Assume $x > 0$)

Using loss of precision theorem,

$$2^{-q} \leq 1 - y/x \leq 2^{-p}$$

where $x > y$ and at most q, and at least p binary bits lost in subtraction.

Want at most 1 digit lost $\Rightarrow q = 1$

$\Rightarrow \quad 2^{-1} \leq 1 - y/x \qquad \sqrt{1+\frac{1}{x}} > 1$ (for large x + positive)

so we have $1 - \dfrac{1}{\sqrt{1+1/x}} \geq 1/2$

$\Rightarrow 1/2 \geq \dfrac{1}{\sqrt{1+1/x}}$

$\Rightarrow \dfrac{1}{4} \geq \dfrac{1}{1+1/x} \Rightarrow 1 + 1/x \geq 4$

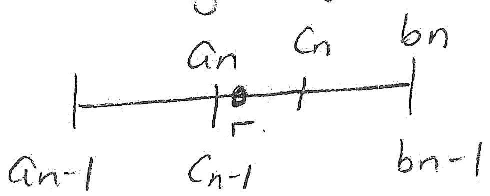
$\Rightarrow \dfrac{1}{x} \geq 3 \Rightarrow x \leq \dfrac{1}{3}$

$\boxed{\text{So use original formula when } x \leq 1/3 \text{ otherwise use alternate formula } (g(x))}$

[18 pts](3) If the bisection method generates intervals $[a_0, b_0]$, $[a_1, b_1]$ and so on while calculating the root r, which of these statements are true? Give proofs or counterexamples in each case.
(a) $a_0 = a_1 = a_2 = \ldots$
(b) $|r - c_n| < |r - c_{n-1}|$
(c) $|r - b_n| \le 2^{-n}(b_0 - a_0)$

a) cannot be true because $f(a)f(b) < 0$ to use bisection (ie, must be a root in interval $[a, b]$.)
The only way $a_0 = a_1 = a_2 \ldots$ is if $r = a$
But then $f(a)f(b) = 0$ $\rightarrow\leftarrow$

b) $|r - c_n| < |r - c_{n-1}|$ is also not always true because in following example:



here $|r - c_n| > |r - c_{n-1}|$

c) Is true $|r - b_n| \le b_n - a_n = 2^{-n}(b_0 - a_0)$

[15 pts](4a) Estimate $f(3.5)$ using the Newton interpolating polynomial for this dataset:

x	2	3	4
y	7.389	20.09	54.60

x	f[x]	f[,]	f[, ,]
2	7.389		
		12.701	
3	20.09		10.9045
		34.51	
4	54.60		

$$p(x) = 7.389 + 12.701(x-2) + 10.9045(x-2)(x-3)$$

$$p(3.5) = 7.389 + 12.701(1.5) + 10.9045(1.5)(.5)$$

$$\boxed{p(3.5) = 34.619}$$

(b) If you are now told that the function being interpolated is $f(x) = e^x$, calculate the exact error **and an upper bound on the error** in your approximation from the theory.

$f(3.5) = e^{3.5} = 33.115$

true error is $|f(3.5) - p(3.5)| = 34.619 - 33.115$
$= 1.504$

Theoretical bound on error gives

① $f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i)$

or: ② $f(t) - p(t) = f[x_0, x_1, .., x_n, t] \prod_{i=0}^{n} (t - x_i)$

using ① gives $f(x) = e^x$, $f^{(n+1)}(x) = e^x$

and on interval $x \in [2,4]$ $f^{(n+1)}(x)$ is maximized at $x = 4$ so

$|f^{(n+1)}(\xi)| \le e^4$.

$\frac{1}{(n+1)!} = \frac{1}{3!} = \frac{1}{6}$ so $|f(x) - p(x)| \le \frac{e^4}{6} |(x-2)(x-3)(x-4)|$

Q: where is $g(x) = (x-2)(x-3)(x-4)$ maximized in $[2,4]$?

$g(x) = (x^2 - 5x + 6)(x-4) = x^3 - 5x^2 + 6x - 4x^2 + 20x - 24$
$= x^3 - 9x^2 + 26x - 24$

$g'(x) = 3x^2 - 18x + 26 = 0 \Rightarrow x = \frac{18 \pm \sqrt{(18)^2 - 4(3)(26)}}{6}$

$= \frac{18 \pm \sqrt{12}}{6} = \frac{18 \pm 2\sqrt{3}}{6} = \frac{9 \pm \sqrt{3}}{3} = 3 \pm \sqrt{3}/3$

$\rightarrow$

$g(x) = (x-2)(x-3)(x-4)$

optimum occurs when $x = 3 \pm \sqrt{3}/3$

$$g(3+\sqrt{3}/3) = (3+\sqrt{3}/3-2)(3+\sqrt{3}/3-3)(3+\sqrt{3}/3-4)$$
$$= (1+\sqrt{3}/3)(\sqrt{3}/3)(-1+\sqrt{3}/3)$$
$$= -.3849$$

$$g(3-\sqrt{3}/3) = (3-\sqrt{3}/3-2)(3-\sqrt{3}/3-3)(3-\sqrt{3}/3-4)$$
$$= (1-\sqrt{3}/3)(-\sqrt{3}/3)(-1-\sqrt{3}/3)$$
$$= .3849$$

So $\boxed{|f(x)-p(x)| \leq \frac{e^4}{6}(.3849) = 3.5025}$

Plugging in $x = 3.5$ gives

$$\text{error} \leq \frac{e^4}{6}(3.5-2)(3.5-3)(3.5-4)$$
$$\approx 3.412$$

[15 pts](5a) Give the Taylor series derivation of the scalar version of Newton's Method and from this derivation indicate why Newton's Method is quadratically convergent.

$$f(x) = f(x_n) + (x - x_n) f'(x_n) + \frac{(x-x_n)^2}{2} f''(\xi)$$

ξ between x and x_n.

Let x = root r, so

$$f(r) = f(x_n) + (r - x_n) f'(x_n) + \frac{(r-x_n)^2}{2} f''(\xi)$$

$$0 = f(x_n) + (r - x_n) f'(x_n) + \frac{(r-x_n)^2}{2} f''(\xi)$$

$$\Rightarrow -(r - x_n) f'(x_n) = f(x_n) + \frac{(r-x_n)^2}{2} f''(\xi)$$

$$\Rightarrow r - x_n = -\frac{f(x_n)}{f'(x_n)} - \frac{(r-x_n)^2}{2} \frac{f''(\xi)}{f'(x_n)}$$

(b) Under what conditions will Newton's Method converge?

$$\Rightarrow r = x_n - \frac{f(x_n)}{f'(x_n)} - \underbrace{\frac{(r-x_n)^2}{2} \frac{f''(\xi)}{f'(x_n)}}$$

ξ between r and x_n.

Newton: $x_{n+1} =$ $r \approx x_n - \frac{f(x_n)}{f'(x_n)}$ $n \geq 0$ with error given by this term.

If we let $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ then

$$|r - x_{n+1}| \leq M |r - x_n|^2 \text{ where } M = \frac{f''(\xi)}{2f'(x_n)}$$

$\Downarrow$

quadratic convergence.

Conditions: x_0 sufficiently close to r and r not a double root.

[12 pts] (6) Will either of the following iterations converge to the indicated fixed point α (provided x_0 is sufficiently close to α?) If the iteration does converge, does it converge quickly or slowly to the fixed point?

(a) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, \quad \alpha = 2$
(b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, \quad \alpha = 3^{1/3}$

check $g'(\alpha)$,

a) $g'(x) = 6 - \frac{12}{x_n^2}$ $\qquad g'(2) = 6 - \frac{12}{4} = 6 - 3 = 3 > 1$

so a) will not converge

b) $g'(x) = 2/3 - \frac{2}{x_n^3}$

$g'(3^{1/3}) = 2/3 - \frac{2}{3} = 0 < 1$

so b) will converge and fast

Please sign the following honor statement: *On my honor, I pledge that I have neither given nor received any aid on this exam.*