Math 6313 Midterm Exam 10/13/16
Dr. Minkoff

Name:

Instructions: You may not use notes or books on this exam. Don't spend too much time on any one problem. Show your work!

NAME:

1 /15	2 /15	3 /15	4 /15
5 /15	6 /10	T /85	

[15 pts](1a) Find the machine numbers x_+ and x_- in the Marc-32 that are just to the right and left of 3/5.

$3/5 = .6_{10}$

$$\begin{array}{r} .6 \\ \times 2 \\ \hline 1.2 \end{array} \quad \begin{array}{r} .2 \\ \times 2 \\ \hline .4 \end{array} \quad \begin{array}{r} .4 \\ \times 2 \\ \hline .8 \end{array} \quad \begin{array}{r} .8 \\ \times 2 \\ \hline 1.6 \end{array}$$

$a_0 = 1$
$a_1 = 0$
$a_2 = 0$
$a_3 = 1$

$$\boxed{\text{so } 3/5 = .\overline{1001}_2}$$

$$\boxed{\begin{array}{l} \text{so } x_- = .100110011001100110011001 \\ \phantom{\text{so }} x_+ = .100110011001100110011010 \end{array}}$$

(b) Which machine number (x_+ or x_-) is closer to the real number x?

$$x - x_- = \begin{array}{r} .1001\,1001\,1001\,1001\,1001\,1001\,1001\ldots \\ .1001\,1001\,1001\,1001\,1001\,1001 \\ \hline \end{array}$$

$$= .\overline{1001} \times 2^{-24} = 3/5 \times 2^{-24}$$

$$x_+ - x_- = (x_+ - x) + (x - x_-) \Rightarrow (x_+ - x_-) - (x - x_-) = (x_+ - x)$$

$$x_+ - x_- = \begin{array}{r} .1001\,1001\,1001\,1001\,1001\,1010 \\ -\ .1001\,1001\,1001\,1001\,1001\,1001 \\ \hline .0000\,0000\,0000\,0000\,0000\,0001 \end{array} = 1 \times 2^{-24}$$

$$\text{so} \quad x_+ - x = 1 \times 2^{-24} - 3/5 \times 2^{-24} = 2/5 \times 2^{-24}$$

$$\boxed{\text{So } x_+ \text{ is closer to } x}$$

[15 pts](2a) For what values of x will

$$y(x) = \ln x - 1$$

result in loss of significance?

if $x \approx e$

(b) Devise an alternate way to calculate $y(x)$ for these values which does not result in loss of significance.

$$y(x) = \ln x - \ln e = \ln(x/e)$$

(c) If at most 2 binary bits of precision are to be lost in the computation of $y(x)$ using the formula in Part (a), what restriction must be placed on x? (Hint: assume $x > e$.)

Thm says:

$$2^{-2} \leq 1 - y/x \qquad x > y > 0$$

if $x > e$, $\ln x > \ln e = 1$

so $2^{-2} \leq 1 - \frac{1}{\ln x} \Rightarrow$

$$\frac{1}{\ln x} \leq 1 - 1/4 = 3/4 \Rightarrow \frac{4}{3} \leq \ln x$$

$$\Rightarrow x \geq e^{4/3} \doteq 3.7937$$

so require $\boxed{x \geq 3.7937}$

[15 pts](3a) Estimate $f(3.5)$ using the Newton interpolating polynomial (and a divided difference table) for this dataset:

x	2	3	4
y	7.389	20.09	54.60

x	f[x]	f[,]	f[,,]
2	7.389		
		12.701	
3	20.09		10.9045
		34.51	
4	54.60		

$$p(x) = 7.389 + 12.701(x-2) + 10.9045(x-2)(x-3)$$

$$p(3.5) = 7.389 + 12.701(1.5) + 10.9045(1.5)(.5)$$

$$\boxed{p(3.5) = 34.619}$$

(b) If you are now told that the function being interpolated is $f(x) = e^x$, calculate the exact error **and an upper bound on the error** in your approximation from the theory.

$$f(3.5) = e^{3.5} = 33.115$$

true error is $|f(3.5) - p(3.5)| = 34.619 - 33.115$
$= 1.504$

Theoretical bound on error gives

① $f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n}(x - x_i)$

or: ② $f(t) - p(t) = f[x_0, x_1, \dots, x_n, t] \prod_{i=0}^{n}(t - x_i)$

using (1) gives $f(x) = e^x$, $f^{(n+1)}(x) = e^x$

and on interval $x \in [2, 4]$ $f^{(n+1)}(x)$ is maximized at $x = 4$ so

$|f^{(n+1)}(\xi)| \le e^4$.

$\frac{1}{(n+1)!} = \frac{1}{3!} = \frac{1}{6}$ so $|f(x) - p(x)| \le \frac{e^4}{6} |(x-2)(x-3)(x-4)|$

Q: where is $g(x) = (x-2)(x-3)(x-4)$ maximized in $[2, 4]$?

$g(x) = (x^2 - 5x + 6)(x - 4) = x^3 - 5x^2 + 6x - 4x^2 + 20x - 24$

$= x^3 - 9x^2 + 26x - 24$

$g'(x) = 3x^2 - 18x + 26 = 0 \Rightarrow x = \frac{18 \pm \sqrt{(18)^2 - 4(3)(26)}}{6}$

$= \frac{18 \pm \sqrt{12}}{6} = \frac{18 \pm 2\sqrt{3}}{6} = \frac{9 \pm \sqrt{3}}{3} = 3 \pm \sqrt{3}/3$

$g(x) = (x-2)(x-3)(x-4)$

optimum occurs when $x = 3 \pm \sqrt{3}/3$

$$g(3+\sqrt{3}/3) = (3+\sqrt{3}/3-2)(3+\sqrt{3}/3-3)(3+\sqrt{3}/3-4)$$
$$= (1+\sqrt{3}/3)(\sqrt{3}/3)(-1+\sqrt{3}/3)$$
$$= -.3849$$

$$g(3-\sqrt{3}/3) = (3-\sqrt{3}/3-2)(3-\sqrt{3}/3-3)(3-\sqrt{3}/3-4)$$
$$= (1-\sqrt{3}/3)(-\sqrt{3}/3)(-1-\sqrt{3}/3)$$
$$= .3849$$

So $\boxed{|f(x)-p(x)| \le \frac{e^4}{6}(.3849) = 3.5025}$

Plugging in $x = 3.5$ gives

$$\text{error} \le \frac{e^4}{6}(3.5-2)(3.5-3)(3.5-4)$$
$$\approx 3.412$$

[15 pts](4a) Show that the function $f(x) = x - 3 - 2\cos x$ has at least one real root. (Hint: use the hypotheses required for the bisection algorithm.)

$f(x)$ is continuous as it is a sum of a continuous trig function $(2\cos x)$ and a polynomial $(x-3)$. (both of which are continuous so their sum is continuous).

if $x = \frac{\pi}{2}$ $f(x) = \frac{\pi}{2} - 3 - 2\cos\frac{\pi}{2} \doteq -1.43 < 0$

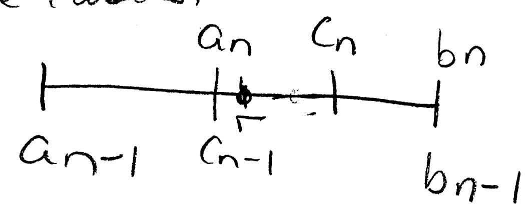
if $x = \pi$ $f(x) = \pi - 3 - 2\cos\pi = 3.14 - 3 + 2 \doteq 2.14 >$

so Intermediate Value Thm $\Rightarrow f(x)$ has a root betwe $\frac{\pi}{2}$ and π

(b) For the bisection method determine which of these assertions are true and which can be false. (Hint: it might be helpful to draw a good picture. **You must justify your answer to receive credit.**)

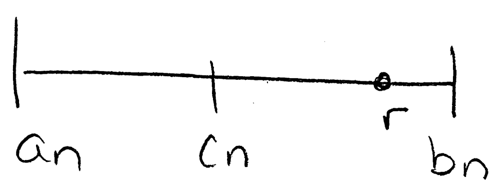
1. $|r - c_n| < |r - c_{n-1}|$

False in some cases:



2. $a_n \le r \le c_n$

False if



3. $|r - a_n| \le 2^{-n}$

False, we do however have

$|r - a_n| \le |b_n - a_n| = 2^{-n}(b_0 - a_0)$

[15 pts](5a) Give the Taylor series derivation of the scalar version of Newton's Method and from this derivation indicate why Newton's Method is quadratically convergent.

Newton's Method comes from the Taylor series of f about x_n.

$$f(x) = f(x_n) + (x - x_n) f'(x_n) + \frac{(x-x_n)^2}{2} f''(\xi)$$

if $x =$ root r we get

$$0 = f(r) = f(x_n) + (r - x_n) f'(x_n) + \frac{(r-x_n)^2}{2} f''(\xi)$$

$$r - x_n = -\frac{f(x_n)}{f'(x_n)} - \frac{(r-x_n)^2}{2} \frac{f''(\xi)}{f'(x_n)}$$

$$\Rightarrow \quad r = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(r-x_n)^2}{2} \frac{f''(\xi)}{f'(x_n)}$$

From Newton's method $-\frac{f(x_n)}{f'(x_n)} = x_{n+1} - x_n$

so $$r = x_n + (x_{n+1} - x_n) - \frac{(r-x_n)^2}{2} \frac{f''(\xi)}{f'(x_n)}$$

$$r - x_{n+1} = -\frac{(r-x_n)^2}{2} \frac{f''(\xi)}{f'(x_n)}$$

(b) What is the main assumption under which Newton's Method will converge?

$$\Rightarrow \quad \frac{e_{n+1}}{e_n^2} = \frac{-f''(\xi)}{2f'(x_n)}$$ $\Rightarrow$ quadratic convergence if we bound quantity on rhs.

(b) Need to have a good starting guess x_0 for root.

[10 pts](6) Consider the iteration

$$x_{n+1} = 6.28 + \sin(x_n)$$

where $n \geq 0$. The true root is $\alpha = 6.01550307297$. The results of applying the functional iteration algorithm are given in the table below.

(a) Does the functional iteration algorithm converge for this problem? **(Note: to receive credit you must use the theory to justify your answer!)**
(b) Will the iterates converge quickly or slowly to the root α? Why?

n	Computed x_n	$\alpha - x_n$
0	6.0000000	1.55E–2
1	6.0005845	1.49E–2
2	6.0011458	1.44E–2
3	6.0016848	1.38E–2
4	6.0022026	1.33E–2
5	6.0027001	1.28E–2
6	6.0031780	1.23E–2
7	6.0036374	1.18E–2

$X_{n+1} = f(X_n)$

we look at $f'(X_n)$ at root α

$f'(X_n) = \frac{d}{dx}[6.28 + \sin(X_n)]$

$= \cos(X_n)$

$f'(\alpha) = \cos(6.01550307297) = .9644$
< 1

(a) Yes, the functional iteration converges for this problem.

(b) Iteration will converge very slowly to root α. Rate of convergence is close to 1 (rate $= .9644$).

Please sign the following honor statement: *On my honor, I pledge that I have neither given nor received any aid on this exam.*